Structure of Tempered Homogeneous Spaces

II. Combinatorics Approach

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Minicourses

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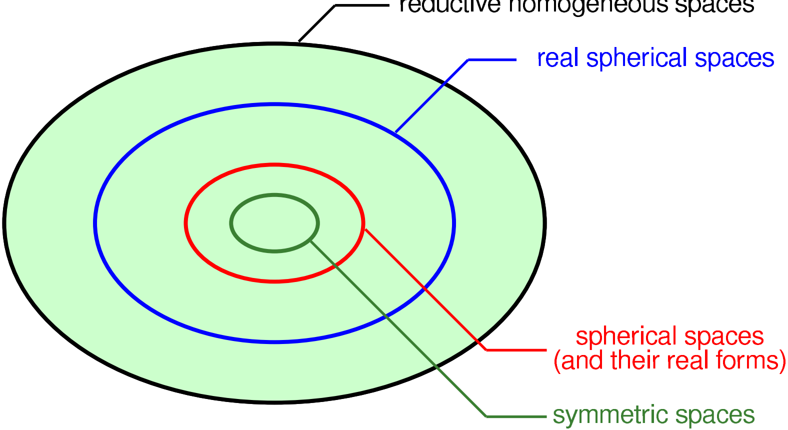
Reductive homogeneous space G/H

G: real reductive groups

H: reductive subgroup



Plan of Lectures

- Talk 1: (February 17, 2025)
  Tempered homogeneous spaces
  —Dynamical approach : L^q estimate of $\mathrm{vol}(gS \cap S)$

- Talk 2: ~~(February 19, 2025)~~ (February 20, 2025)
  Classification theory of "tempered space" G/H
  —Combinatorics of convex polyhedra

- Talk 3: (February 21, 2025)
  Tempered homogeneous spaces
  —Interaction with topology and geometry

References

The theme of the mini-course is joint with Yves Benoist.

Tempered Homogeneous Spaces:

- ——— I. (J. Euro Math., 2015)
  Method (Dynamical System)
- ——— II. (Margulis Festschrift, 2022, Chicago Univ. Press)
  Representation Theory
- ——— III. (J. Lie Theory, 2021)
  Classification Theory (Combinatorics)
- ——— IV. (J. Inst. Math. Jussieu, 2023)
  Limit algebra, geometric quantization

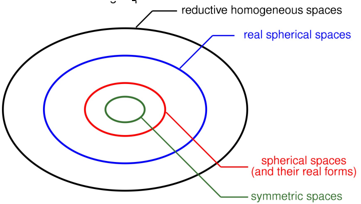
Tensor product of GL_n (J. Algebra, 2023)

Main goal for today — Classification theory

Quite surprisingly, it turns out that a complete description of tempered reductive homogeneous spaces G/H is realistic.

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is tempered.

Why ? How ?



* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Reminder: Tempered spaces and tempered subgroups

$G \supset H$ Lie groups

- Induction $H \uparrow G$ $\cdots$ $L^2(G/H) \ll L^2(G)$.

Definition We say G/H is a tempered homogeneous space if $L^2(G/H)$ is a tempered rep of G.

- Restriction $G \downarrow H$ $\cdots$ $\pi|_H \ll L^2(H)$

Definition We say H is a G-tempered subgroup if $\pi|_H$ is a tempered rep of H for any $\pi \in \widehat{G} \setminus \{\mathbf{1}\}$.

cf. Margulis used "G-tempered subgroup" in a different sense.

Plan of Lectures

- Talk 1: (February 17, 2025)

  Tempered homogeneous spaces

  —Dynamical approach

- Talk 2: ~~(February 19, 2025)~~ (February 20, 2025)

  Classification theory of tempered G/H

  — Combinatorics of convex polyhedra

  Definition** (Lecture 1)

  $$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$$

- Talk 3: (February 21, 2025)

  Tempered homogeneous spaces

  —Interaction with topology and geometry

Reminder $p_V \in \mathbb{R}_{>0}$

Let $\mathfrak{h}$ be a Lie algebra, and $\mathfrak{a}$ its max split abelian subalgebra.

For a finite-dimensional rep $\tau\colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduced:

Definition* (Lecture 1: piecewise linear function ρ_V)

$$\rho_V : \mathfrak{a} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum | \text{eigenvalues of } Y \curvearrowright V |.$$

Definition** (Lecture 1)

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalues of } Y \curvearrowright \mathfrak{h}|}{\sum |\text{eigenvalues of } Y \curvearrowright V|}.$$

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

** Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Reminder: Main results in Lecture 1

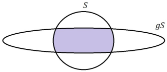
Let H be a semisimple Lie group.
Consider $H \to SL_{\mathbb{R}}(V)$ and $H \subset G$ (reductive).

Theorems A and B* (Lecture 1) ($L^2(G/H)$)

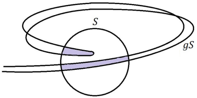
$p_V \leq 2 \iff H \curvearrowright L^2(V)$ is tempered.

$p_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff G/H$ is a tempered homogeneous space.

$\Longleftarrow$ easier (local estimate)



$\Longrightarrow$ more difficult (global estimate)



* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

Reminder from Lecture 1

p_V (combinatorics) $\Longleftrightarrow$ Analytic Rep Theory

Theorems A and B* (Lecture 1) ($L^2(G/H)$)

$p_V \leq 2 \quad \Longleftrightarrow \quad H \curvearrowright L^2(V)$ is tempered.

$p_{\mathfrak{g}/\mathfrak{h}} \leq 1 \quad \Longleftrightarrow \quad G/H$ is a tempered homogeneous space.

Theorem E** ($G \downarrow H$) Let $G := SL(n, \mathbb{R})$ and H a reductive subgp.
Let $H \curvearrowright V := \mathbb{R}^n$ be the natural rep.
Then one has the equivalence:

(1) $p_V < 1 \quad \Longleftrightarrow \quad H$ is a Margulis G-tempered subgroup***.

(2) $p_V \leq 2 \quad \Longleftrightarrow \quad H$ is a tempered subgroup.

* Y. Benoist–T. Kobayashi, Tempered reductive homogeneous spaces, J. Eur. Math. Soc. **17** (2015), 3015–3036.

** K–, (to appear).

*** G. Margulis, Bull. Soc. Math. France **125** (1997), 447–456.

Main theme of Lecture 2

Basic Problem Classify all non-tempered homogeneous spaces.

⇑ (wavy arrow)

Combinatorics

Understand the number p_V associated to $\tau\colon H \to GL_{\mathbb{R}}(V)$.

Combinatorics for p_V

Very special cases of combinatorics for p_V have already interactions with

- Kazhdan's estimate $(SL(3,\mathbb{R}) \downarrow SL(2,\mathbb{R}) \ltimes \mathbb{R}^2)$,
- Tempered subgroup a la Margulis,
- Minimal K-type theory of Vogan,

  (G, H) symemtric pair, H split
- Plancherel formula for G/H,

  (G, H) semisimple symemtric pair
- Vanishing condition of gen. Borel–Weil–Bott theorem,

  Zuckerman's module $A_{\mathfrak{q}}(\lambda)$ with singular parameter λ,

and more.

Want to understand $\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}$ and $p_V \in \mathbb{R}_{>0}$

Let $\mathfrak{h}$ be a Lie algebra, and $\mathfrak{a}$ its max split abelian subalgebra.

For a finite-dimensional rep $\tau : \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we introduced:

Definition* (Lecture 1: piecewise linear function ρ_V)

$$\rho_V : \mathfrak{a} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\text{eigenvalues of } Y \curvearrowright V|.$$

Definition** (Lecture 1)

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} = \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\sum |\text{eigenvalues of } Y \curvearrowright \mathfrak{h}|}{\sum |\text{eigenvalues of } Y \curvearrowright V|}.$$

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Basic properties of ρ_V

- For an exact sequence $0 \to W \to V \to V/W \to 0$ of $\mathfrak{h}$-modules, one has

$$\rho_V = \rho_W + \rho_{V/W}.$$

- (contragredient rep) $\rho_V = \rho_{V^*}$

Example ($\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$)

For $0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h} \to 0$ as $\mathfrak{h}$-modules, one sees

$$p_{\mathfrak{g}/\mathfrak{h}} \le 1 \iff \rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}/\mathfrak{h}} \iff 2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}$$

($\underset{\text{Theorem B}}{\iff}$ $G \curvearrowright L^2(G/H)$ is tempered rep)

Definition** (Lecture 1)

$$p_V := \max_{Y \in \mathfrak{a} \setminus \{0\}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)}$$

Elementary example: computation of ρ_V

Definition* (Lecture 1: piecewise linear function ρ_V)

$\rho_V : \mathfrak{a} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |$ eigenvalues of $Y \curvearrowright V$ $|$.

$$\mathfrak{h} := \mathfrak{sl}(p, \mathbb{R}) \to \mathrm{End}_{\mathbb{R}}(V)$$

$$\mathfrak{a} := \{X = \mathrm{diag}(x_1, \ldots, x_p) : \sum x_i = 0\}$$

Example 1) $V = \mathbb{R}^p$

{Eigenvalues of $X \curvearrowright \mathbb{R}^p\} = \{x_i : 1 \leq i \leq p\}$

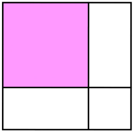
$$\rho_V = \frac{1}{2} \sum_{i=1}^{p} |x_i|$$

Example 2) $V = \mathfrak{h}$ (adjoint representation)

{Eigenvalues of $\mathrm{ad}(X)\} = \{x_i - x_j : 1 \leq i \neq j \leq p\}$

$$\rho_{\mathfrak{h}} = \sum_{1 \leq i < j \leq p} |x_i - x_j|$$

Example $G = SL(3,\mathbb{R}) \supset H = SL(2,\mathbb{R})$



$$\mathfrak{a} = \{\mathrm{diag}(x_1, x_2, 0) : x_1 + x_2 = 0\}$$

$\mathfrak{h} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{h}$ $\qquad \rho_{\mathfrak{h}} = |x_1 - x_2| = 2|x_1|$

Example 1 (G/H is a tempered space.)

Proof $\mathfrak{h} \overset{\mathrm{ad}}{\curvearrowright} \mathfrak{g}/\mathfrak{h}$ $\qquad \rho_{\mathfrak{g}/\mathfrak{h}} = |x_1| + |x_2| = 2|x_1| \qquad \therefore\ p_{\mathfrak{g}/\mathfrak{h}} = 1$

$L^2(G/H)$ is tempered $\overset{\text{Theorem B}}{\Longleftrightarrow}$ $p_{\mathfrak{g}/\mathfrak{h}} \leq 1$ Yes!

Example 2 (H is a tempered subgroup of G)

Proof $\mathfrak{h} \curvearrowright V = \mathbb{R}^3$ $\qquad \rho_V = \frac{1}{2}(|x_1| + |x_2|) = |x_1| \qquad \therefore\ p_V = 2.$

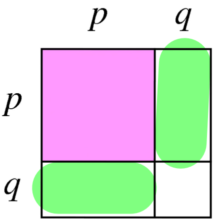
$\pi|_H$ is tempered $^\forall\pi \in \widehat{G} \setminus \{\mathbf{1}\}$ $\overset{\text{Theorem E}}{\Longleftrightarrow}$ $p_V \leq 2$ Yes!

$(G, H) = (SL(p+q,\mathbb{R}), SL(p,\mathbb{R}))$

$\mathfrak{h} = \mathfrak{sl}(p,\mathbb{R})$

$\cup$

$\mathfrak{a} = \{x = \mathrm{diag}(x_1, \ldots, x_p) : x_1 + \cdots + x_p = 0\}.$



$$\rho_{\mathfrak{h}}(x) = \sum_{1 \le i < j \le p} |x_i - x_j|$$

$$\rho_{\mathfrak{g}/\mathfrak{h}}(x) = q \sum_{i=1}^{p} |x_i|$$

$L^2(G/H)$ is tempered (*i.e.*, G/H is a tempered space)

$$\underset{\text{Theorem B}}{\iff} \rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}/\mathfrak{h}}$$

$$\iff \sum_{1 \le i < j \le p} |x_i - x_j| \le q \sum_{i=1}^{p} |x_i| \quad \text{whenever } \sum_{i=1}^{p} x_i = 0.$$

For which (p, q) does this happen?

Combinatorial problem $p_{\mathfrak{g}/\mathfrak{h}} \leq 1$

Question Find a necessary and sufficient condition on (p, q) such that

$$\sum_{1 \leq i < j \leq p} |x_i - x_j| \leq q \sum_{i=1}^{p} |x_i| \qquad (*)$$

for all $(x_1, \ldots, x_p) \in \mathbb{R}^p$ with $x_1 + \cdots + x_p = 0$.

This is an inequality for piecewise linear functions.
$\cdots$ Enough to check finitely many inequalities at the edges of convex polyhedral cones.

Answer $p \leq q + 1$

Necessity Let $x = (1, 0, \ldots, 0, -1)$ (witness vector).

Then $(*) \iff 2 + 2(p-2) \leq 2q \iff p - 1 \leq q$.

Main theme of Lecture 2

Basic Problem Classify all non-tempered homogeneous spaces.

$\Uparrow$

Combinatorics

Understand the number p_V associated to $\tau\colon H \to GL_{\mathbb{R}}(V)$.

Plan of Lecture 2

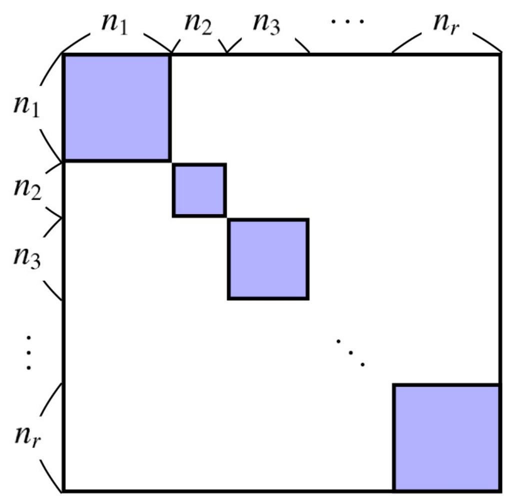
1. Reminder from Lecture 1:
   - Criterion for $L^2(X)$ to be almost L^p representation

2. Example. Computation of p_V

3. Example. $SL(p+q+r)/SL(p) \times SL(q) \times SL(r)$

4. Classification theory of reductive tempered homogeneous spaces

$$G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$
$$n_1 + n_2 + \cdots + n_r = n$$



$$G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$
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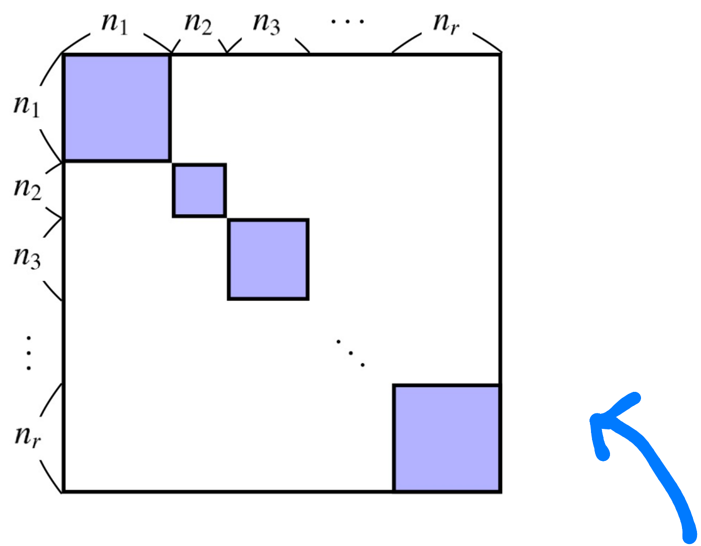
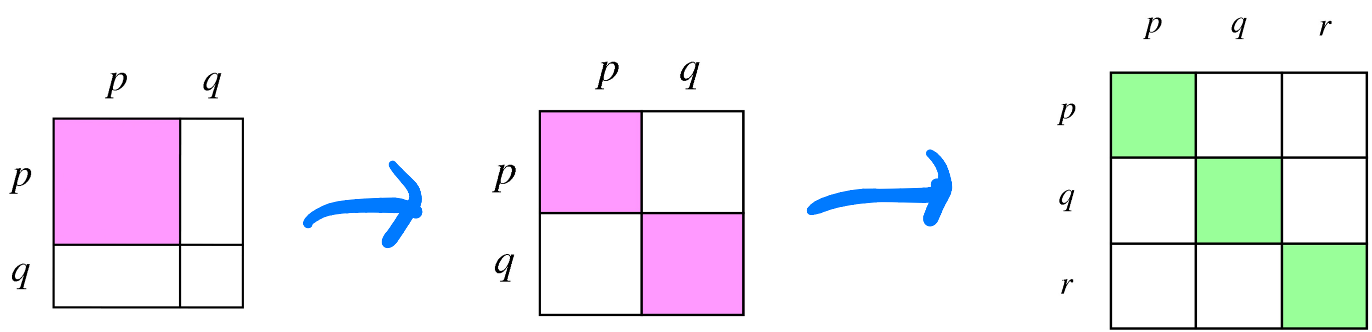


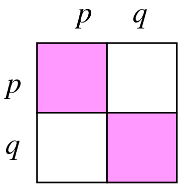


Combinatorics of temperedness criterion — Example

When is the unitary rep $L^2(G/H)$ tempered ($\Leftrightarrow$ almost L^2)?

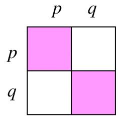
Consider an example with 2 parameters:

$$G/H = SL(p+q,\mathbb{R})/SL(p,\mathbb{R}) \times SL(q,\mathbb{R}).$$



Combinatorics of temperedness criterion — Example

Find a condition on (p, q) such that $G \curvearrowright L^2(G/H)$ is tempered



$$G/H = SL(p+q,\mathbb{R})/SL(p,\mathbb{R}) \times SL(q,\mathbb{R}).$$

Our temperedness criterion $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ amounts to the following:

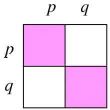
$$\sum_{1 \leq i < j \leq p} \left| x_i - x_j \right| + \sum_{1 \leq i < j \leq q} \left| y_i - y_j \right| \leq \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \left| x_i - y_j \right|$$

for all $(x_1, \cdots, x_p, y_1, \cdots, y_q) \in \mathbb{R}^{p+q}$ with $\sum x_i = 0$, $\sum y_j = 0$.

Combinatorics of temperedness criterion — Example

$$\sum_{1\le i<j\le p} \left| x_i - x_j \right| + \sum_{1\le i<j\le q} \left| y_i - y_j \right| \le \sum_{\substack{1\le i\le p \\ 1\le j\le q}} \left| x_i - y_j \right|$$

for all $(x_1, \cdots, x_p, y_1, \cdots, y_q) \in \mathbb{R}^{p+q}$ with $\sum x_i = 0$, $\sum y_j = 0$.



Evaluations at very special edges:

$(x_1, \cdots, x_p, y_1, \cdots, y_q) = (1, 0, \ldots, 0, -1;\ 0, \ldots, 0)$ yields $p - q \le 1$,

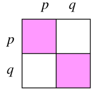
$(x_1, \cdots, x_p, y_1, \cdots, y_q) = (0, \ldots, 0;\ 1, 0, \ldots, 0, -1)$ yields $-1 \le p - q$.

Hence $|p - q| \le 1$ is a necessary condition. However, we still need to check finite but "huge number" of edges.

Combinatorics of temperedness criterion — Example

Find a condition on (p, q) such that $G \curvearrowright L^2(G/H)$ is tempered

$$G/H = SL(p+q, \mathbb{R})/SL(p, \mathbb{R}) \times SL(q, \mathbb{R}).$$



Our temperedness criterion $\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}/\mathfrak{h}}$ amounts to the following:

$$\sum_{1 \le i < j \le p} |x_i - x_j| + \sum_{1 \le i < j \le q} |y_i - y_j| \le \sum_{\substack{1 \le i \le p \\ 1 \le j \le q}} |x_i - y_j|$$

for all $(x_1, \cdots, x_p, y_1, \cdots, y_q) \in \mathbb{R}^{p+q}$ with $\sum x_i = 0$, $\sum y_j = 0$.

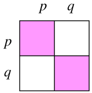
$\Longleftrightarrow |p - q| \le 1.$

We have two interpretations.

$\Longleftrightarrow$ (1) $G_{\mathbb{R}} = SU(p, q)$ is quasi-split $\leftrightsquigarrow$ (G, H) symmetric pair.

$\Longleftrightarrow$ (2) $2 \max(p, q) \le p + q + 1.$

$$(G,H) = (GL(p+q,\mathbb{R}), GL(p,\mathbb{R}) \times GL(q,\mathbb{R}))$$



In this very particular case (*i.e.*, H is split & (G,H) is symmetric pair), the function

$$\rho_{\mathfrak{g}} - 2\rho_{\mathfrak{h}}$$

appeared in a different context, namely,

Harish-Chandra's parameter — Blattener parameter

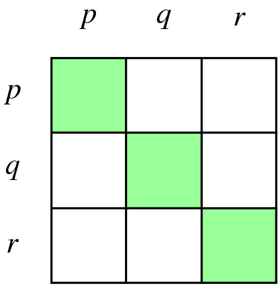
for discrete series representations, and the combinatorial techniques have been developed by many experts including Parthasarathy, Vogan, among others.

Combinatorics of temperedness criterion — Example

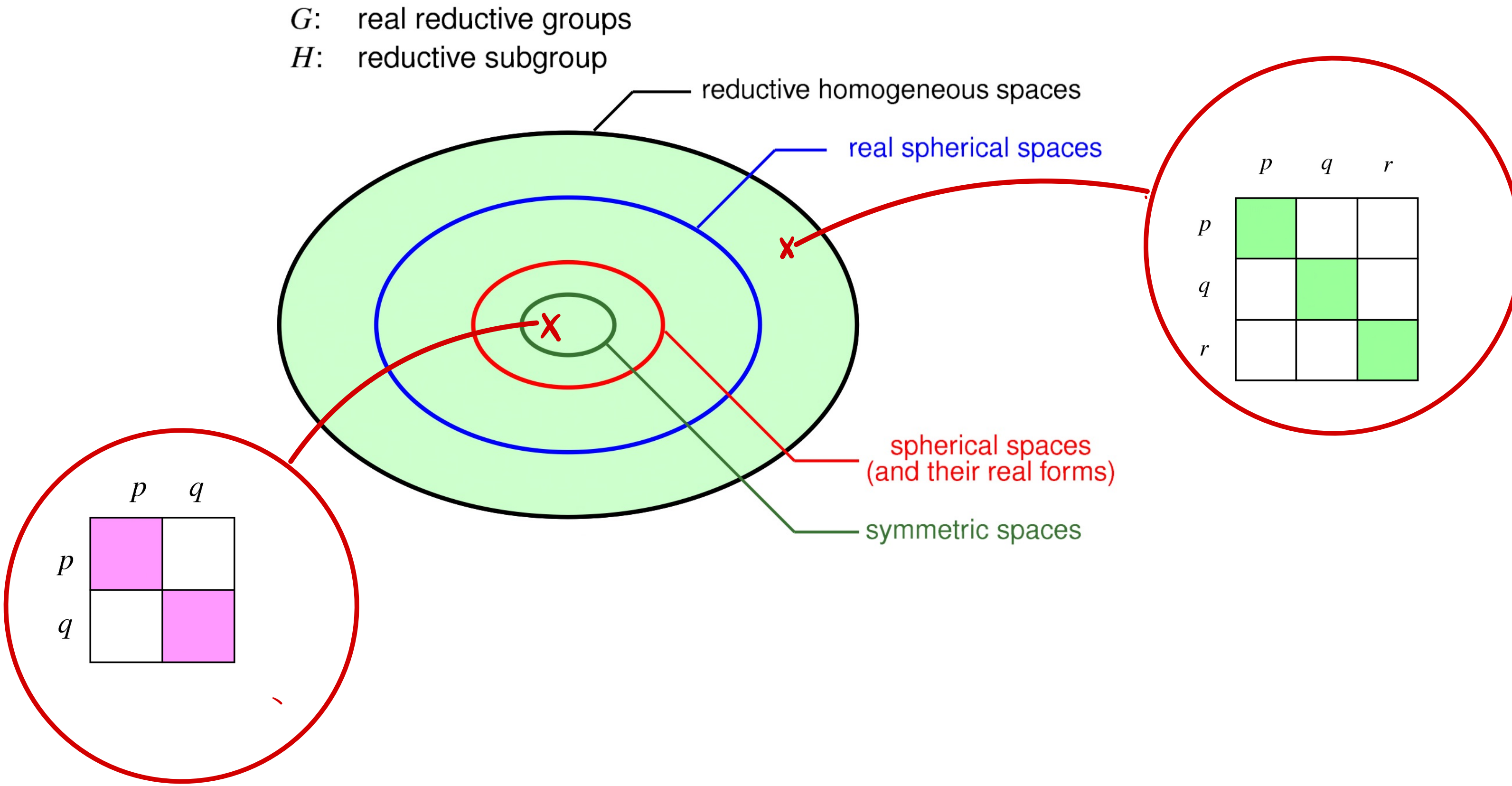
When is $L^2(G/H)$ is tempered ($\Leftrightarrow$ almost L^2)?

Consider a non-symmetric space with three parameters:

$$G/H = SL(p+q+r,\mathbb{R})/SL(p,\mathbb{R}) \times SL(q,\mathbb{R}) \times SL(r,\mathbb{R}).$$



$$G/H = SL(p+q+r,\mathbb{R})/SL(p,\mathbb{R}) \times SL(q,\mathbb{R}) \times SL(r,\mathbb{R}).$$



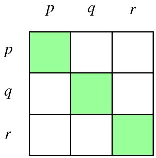
Combinatorics of temperedness criterion — Example

$$G/H = SL(p+q+r,\mathbb{R})/SL(p,\mathbb{R})\times SL(q,\mathbb{R})\times SL(r,\mathbb{R}).$$

Our temperedness criterion $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ amounts to the following:

$$\sum_{1\leq i<j\leq p} |x_i - x_j| + \sum_{1\leq i<j\leq q} |y_i - y_j| + \sum_{1\leq i<j\leq r} |z_i - z_j|$$
$$\leq \sum_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} |x_i - y_j| + \sum_{\substack{1\leq j\leq q\\ 1\leq k\leq r}} |y_j - z_k| + \sum_{\substack{1\leq k\leq r\\ 1\leq i\leq p}} |z_k - x_i|$$

for all $(x_1,\cdots,x_p, y_1,\cdots,y_q, z_1,\cdots,z_r) \in \mathbb{R}^{p+q+r}$ with $\sum x_i = 0$, $\sum y_j = 0$, $\sum z_k = 0$.



Combinatorics of temperedness criterion — Example

$$G/H = SL(p+q+r,\mathbb{R})/SL(p,\mathbb{R}) \times SL(q,\mathbb{R}) \times SL(r,\mathbb{R}).$$

Our temperedness criterion $\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}$ amounts to the following:

$$\sum_{1\leq i<j\leq p} |x_i - x_j| + \sum_{1\leq i<j\leq q} |y_i - y_j| + \sum_{1\leq i<j\leq r} |z_i - z_j|$$

$$\leq \sum_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} |x_i - y_j| + \sum_{\substack{1\leq j\leq q\\ 1\leq k\leq r}} |y_j - z_k| + \sum_{\substack{1\leq k\leq r\\ 1\leq i\leq p}} |z_k - x_i|$$

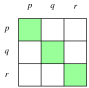
for all $(x_1,\cdots,x_p, y_1,\cdots,y_q, z_1,\cdots,z_r) \in \mathbb{R}^{p+q+r}$ with $\sum x_i = 0$, $\sum y_j = 0$, $\sum z_k = 0$.

$$\Longleftrightarrow 2\max(p,q,r) \leq p+q+r+1.$$

$\cdots$ combinatorics on convex polyhedral cones

Example: $H := SL(p,\mathbb{R}) \times SL(q,\mathbb{R}) \times SL(r,\mathbb{R})$

Consider two homomorphisms:



$$H \hookrightarrow SL(p+q+r,\mathbb{R}) =: G, \qquad (1)$$

$$H \to SL(pq+qr+rp,\mathbb{R}) =: \widetilde{G}. \qquad (2)$$

(2) is defined via $H \curvearrowright V := \mathrm{Hom}(\mathbb{R}^q,\mathbb{R}^p) \oplus \mathrm{Hom}(\mathbb{R}^r,\mathbb{R}^p) \oplus \mathrm{Hom}(\mathbb{R}^r,\mathbb{R}^q)$.

Consider 3 unitary reps $H \curvearrowright L^2(V)$, $\widetilde{G} \downarrow H$, and $G \curvearrowright L^2(G/H)$:

Example One has an equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv):

(i) $H \curvearrowright L^2(V)$ is a tempered rep of H.

(ii) For any irred unitary rep π $(\neq \mathbf{1})$ of $\widetilde{G} = SL(pq+qr+rp,\mathbb{R})$,
the restriction $\pi|_H$ via (2) is a tempered representation of H.

(iii) $L^2(G/H)$ is a tempered rep of $G = SL(p+q+r,\mathbb{R})$.

(iv)* $2\max(p,q,r) \leq p+q+r+1$.

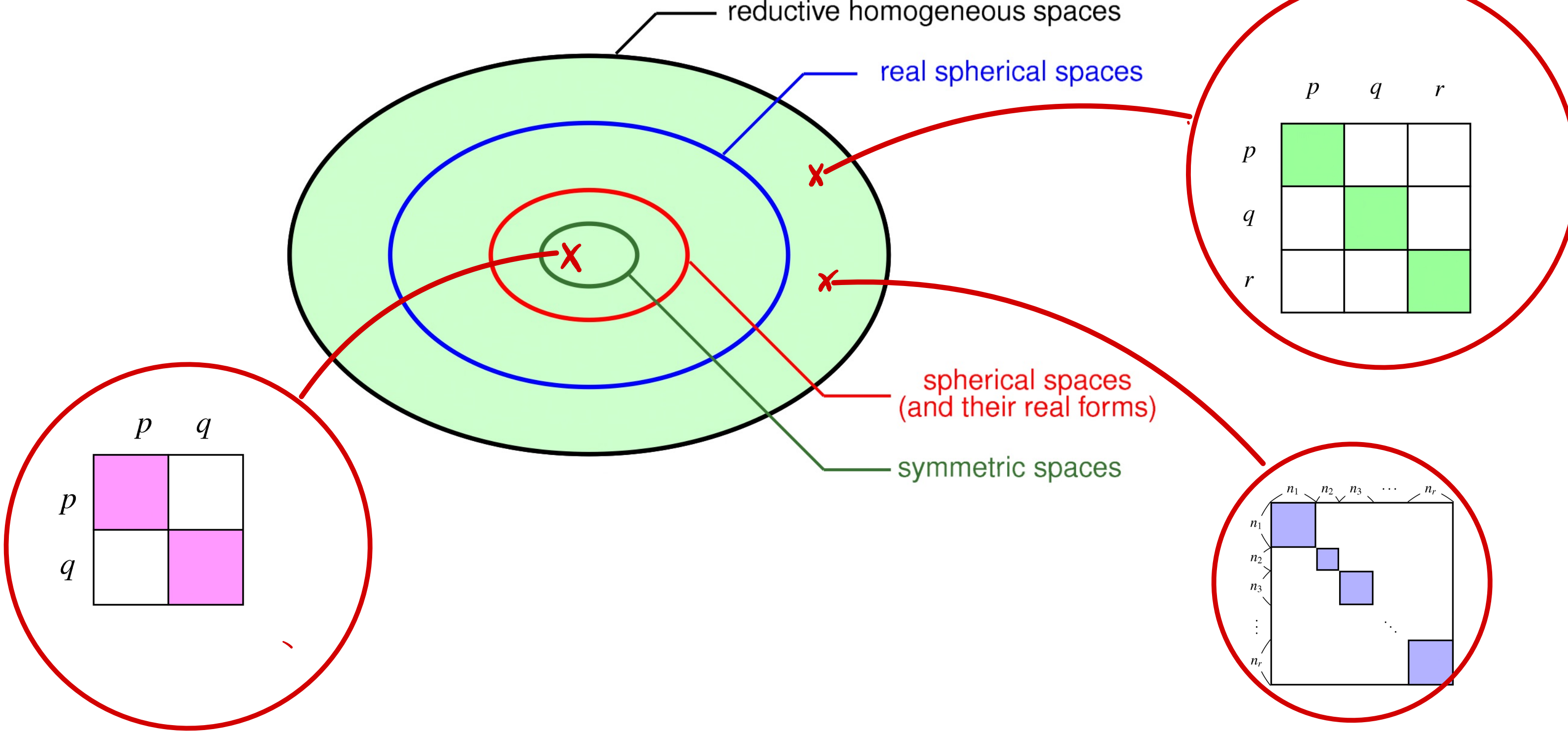
(i) and (iii) $\cdots$ Theorem B; (ii) $\cdots$ Theorem E (to appear).

* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021) for the combinatorics (iii) $\Leftrightarrow$ (iv).

$$G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$

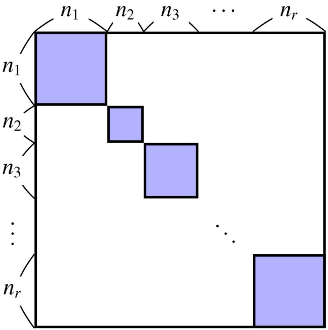
$$n_1 + n_2 + \cdots + n_r = n$$

G: real reductive groups
H: reductive subgroup



$$X = G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$

Find the optimal constant $q(G;X)$ such that $\mathrm{vol}(gS \cap S)$ is almost $L^q(G)$, for any compact set $S \subset X$.



* Y. Benoist–Y. Inoue–T. Kobayashi, J. Algebra (2023).

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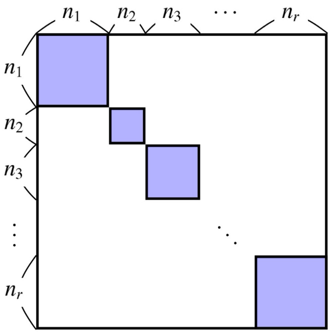
<u>Theorem</u> ('23)* Let $m := \max(n_1, \ldots, n_r)$. Then

$$q(G;X) \doteq \frac{n-1}{n-m}.$$



* Y. Benoist–Y. Inoue–T. Kobayashi, J. Algebra (2023).

$$X = G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$

Find the optimal constant $q(G;X)$ such that $\mathrm{vol}(gS \cap S)$ is almost $L^q(G)$, for any compact set $S \subset X$.

<u>Theorem</u> ('23)* Let $m := \max(n_1, \ldots, n_r)$. Then

$$q(G;X) = \frac{n-1}{n-m}.$$

$$\frac{1}{1 + \frac{\rho_{\mathfrak{g}/\mathfrak{h}}}{\rho_{\mathfrak{h}}}} = \frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}}} = \frac{\sum_{i=1}^{r} \sum_{a,b \in \{i\text{-th block}\}} |x_a - x_b|}{\sum_{1 \le a < b \le n} |x_a - x_b|}$$

Idea: Where is the maximum of $\frac{\rho_{\mathfrak{h}}}{\rho_{\mathfrak{g}/\mathfrak{h}}}$ attained?

$\rightsquigarrow$ Candidate: 2^n edges of convex polyhedral cone.

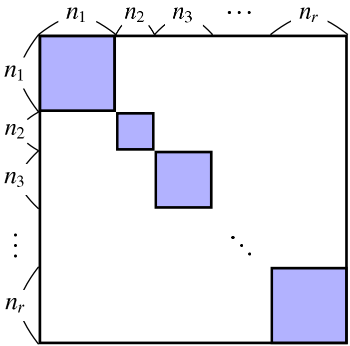
$\rightsquigarrow$ (trick) n edges of convex polyhedral cone.

* Y. Benoist–Y. Inoue–T. Kobayashi, J. Algebra (2023).

Non-tempered reductive homogeneous space

What is the best p for which $L^2(G/H)$ is almost L^p?

$$G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$$
$$n_1 + n_2 + \cdots + n_r = n$$



almost L^p criterion (recall from Lecture 1)

Let G be a semisimple Lie group, H a reductive subgroup, and $X = G/H$.

Theorem B (Lecture 1) The optimal constant $q(G;X)$ such that $\mathrm{vol}(gS \cap S)$ is almost L^q for any compact subset S in X is given by

$$q(G;X) = 1 + p_{\mathfrak{g}/\mathfrak{h}}.$$

Concerning the regular rep $G \curvearrowright L^2(X)$ for p even,

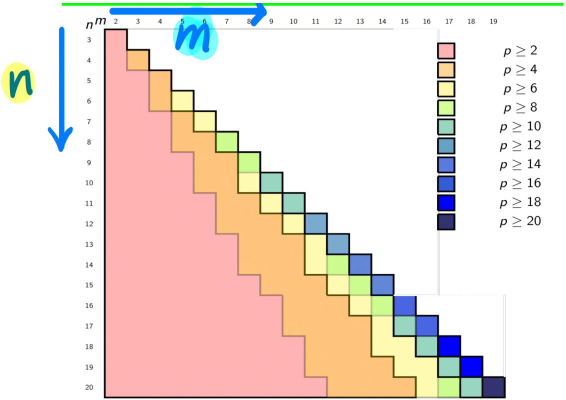
$$L^2(X) \text{ is almost } L^p \iff 1 + p_{\mathfrak{g}/\mathfrak{h}} \leq p \iff \rho_{\mathfrak{h}} \leq (p-1)\rho_{\mathfrak{g}/\mathfrak{h}}.$$

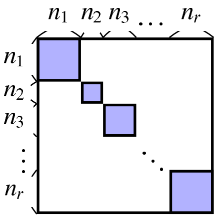
$$L^2(X) \text{ is tempered} \iff p_{\mathfrak{g}/\mathfrak{h}} \leq 1 \iff \rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}/\mathfrak{h}}.$$

The temperedness criterion holds also for a non-reductive subgroup H.

Almost L^p representation

Example $G/H = GL(n,\mathbb{R})/GL(n_1,\mathbb{R}) \times \cdots \times GL(n_r,\mathbb{R})$
The smallest even integer p for which $L^2(G/H)$ is almost L^p amounts to $p = 2[\frac{n-1}{2(n-m)}]$ with $m = \max(n_1, \cdots, n_r)$.





* Y. Benoist–Y. Inoue–T. Kobayashi, J. Algebra (2023).

Plan of Lecture 2

1. Reminder from Lecture 1:
   - Criterion for $L^2(X)$ to be almost L^p representation

2. Example. Computation of p_V

3. Example. $SL(p+q+r)/SL(p) \times SL(q) \times SL(r)$

4. Classification of reductive tempered homogeneous spaces

Classification theory — theorem

Quite surprisingly, it turns out that a complete description of non-tempered reductive homogeneous spaces G/H is realistic.

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

Example For $n_1 + \cdots + n_r \leq n$, we consider

$$G/H := GL(n, \mathbb{R})/GL(n_1, \mathbb{R}) \times \cdots \times GL(n_r, \mathbb{R}).$$

$L^2(G/H)$ is non-tempered $\iff \max_i n_i > \frac{1}{2}(n+1)$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory — theorem

Quite surprisingly, it turns out that a complete description of non-tempered reductive homogeneous spaces G/H is realistic.

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

Example For $p_1 + \cdots + p_r \leq p$ and $q_1 + \cdots + q_r \leq q$, we consider

$$G/H := SO(p,q)/(SO(p_1,q_1) \times SO(p_2,q_2) \times \cdots \times SO(p_r,q_r)).$$

$L^2(G/H)$ is non-tempered $\iff \max_{p_i q_i \neq 0} (p_i + q_i) > \frac{1}{2}(p+q+2)$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory of non-tempered G/H — Strategy

Setting: $G \supset H$ both real reductive.

Step 1. Reduction

1.A. G reductive $\Longrightarrow$ G simple (perfect)

1.B. (G, H) real $\Longrightarrow$ $(G_{\mathbb{C}}, H_{\mathbb{C}})$ (useful)

Step 2. Classify non-tempered $G_{\mathbb{C}}/H_{\mathbb{C}}$ when $G_{\mathbb{C}}$ is complex simple.

2.A. Combinatorics for p_V for simple $H \curvearrowright V$ (irreducible)

2.B. Combinatorics for p_V for reductive $H \curvearrowright V$(reducible)

Step 3. Understand non-tempered $G_{\mathbb{C}}/H_{\mathbb{C}}$ for complex simple $G_{\mathbb{C}}$.

Step 4. Determine which real forms of $G_{\mathbb{C}}/H_{\mathbb{C}}$ are non-tempered.

Classifying non-tempered G/H — Step 1. Reduction

Setting: $G \supset H$ both real reductive.

Step 1.A. G reductive $\Rightarrow$ G simple

For $H \subset G = G_1 \times \cdots \times G_n$, we set $H_i := H \cap G_i$.

$$L^2(G/H) \text{ is tempered } \underset{\text{difficult}}{\overset{\text{easy}}{\rightleftarrows}} L^2(G_i/H_i) \text{ is tempered } \ {}^\forall i.$$

Criterion $\Updownarrow$ (Lecture 1) $\qquad\qquad$ $\Updownarrow$ Criterion (Lecture 1)

$$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}} \qquad \Longleftrightarrow \qquad 2\rho_{\mathfrak{h}_i} \leq \rho_{\mathfrak{g}_i} \ {}^\forall i.$$

Example If $\mathfrak{h} \cap \mathfrak{g}_i = \{0\}$ ${}^\forall i$, then $L^2(G/H)$ is tempered.

Classification theory of non-tempered G/H — Strategy

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Classifying non-tempered G/H — Step 1. Reduction

Setting: $G \supset H$ both real reductive.

Step 1.A. G reductive $\Rightarrow$ G simple

Step 1.B. (G, H) real $\Longrightarrow$ $(G_{\mathbb{C}}, H_{\mathbb{C}})$

$$
\begin{array}{ccc}
L^2(G_{\mathbb{C}}/H_{\mathbb{C}}) \text{ is tempered} & \Longrightarrow & L^2(G/H) \text{ is tempered.} \\
\text{Criterion} \Updownarrow & & \Updownarrow \text{Criterion} \\
2\rho_{\mathfrak{h}_{\mathbb{C}}} \le \rho_{\mathfrak{g}_{\mathbb{C}}} & \Longrightarrow & 2\rho_{\mathfrak{h}} \le \rho_{\mathfrak{g}}.
\end{array}
$$

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Step 3.

Step 4.

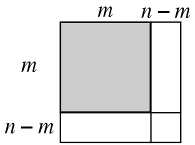
Classification — feature: “huge factors” in $H_{\mathbb{C}}$

Point $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is non-tempered only if $H_{\mathbb{C}}$ has a “huge factor”.

Theorem H (“huge factor”) * Let $G_{\mathbb{C}}$ be a simple Lie group, and $H_{\mathbb{C}}$ a reductive subgroup. If $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is non-tempered, then $H_{\mathbb{C}}$ is “huge” in the following sense.

(Type A) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$, then $\mathfrak{h}_{\mathbb{C}}$ contains

- $\mathfrak{sl}(m, \mathbb{C})$ with $m > \frac{1}{2}(n+1)$ or
- $\mathfrak{sp}(m, \mathbb{C})$ with $n = 2m$.



. . .

(Type E_7) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}_7^{\mathbb{C}}$, then $\mathfrak{h}_{\mathbb{C}}$ contains $\mathfrak{d}_6^{\mathbb{C}}$ or $\mathfrak{e}_6^{\mathbb{C}}$.

(Type E_8) If $\mathfrak{g}_{\mathbb{C}} = \mathfrak{e}_8^{\mathbb{C}}$, then $\mathfrak{h}_{\mathbb{C}}$ contains $\mathfrak{e}_7^{\mathbb{C}}$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Tool

Let $\mathfrak{g}$ be a complex simple Lie algebra.
Want to find a subalgebra $\mathfrak{h}$ s.t. $p_{\mathfrak{g}/\mathfrak{h}} \leq 1$ (temperedness criterion).

For a representation $\tau \colon \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(V)$, we defined

$$p_V = \max_{Y \in \mathfrak{h}} \frac{\rho_{\mathfrak{h}}(Y)}{\rho_V(Y)} \quad (\geq 0).$$

Preparation in a more general setting:

- Analyze when $p_V > 1$ for a representation (τ, V).

$\cdots$ Finite inequalities on generators of covex polyhedral cones.

("exponential time" $\Rightarrow$ "polynomial time")

Case 1 $\mathfrak{h}$ simple, (τ, V) irreducible.

Case 2 $\mathfrak{h} \curvearrowright V_1 \oplus V_2$.

Case 3 $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \curvearrowright V = V_1 \otimes V_2, \cdots$.

Example of p_V with $p_V > 1$

$$H \curvearrowright V \quad \text{(linear)} \quad \rightsquigarrow \quad p_V \in \mathbb{R}_{>0}.$$

<u>Example</u> Consider $H = SL(4,\mathbb{R}) \curvearrowright V$ irreducible

(1) $V = \mathbb{C}^4 \quad \Rightarrow p_V = 6.$

(2) $V = S^2(\mathbb{C}^4) \quad \Rightarrow p_V = \frac{3}{2}.$

(3) $V = \Lambda^2(\mathbb{C}^4) \quad \Rightarrow p_V = 3.$

(4) $V = \Lambda^3(\mathbb{C}^4) \quad \Rightarrow p_V = 6.$

If V or V^* is not in (1)–(4), then $p_V \leq 1$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory of non-tempered G/H — Strategy

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Classification theory: generic stabilizers of $H \curvearrowright \mathfrak{g}/\mathfrak{h}$

For a representation $\tau\colon H \to GL(V)$, we set $(V)_{\mathrm{Ab}} \subset (V)_{\mathrm{Am}}$ by

$$(V)_{\mathrm{Ab}} := \{x \in V : \text{the stabilizer } H_x \text{ is abelian}\},$$
$$(V)_{\mathrm{Am}} := \{x \in V : \text{the stabilizer } H_x \text{ is amenable}\}.$$

Classification theory includes:

Theorem I* $G \supset H$ be pairs of real reductive algebraic groups.
One has the implication (i) ⇒ (ii) ⇒ (iii).
(i) $(\mathfrak{g}/\mathfrak{h})_{\mathrm{Ab}}$ is dense in $\mathfrak{g}/\mathfrak{h}$.
(ii) $L^2(G/H)$ is a tempered unitary representation of G.
(iii) $(\mathfrak{g}/\mathfrak{h})_{\mathrm{Am}}$ is dense in $\mathfrak{g}/\mathfrak{h}$.

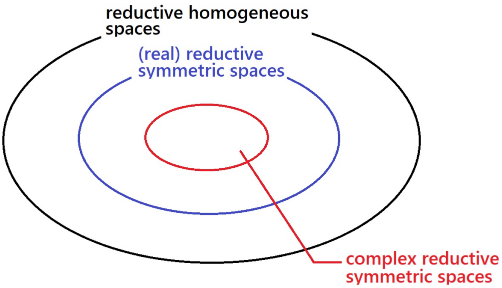
Corollary J $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is tempered iff $(\mathfrak{g}_{\mathbb{C}}/\mathfrak{h}_{\mathbb{C}})_{\mathrm{Ab}}$ is dense in $\mathfrak{h}_{\mathbb{C}}$.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

Classification theory: side remarks

Theorem G* One can give a complete description of pairs $G \supset H$ of real reductive algebraic groups for which $L^2(G/H)$ is not tempered.

- Special cases are already non-trivial



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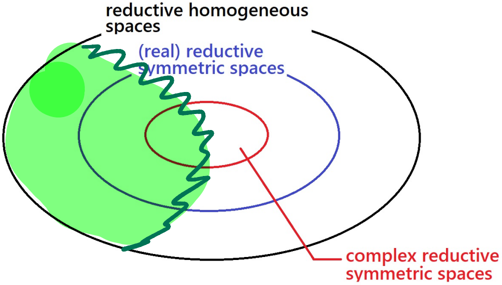


$L^2(G/H)$ is not tempered.

* Benoist–Kobayashi, Tempered homogeneous spaces III, J. Lie Theory **31** (2022), 833–869.

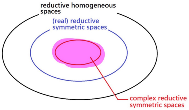
Combinatorics for p_V

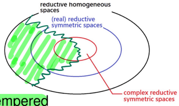
Very special cases of combinatorics for p_V have already interactions with

- Kazhdan's estimate ($SL(3,\mathbb{R}) \downarrow SL(2,\mathbb{R}) \ltimes \mathbb{R}^2$),
- Tempered subgroup a la Margulis,
- Minimal K-type theory of Vogan,
  (G, H) symemtric pair, H split
- Plancherel formula for G/H,
  (G, H) semisimple symemtric pair
- Vanishing condition of gen. Borel–Weil–Bott theorem,
  Zuckerman's module $A_{\mathfrak{q}}(\lambda)$ with singular parameter λ,

and more.

B. Classification theory: side remarks 1

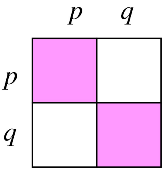




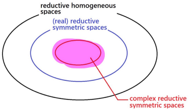
$L^2(G/H)$ is not tempered

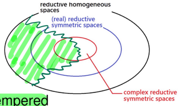
(a) Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be a complex reductive symmetric space.
Take a real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ such that $G_{\mathbb{R}} \cap H_{\mathbb{C}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$.

Example $G_{\mathbb{C}}/H_{\mathbb{C}} = GL(p+q,\mathbb{C})/GL(p,\mathbb{C}) \times GL(q,\mathbb{C})$
$\rightsquigarrow$ $G_{\mathbb{R}} = U(p,q)$.



B. Classification theory: side remarks 1





$L^2(G/H)$ is not tempered

(a) Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be a complex reductive symmetric space.
Take a real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ such that $G_{\mathbb{R}} \cap H_{\mathbb{C}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Corollary J in this special case implies that

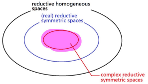
$$L^2(G_{\mathbb{C}}/H_{\mathbb{C}}) \text{ is } G_{\mathbb{C}}\text{-tempered} \iff G_{\mathbb{R}} \text{ is quasi-split.}$$

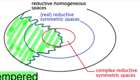
Vogan's minimal K-type theory tells us that

$$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}} \iff \mathfrak{g}_{\mathbb{R}} \text{ is quasi-split.}$$

Since $L^2(G_{\mathbb{C}}/H_{\mathbb{C}})$ is G-tempered $\iff 2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ (Lecture 1), this gives an alternative proof of Corollary J in this special case.

Classification theory: side remarks 1





$L^2(G/H)$ is not tempered

(a) Let $G_{\mathbb{C}}/H_{\mathbb{C}}$ be a complex reductive symmetric space.

Take a real form $G_{\mathbb{R}}$ of $G_{\mathbb{C}}$ such that $G_{\mathbb{R}} \cap H_{\mathbb{C}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$. Corollary J in this special case implies that

$$L^2(G_{\mathbb{C}}/H_{\mathbb{C}}) \text{ is } G_{\mathbb{C}}\text{-tempered} \iff G_{\mathbb{R}} \text{ is quasi-split.}$$

Vogan's theory on minimal K-types gives an alternative proof:

$2\rho_{\mathfrak{k}_{\mathbb{C}}} \leq \rho_{\mathfrak{g}_{\mathbb{C}}}$ $\underset{\text{Vogan}}{\iff}$ $G_{\mathbb{R}}$ is quasi-split

Lecture 1 $\Updownarrow$ Dynamics — combinatorics / Corollary J $\Updownarrow$ — $\Updownarrow$ definition

$L^2(G_{\mathbb{C}}/K_{\mathbb{C}})$ is tempered — $(\mathfrak{p}_{\mathbb{C}})_{\mathrm{Ab}}$ is dense in $\mathfrak{p}_{\mathbb{C}}$.

Classification theory: side remarks 2

- Special cases are already non-trivial.

(b) Let G/H be a reductive symmetric space.

The Plancherel theorem* for G/H:

$$L^2(G/H) \simeq \bigoplus_{j=1}^{N} \int_{\nu}^{\oplus} \sum_{\lambda}^{\oplus} \operatorname{Ind}_{L_j N_j}^{G}(\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}) d\nu.$$

$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)} \cdots$ relative discrete series for $L_j/(L_j \cap H)$.

- Delicate issues arise from $\tau_\lambda^{(j)}$ with "singular" λ.

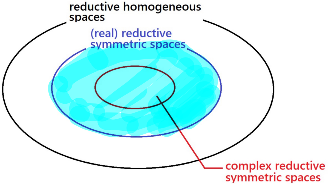
$L^2(G/H)$ is tempered	$\overset{\text{Plancherel}}{\Longleftrightarrow}$	$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j for all λ, a.e. ν
$\Downarrow$ Theorem I		$\Downarrow$ obvious
$(\mathfrak{g}/\mathfrak{h})_{\mathrm{Am}}$ is dense in $\mathfrak{g}/\mathfrak{h}$	$\overset{\text{Quantization}}{\Longleftrightarrow}$	$\tau_\lambda^{(j)} \otimes \mathbb{C}_\nu^{(j)}$ is a tempered rep of L_j, ${}^\forall\lambda \gg 0$, a.e. ν

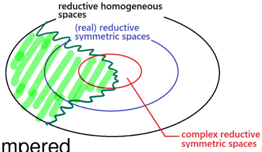
* T. Oshima (1980s); Delorme, Ann. Math. 1998; van den Ban–Schlichtkrull, Invent. Math. 2005.

** Y. Benoist–T. Kobayashi, Tempered homogeneous spaces III, J. Lie Theory (2021).

Classification theory: side remarks 2

- Special cases are already non-trivial.





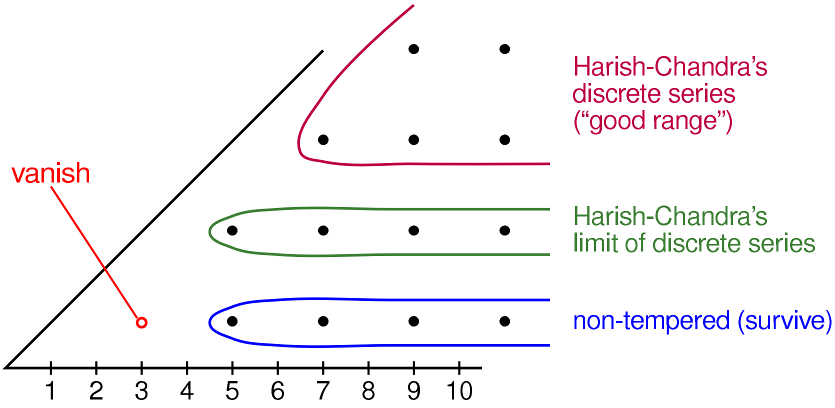
$L^2(G/H)$ is not tempered

(b) Let G/H be a (real) reductive symmetric space.

Our classification in this special setting (b) singles out a small number of reductive symmetric spaces such that a "large part" of the spectra in $L^2(G/H)$ (*e.g.*, induced from discrete series of Flensted-Jensen type) are tempered but $L^2(G/H)$ itself is not tempered.

E.g. For $p_1 \geq 1, q_1 \geq 1, p_1 + q_1 = p_2 + q_2 + 1$,
$Sp(p_1 + p_2, q_1 + q_2)/(Sp(p_1, q_1) \times Sp(p_2, q_2))$ is NOT tempered,
although "most of" the spectra are tempered.

Discrete series for $Sp(4,1)/Sp(1) \times Sp(3,1)$



Plan of Lectures

- Talk 1: Tempered homogeneous spaces
  —Dynamical approach

- Talk 2: Classification theory of tempered G/H
  — Combinatorics of convex polyhedra

- Talk 3: Tempered homogeneous spaces
  —Interaction with topology and geometry

Thank you for your attention!