Branching Problems and Global Analysis of Locally Symmetric Spaces with Indefinite-Metric

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International Congress of Basic Science
at Beijing, July 25, 2024

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}$$

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H \\ & \searrow & & \swarrow & \\ & & \Gamma\backslash G/H & & \end{array}$$

$$\underbrace{G/H}_{\text{local geometric structure}} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\text{global}}\backslash G/H$$

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Consider "intrinsic differential operators" (*e.g.,* Laplacian) on $\Gamma\backslash G/H$.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$\Gamma \subset G \supset H$
discrete subgp, Lie group, subgroup

$SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R}) \supset SO(2)$

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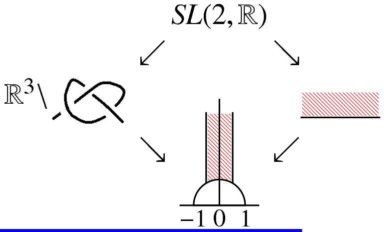

Special cases are already deep and rich.

- $\Gamma = \{e\}$
- H compact
- $G = \mathbb{R}^{p,q}$

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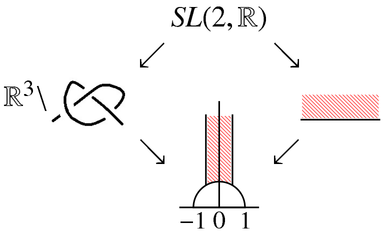

Special cases are already deep and rich.

- $\Gamma = \{e\} \cdots$ non-commutative harmonic analysis on $L^2(G/H)$
  Gelfand, Harish-Chandra, Helgason, Flensted-Jensen, T. Oshima, Delorme, ...
- H compact, Γ arithmetic $\cdots$ automorphic forms (local theory)
  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, ...
- $G = \mathbb{R}^{p,q}$ (abelian, but non-Riemannian), $\Gamma = \mathbb{Z}^{p+q}$, $H = \{e\}$
  Oppenheim conjecture, Dani, Margulis, Ratner, Eskin, Mozes, ...

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Γ	$\subset$	G	$\supset$	H
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New challenge: Spectral analysis on $\Gamma\backslash G/H$ by $\mathbb{D}_G(G/H)$
beyond the traditional Riemannian setting

$\cdots$ non-abelian G, non-trivial Γ and non-compact H.

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Further difficulties arise

- (geometry)
- (analysis)
- (representation theory)

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  $\cdots$ "local to global" beyond Riemannian setting
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  $\cdots$ "local to global" beyond Riemannian setting
- (analysis) Laplacian is no more elliptic. No "Weyl's law."
- (representation theory) $\mathrm{vol}(\Gamma\backslash G) = \infty$ even when $\Gamma\backslash G/H$ is compact

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Further difficulties arise

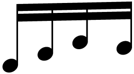
$\rightsquigarrow$ construct new geometry for the study:

- Standard quotients $\Gamma\backslash G/H$ (Topic A)
- Deforming $\Gamma\backslash G/H$ (Topic B)

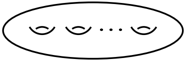
$\rightsquigarrow$ change methods for the study!

- Counting Γ-orbits (uniform estimate) (Topic C)
- Branching of unitary rep $G \downarrow \overline{\Gamma}$ (Topic D)

locally symmetric spaces beyond Riemannian setting



Geometry $\rightsquigarrow$ Analysis



Spectrum of Laplacian

$\Downarrow$

go beyond the Riemannian case
(e.g. Lorentzian manifold)

$\dashrightarrow$?

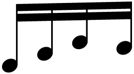
Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

Geometry

Construct ?

Analysis

Hear ?



A. Construct $\Gamma\backslash G/H$

(existence *vs.* obstruction
of compact forms)

B. Deform $\Gamma \curvearrowright G/H$

(rigidity *vs.* deformation)

C. Universal sound

(analysis on G/H
+ "counting" Γ-orbits)

D. Capture all sounds

(branching $G \downarrow \overline{\Gamma}$ (Zariski closure)
+ nonsymmetric spherical sp)

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Local to global in Riemannian geometry

The study “ from local to global in Riemannian geometry ” has been a major trend in geometry since the 20th century with remarkable developments.

As a warming up, one may recall one of classic theorems by Bonnet and Myers:

> Example (Myers 1941) A complete Riemannian manifold with Ricci curvature $\geq c (> 0)$ is compact.

How about “ local to global ” in pseudo-Riemannian geometry such as Lorentzian geometry?

Reminder : pseudo-Riemannian manifold

Definition A pseudo-Riemannian manifold (X, g) is a manifold equipped with non-degenerate bilinear form

$$g_x : T_xX \times T_xX \to \mathbb{R} \quad (x \in X)$$

depending smoothly on $x \in X$.

The signature (p, q) of g_x is locally constant.

(X, g) is a Riemannian manifold if $q = 0$,
a Lorentzian manifold if $q = 1$.

- The Laplacian Δ is a second-order differential operator on X defined by

$$\Delta = \operatorname{div} \circ \operatorname{grad} .$$

This is not an elliptic differential operator if g is indefinite.

The Calabi–Markus phenomenon (1962)

"Local to Global" in pseudo-Riemannian manifolds:

Fact (Calabi–Markus, 1962*)
Any de Sitter manifold is non-compact.

de Sitter mfd = Lorentzian manifold with sectional curvature $\equiv 1$

This is an example of space form in pseudo-Riemannian geometry (the $q = 1$ case in the next slide).

* E. Calabi–L. Markus, Relativistic space forms, Ann. Math., **75**, (1962), 63–76.

Space forms in pseudo-Riemannian geometry (definition)

(M, g) : pseudo-Riemannian manifold of signature (p, q).

Def* (M, g) is a space form if sectional curvature κ is constant.

Model space The hypersurface

$$\{(x_1, \ldots, x_{p+q+1}) : \sum_{i=1}^{p+1} x_i^2 - \sum_{j=1}^{q} y_j^2 = 1\}$$

in $\mathbb{R}^{p+1,q} = (\mathbb{R}^{p+q+1}, dx_1^2 + \cdots + dx_{p+1}^2 - dy_1^2 - \cdots - dy_q^2)$
has a pseudo-Riemannian structure of signature (p, q) with $\kappa \equiv 1$
or that of signature (q, p) with $\kappa \equiv -1$.

The model space is identified with the homogeneous space

$$G/H = O(p+1, q)/O(p, q).$$

* J. A. Wolf, Spaces of Constant Curvature, 6th ed. AMS, 2011

Space forms (examples)

Space form $\cdots$

$$\begin{cases} \text{Signature } (n-q, q) \text{ of pseudo-Riemannian metric} \\ \text{Curvature } \kappa \in \{+, 0, -\} \end{cases}$$

<u>Example</u> $q = 0$ (Riemannian manifold)

sphere S^n	$\mathbb{R}^n$	hyperbolic sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

$O(n+1)/O(n)$ $\qquad$ $O(n,1)/O(n)$

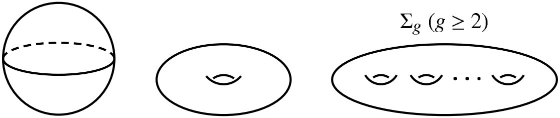
<u>Example</u> $q = 1$ (Lorentzian manifold)

de Sitter sp	Minkowski sp	anti-de Sitter sp
$\kappa > 0$	$\kappa = 0$	$\kappa < 0$

$O(n,1)/O(n-1,1)$ $\qquad$ $\mathbb{R}^{n-1,1}$ $\qquad$ $O(n-1,2)/O(n-1,1)$

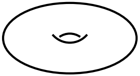
2-dim'l compact space forms

Riemannian case ($\Longleftrightarrow$ signature $(2,0)$)



curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

Lorentzian case ($\Longleftrightarrow$ signature $(1,1)$)

—  —

curvature $\kappa > 0$ $\kappa = 0$ $\kappa < 0$

There does NOT exist a compact form if $\kappa > 0$ and $\kappa < 0$

Local to global problem in pseudo-Riemannian geometry

Space Form Problem* in pseudo-Riemannian geometry

Local Assumption

signature (p, q), curvature $\kappa \in \{+, 0, -\}$

$$\Downarrow$$

Global Results

- Do compact forms exist?
- What groups can arise as their fundamental groups?

* T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes in Math. **2313**, 217–231, Springer, 2023.

See also T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces. Mathematics Unlimited — 2001 and Beyond, pages 723–747. Springer-Verlag, 2001.

Formulation in group language (special case)

Uniformization theorem*: Any complete pseudo-Riemannian manifold M of signature (q,p) with $\kappa \equiv -1$ and $p \neq 1$ is of the form

$$\Gamma \backslash G/H$$

where $G = O(p+1,q)$, $H = O(p,q)$, and Γ is a discrete subgroup of G such that Γ acts properly discontinuously and freely on G/H.

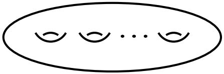
$$\underbrace{G/H}_{\text{local geometric structure}} \quad \to \quad \underbrace{\Gamma \backslash G/H}_{\text{global}}$$

Example (Klein–Poincaré–Koebe)

$(p,q) = (0,2)$. Set $\Gamma := \pi_1(\Sigma_g)$.

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\} \simeq SO(1,2)/O(2)$

$\downarrow$ covering

$\Sigma_g =$  $(g \geq 2) \simeq \pi_1(\Sigma_g) \backslash \mathbb{H}$

surface group

* J. A. Wolf, Spaces of Constant Curvature, 6th ed. AMS, 2011

Generalities: Proper action & properly discontinuous action

We recall a continuous action $L \curvearrowright X$ is called proper if

$$L \times X \to X \times X, \quad (g, x) \mapsto (x, gx)$$

is a proper map, *i.e.*, the full inverse of a compact set is compact.

properly discontinuous action = proper action when L is a discrete goup.

General Problem 1 Let $L \subset G \supset H$. Detect whether the L-action on $X := G/H$ is proper or not.

Symbolically written as " $L \pitchfork H$ in G " when $L \curvearrowright G/H$ properly*.

* T. Kobayashi, Criterion for proper actions on homogeneous spaces of reductive groups, J. Lie Theory, **6** (1996), 147–163.

Properness Criterion

G: real reductive Lie group

$G = K \exp(\mathfrak{a}) K$: Cartan decomposition

$\nu\colon G \to \mathfrak{a}$: Cartan projection (up to Weyl group)

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E.g. $\nu: GL(n,\mathbb{R}) \to \mathbb{R}^n$

$$g \mapsto \frac{1}{2}(\log\lambda_1, \cdots, \log\lambda_n)$$

Here, $\lambda_1 \geq \cdots \geq \lambda_n\ (> 0)$ are the eigenvalues of ${}^t g g$.

$$G = GL(n,\mathbb{R})$$
$$K = O(n)$$
$$\mathfrak{a} \simeq \mathbb{R}^n$$
$$\text{Weyl group} \simeq \mathcal{S}_n$$

Properness Criterion

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Theorem A (properness criterion) * Let $L \subset G \supset H$.

$L \pitchfork H$ in $G \iff \nu(L) \pitchfork \nu(H)$ in $\mathfrak{a}$.

non-abelian abelian

** Kobayashi, Math. Ann., '89, J. Lie Theory, '96. Benoist, Ann. Math., '96.

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non-abelian — abelian

Criterion for proper actions.

- Special case: for L, H reductive subgp, K– '89
- Quantitative estimate for properness

$\Rightarrow$
- deformation theory of Γ (Topic B)
- "counting" Γ-orbits (Topic C)

** Kobayashi, Math. Ann., '89, J. Lie Theory, '96. Benoist, Ann. Math., '96.

Basic notion: Discontinuous group Γ for $X = G/H$

Setting	Γ	$\subset$	G	$\supset$	H
	discrete		Lie group		closed subgroup

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Setting Γ (discrete) $\subset$ G (Lie group) $\supset$ H (closed subgroup)

$$\begin{array}{ccccc} & & G & & \\ & \text{covering} \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H = X \\ & \searrow & p_\Gamma & \swarrow & \\ & & \Gamma\backslash G/H = \Gamma\backslash X & & \end{array}$$

- $\Gamma\backslash G$ and $X = G/H$ are C^∞ manifolds.

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- $\Gamma\backslash G$ and $X = G/H$ are C^∞ manifolds.
- The quotient $\Gamma\backslash X = \Gamma\backslash G/H$ is not necessarily Hausdorff.

It becomes a Hausdorff C^∞ manifold with p_Γ being a covering, if Γ acts properly discontinuously and freely on X.

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Definition Such Γ is called a discontinuous group for $X = G/H$.

Global properties in pseudo-Riemannian geometry

Space form problem for $\kappa \equiv -1$, signature (q, p).

Fact (1962–2024)[*,**] Let $G/H = O(p+1, q)/O(p, q)$.

(1) (Calabi–Markus phenomenon) G/H admits an infinite discontinuous group if and only if $p < q$.

(2) If G/H admits a cocompact discontinuous group, then either $pq = 0$ or "$p < q$ and q is even".

(3) G/H admits a cocompact discontinuous group for (p, q) below.

p	$\mathbb{N}$	0	1	3	7
q	0	$\mathbb{N}$	$2\mathbb{N}$	$4\mathbb{N}$	8

[*] Calabi–Markus (1962), Wolf (1962), Kulkarni (1981), Kobayashi (1996), Tholozan (2015), Morita (2017), $\cdots$.

[**] T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes Math. **2313**, 217–231, Springer, 2023.

Basic question on discontinuous groups for G/H

$$\underbrace{G/H}_{\text{local geometric structure}} \quad \rightarrow \quad \underbrace{\Gamma\backslash G/H}_{\text{global}}$$

<u>General Problem 3</u>* Which homogeneous space G/H admits 'large' discontinuous groups Γ with the following properties?

1) $\#\Gamma = \infty$;

2) $\Gamma\backslash G/H$ is compact (or of finite volume) (topic A);

3) $\Gamma \curvearrowright G/H$ is "deformable"/ "rigid" (topic B).

- The first problem (Calabi–Markus phenomenon 1962) is solved in the reductive case (K- 1989) by Thm A (properness criterion).

* **Problems** in T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces. **Mathematics Unlimited — 2001 and Beyond**, pages 723–747. Springer-Verlag, 2001.

Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

Geometry

Analysis

Prepare 

Hear

A. Construct $\Gamma\backslash G/H$
(existence *vs.* obstruction
of compact forms)

C. Universal sound
(analysis on G/H
+ "counting" Γ-orbits)

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(branching $G \downarrow G'$
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$\Gamma\backslash G/H$ is not Hausdorff if Γ is an arithmetic subgp of G.

- Borel–Harish-Chandra, Mostow–Tamagawa (1960~)

For a linear semisimple Lie group G, one can define arithmetic subgroups Γ of G (*e.g.*, $\Gamma = SL(n,\mathbb{Z})$ in $G = SL(n,\mathbb{R})$)

- $\mathrm{vol}(\Gamma\backslash G) < \infty$ for any arithmetic subgroup Γ,
- ${}^{\exists}\Gamma$ such that $\Gamma\backslash G$ is compact (Borel 1962).

However, $\cdots$

- If H is a non-compact subgroup of G, then

  $\Gamma\backslash G/H$ is not Hausdorff in the quotient topology for any arithmetic subgroup Γ.

Idea of "standard quotient" $\Gamma\backslash G/H$

$$\underset{}{\Gamma} \underset{\text{discrete}}{\subset} \underset{\text{reductive Lie gp}}{G} \supset \underset{\text{reductive subgp}}{H}$$

Difficulties $\Gamma\backslash G/H$ is not always Hausdorff.

$$\Gamma \underset{\text{discrete}}{\subset} G \overset{\text{isometry}}{\curvearrowright} \overset{\text{pseudo-Riemannian}}{X = G/H}$$

- Bad feature: The G-action on $X = G/H$ is not proper whenever H is non-compact.

Construction: Standard quotients $\Gamma\backslash X = \Gamma\backslash G/H$

Let $G \supset H$ be a pair of real reductive Lie groups, and $X := G/H$.

$\Gamma \subset G'$ — properly → X

$\cap$

$G \curvearrowright X = G/H$ (isometry)

pseudo-Riemannian

Definition The resulting C^∞-manifold $\Gamma\backslash X = \Gamma\backslash G/H$ is called a standard quotient of $X = G/H$.

Theorem B* $X = G/H$ admits a cocompact discontinuous group if there exists a reductive subgp G' acting properly and cocompactly on X.

* T. Kobayashi, Proper action on a homogeneous space of reductive type, Math. Ann. (1989).

Local homogeneous structure admitting compact quotients

Application of Theorem B (compact standard quotients)

Example • Riemannian symmetric space G/K,
• odd-dimensional anti-de Sitter space,
• pseudo-Kähler manifolds $SO(2n,2)/U(n,1)$,
• complex sphere $SO(8,\mathbb{C})/SO(7,\mathbb{C})$, $\cdots$.

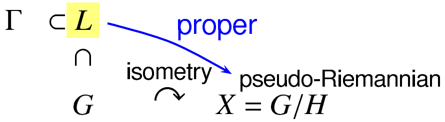
- Criterion*
  $\cdots$ properness criterion (Thm A) and cocompactness criterion.

- List of (G,H) having compact standard quotients $\Gamma\backslash G/H$ **
  $\cdots$ K- '89, K- '96, K–Yoshino (2005),

- Exhaustion***
  $\cdots$ Tojo 2019, 2021; Bochenski–Jastrezebski–Tralle 2022

* T. Kobayashi, Math. Ann. (1989);
** TK, Euro School, (1996); TK–Yoshino, Pure and Appl. Math. Quarterly **1**, (2005);
*** Tojo, Proc. Japan Academy (2019).

Conjecture: compact standard quotient



Conjecture (K– '01)* If $X = G/H$ admits a cocompact discontinuous group, then X admits a compact standard quotient.

- Evidence of Conjecture** $\cdots$ Various methods (1962–2024) for the obstruction to the existence of compact discontinuous groups developed by Borel, Calabi–Markus, Wolf, Kulkarni, K–, K–Ono, Benoist, Labourie, Zimmer, Mozes, Margulis, Shalom, Okuda, Tholozan, Kassel, Morita, $\cdots$

* Conjecture 4.3 in T. Kobayashi, Discontinuous groups for non-Riemannian homogeneous spaces, Mathematics Unlimited — 2001 and Beyond, pages 723–727. Springer-Verlag, 2001;

** T. Kobayashi, Conjectures on reductive homogeneous spaces, Lect. Notes Math. **2313**, 217–231, Springer, 2023.

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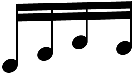

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D. Capture all sounds
(branching $G \downarrow G'$
$+$ nonsymmetric spherical sp)

"Deformation" of a quotient $\Gamma\backslash G/H = \Gamma\backslash X$

Deformation of $\Gamma\backslash G/H$.*

$$\underset{\text{fix}}{\Gamma} \quad \overset{\varphi}{\underset{\text{discrete}}{\hookrightarrow}} \quad \underbrace{G \curvearrowright X = G/H}_{\text{fix}}$$

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$\rightsquigarrow$ Can we say that $\varphi(\Gamma)\backslash X$ is a deformation of $\Gamma\backslash X$?

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Vary an injective homomorphism φ

$\rightsquigarrow$ Can we say that $\varphi(\Gamma)\backslash X$ is a deformation of $\Gamma\backslash X$?

- Two major problems
  — nontrivial deformation of Γ may not exist?
  — deformation Γ may destroy proper discontinuity?

$*$ T. Kobayashi, J. Geometry and Physics 1993; Discontinuous groups for non-Riemannian homogeneous spaces, **Mathematics Unlimited — 2001 and Beyond,** pages 723–747, Springer-Verlag, 2001.

Deformation of standard quotients $\Gamma\backslash G/H$

Formulation: Deformation of standard quotients of $X = G/H$

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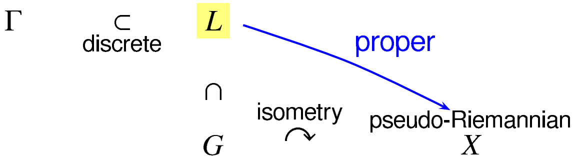


Deformation of standard quotients $\Gamma \backslash G/H$

Formulation: Deformation of standard quotients of $X = G/H$

$\Gamma \underset{\text{cocompact}}{\subset} L$ —proper→ X

$L \cap G$, $G \overset{\text{isometry}}{\curvearrowright} X$ (pseudo-Riemannian)

- There is not enough room for Γ to be deformed inside L.

<u>Fact</u> (Selberg–Weil rigidity) No non-trivial deformation of Γ inside a simple group L except for $L \approx SL(2, \mathbb{R})$.

Deformation of standard quotients $\Gamma\backslash G/H$

Formulation: Deformation of standard quotients of $X = G/H$

$\Gamma \underset{\text{cocompact}}{\subset} L$ — proper → pseudo-Riemannian X

deform ⤵

$\cap$

G $\underset{}{\overset{\text{isometry}}{\curvearrowright}}$ pseudo-Riemannian X

- There is not enough room for Γ to be deformed inside L.
- There exists a larger room for Γ to be deformed in G.

However, after a small deformation
the Γ-action on X may not be properly discontinuous

Small deformation of Γ may distroy proper discontinuity

$$\Gamma \subset G \curvearrowright X = G/H$$

- A small deformation of a discontinuous group Γ may not act properly discontinuously on X:

Example $(\Gamma = \mathbb{Z} \curvearrowright X = \mathbb{R})$

Twisted action $\quad x \mapsto ax + 1 \quad (a \fallingdotseq 1)$

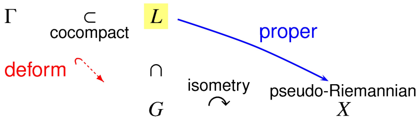
$$G = \mathrm{Aff}(\mathbb{R})_+ \simeq \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a > 0, b \in \mathbb{R}\} \supset H = \{\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} : a > 0\}$$

Then $X := G/H \simeq \mathbb{R}$.

$\Gamma = \langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \rangle \simeq \mathbb{Z} \rightsquigarrow \Gamma \backslash X \simeq \mathbb{Z} \backslash \mathbb{R}$ if $a = 1$

$\Gamma_a = \langle \begin{pmatrix} a & 1 \\ 0 & 1 \end{pmatrix} \rangle \simeq \mathbb{Z} \rightsquigarrow \Gamma_a \backslash X \simeq \mathbb{Z} \backslash \mathbb{R}$ is non-Hausdorff if $a \neq 1$

Goldman's conjecture on 3-dim'l anti-de Sitter mfd



<u>Conjecture</u> (Goldman 1985)* A small <u>deformation</u> of compact standard X_Γ is still a manifold when $X = \mathrm{Ad}S^3 = O(2,2)/O(2,1)$.

<u>Answer</u> (<u>K–1998</u>)* Goldman's conjecture is true.

Proof: Use criterion for proper action. ($\Leftarrow$ <u>Theorem A</u>)

$\rightsquigarrow$ deformation theory for higher dimensional $X_\Gamma = \Gamma\backslash X$

(followed further** work by Kassel, Guéritaud, Kannaka, $\cdots$)

<u>E.g.</u> nontrivial deformations of 3D compact AdS mfd X_Γ is of dimension $= 12g - 12$ if $\operatorname{rank} \Gamma/[\Gamma,\Gamma] = 2g$.

* Goldman, J. Diff. Geom., 1985, K–, Math. Ann., 1998; ** Kassel, Math. Ann. 2012; Kannaka, Selecta Math. 2024.

Deformation of standard quotient $\Gamma\backslash G/H$

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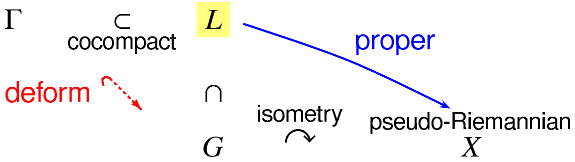

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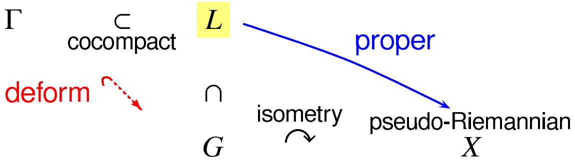


Theorem* (1) (rigidity) Local rigidity of cocompact discontinuous gp Γ for G/H still holds for some (G, H) with H noncompact.

(2) (flexibility) For some other (G, H) with H noncompact, there exist cocompact discontinuous groups Γ that admit non-trivial continuous deformations to Zariski dense subgroups in G, but still keeping proper discontinuity.

* Some classification theory is to appear.

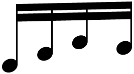
Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

Geometry

Prepare

Analysis

Hear



A. Construct $\Gamma\backslash G/H$

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Formulation: Spectral analysis on $\Gamma\backslash X = \Gamma\backslash G/H$

$$\begin{array}{ccccc} & X & = & G/H \\ \text{covering} & \downarrow & & \\ & \Gamma\backslash X & = & \Gamma\backslash G/H \end{array} \qquad \begin{array}{ccc} \mathbb{D}_G(X) & \ni & D \\ \downarrow & & \downarrow \\ \mathbb{D}(\Gamma\backslash X) & \ni & D_\Gamma \end{array}$$

<u>General Problem</u> Wish to understand joint eigenvalues and (wish to construct) joint eigenfunctions f on $\Gamma\backslash X$:

$$D_\Gamma f = \lambda(D) f \quad \text{for all } D \in \mathbb{D}_G(X),$$

where $\lambda : \mathbb{D}_G(X) \to \mathbb{C}$ is a $\mathbb{C}$-algebra homomorphism.

$\mathbb{D}_G(X)$: ring of G-invariant differential operators on X

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The above formulation makes reasonable sense if $\mathbb{D}_G(X)$ is commutative (e.g. $X = G/H$ is a symmetric space)

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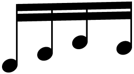
Let us hear 

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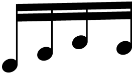
Let us hear 

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“Universal Sound” of anti-de Sitter manifold

classical

Hyperbolic manifold
$\underset{\text{def}}{\Longleftrightarrow}$ Riemannian manifold with sectional curvature $\equiv -1$

beyond Riemannian setting

Def M: anti-de Sitter manifold
$\underset{\text{def}}{\Longleftrightarrow}$ Lorentz manifold with sectional curvature $\equiv -1$

A classical theorem for Riemann surfaces says that eigenvalues of the Laplacian vary as functions on Teichmüller space.

In contrast, a new phenomenon in anti-de Sitter manifolds:

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Kassel and Kobayashi, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), 123–236.

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Remark	X Riemannian	$\Longrightarrow$	Laplacian	Δ	is elliptic
	Lorentzian	$\Longrightarrow$		Δ	hyperbolic

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'stable' $\underset{\text{def}}{=}$ 'does NOT vary under deformation' of anti-de Sitter structure

The deformation space (modulo conjugation) of anti-de Sitter structure has dimension $12g - 12$

Another special case: application to the group case

Theorem C can be generalized* to higher-dimensional symmetric spaces. Special cases include group manifolds $(G \times G)/\operatorname{diag}(G)$.

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Corollary   Assume G is a reductive group with $\operatorname{rank} G = \operatorname{rank} K$. Then, for any torsion-free discrete subgp Γ of G,

$$\operatorname{Hom}_G(\pi_\lambda, L^2(\Gamma \backslash G)) \neq 0$$

for any discrete series rep π_λ of G with sufficiently regular λ.

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Remark Corollary sharpens earlier works given by

Kazhdan '77, J.-S. Li '92,

De George–Wallach '78 (Γ: cocompact),

Clozel '86, Rohlfs–Speh '87 (Γ: non-compact lattice),

which treated an arithmetic subgroup Γ replaced by a congruence subgp Γ' (possibly depending on π_λ).

* Kassel and Kobayashi, Poincaré series for non-Riemannian locally symmetric spaces, Adv. Math. **287**, (2016), 123–236.

Construction of L^2-eigenfunctions of Laplacian on $\Gamma\backslash G/H$

Step 1 Construction of (non-periodic) eigenfunctions

Step 2 Holomorphic extension (to other real forms)

Step 3 Construction of periodic eigenfunctions (Poincaré series)

- Geometric estimate
- Analytic estimate

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$$\Delta f = \lambda f \text{ on } \Gamma\backslash G/H$$

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(integral geometry, Poisson transform)

Step 2 Holomorphic extension (to other real forms)
(Flensted-Jensen's duality)

Step 3 Construction of periodic eigenfunctions (Poincaré series)

- Geometric estimate for proper actions $\Gamma \curvearrowright G/H$
  (Kazhdan–Margulis, K– , Benoist, Kassel–K, . . .)
- Analytic estimate of eigenfunctions on G/H
  (systems of PDEs, micro-local analysis)
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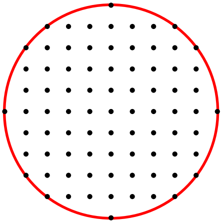
Counting : $\Gamma \cdot x \cap B_R$ in pseudo-Riemannian mfd

Counting (classical) | New setting

Riemannian $\rightsquigarrow$ pseudo-Riemannian G/H



$\Gamma = \mathbb{Z}^2 \curvearrowright X = \mathbb{R}^2$

Count

$\Gamma \cdot x \cap B(R)$

ball with radius R

$\nearrow$ $\searrow$

Eskin–McMullen, Oh, . . .

$\begin{cases} \Gamma : & \text{lattice of } G \\ x : & \text{special} \end{cases}$

Kassel–K[*], Kannaka[**]

$\begin{cases} \Gamma : & \text{discontinuous gp for } X \\ x : & \text{general} \end{cases}$

[*] Kassel–Kobayashi, Adv. Math., 2016; Kannaka, Selecta Math. 2024.

Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

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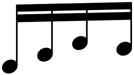
Let us hear 

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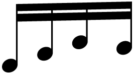
Let us hear 

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Branching problems

$$\begin{array}{ccc} G & \xrightarrow[\text{irreducible}]{\pi} & GL(V) \\ \cup & \nearrow & \\ G' & \pi|_{G'} & \end{array}$$

Branching problems

$$G \xrightarrow[\text{irreducible}]{\pi} GL(V)$$

$$\cup$$

$$G' \quad \pi|_{G'}$$

<u>Example</u> (tensor product of two representations)

$$G_1 \times G_1 \xrightarrow[\text{outer tensor product}]{\pi' \boxtimes \pi''} GL(V' \otimes V'')$$

$$\cup$$

$$\operatorname{diag} G_1 \quad \pi' \otimes \pi''$$

Branching problems

$$G \xrightarrow[\text{irreducible}]{\pi} GL(V)$$

$$\cup$$

$$G' \quad \pi|_{G'}$$

<u>Branching problem</u> (in a wider sense than the usual)

$\cdots$ wish to understand
how the restriction $\pi|_{G'}$ behaves as a G'-module.

Standard quotients $\Gamma\backslash G/H$: spherical assumption

Recall $\Gamma\backslash G/H$ is a standard quotient of $X = G/H$ if

$$
\begin{array}{ccccc}
\Gamma & \underset{\text{discrete}}{\subset} & G' & \xrightarrow{\text{proper}} & \\
 & & \cap & & \\
 & & G & \curvearrowright & X = G/H
\end{array}
$$

Assume $G' \curvearrowright X$ properly. $\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$ (A)

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Further, we assume that $G'_{\mathbb{C}}$ acts spherically on $X_{\mathbb{C}}$. $\cdots\cdots$ (B)

(i.e. Borel subgp of $G'_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$)

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<u>Strategy</u> Capture all the spectrum for

$$\Gamma \backslash X \simeq \Gamma \backslash G/H$$

by analyzing <u>branching laws</u> for the restriction of infinite-dimensional representations of G with respect to $G \downarrow G'$.

<u>Theorem D</u> * (Branching Theory) Assume (A) and (B) are satisfied. Then for any irred rep π of G occurring in $C^\infty(X)$, the restriction $\pi|_{G'}$ decomposes <u>discretely</u> into G'-irreducible modules.

* T. Kobayashi, Global analysis by hidden symmetry, Progr. Math. (2017).

Techniques involved in the proof of Theorems D and E below

Branching of finite-dimensional reps for compact Lie groups

⇝ *

Structure theorem for the ring of differential operators

⇝ **

Branching of infinite-dimensional reps for reductive Lie groups

⇝ Theorem E below

Analysis on $\Gamma\backslash G/H$ with H noncompact

(The first two steps together: Theorem D)

* Kassel–Kobayashi, Invariant differential operators on spherical homogeneous spaces with overgroups, J. Lie Theory 2019;

** Kobayashi, Global analysis with hidden symmetry, Progr. Math. 2017.

Expansion into eigenfns on $\Gamma\backslash X$ with indefinite metric

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$$C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda) := \{f \in C^\infty(\Gamma\backslash X) : D_\Gamma f = \lambda(D) f \ {}^\forall D \in \mathbb{D}_G(X)\}$$

where we recall

$$\begin{array}{ccccc} \mathbb{D}(\Gamma\backslash X) & \hookleftarrow & \mathbb{D}_G(X) & \xrightarrow{\lambda} & \mathbb{C} \\ \Cup & & \Cup & & \Cup \\ D_\Gamma & \hookleftarrow & D & \mapsto & \lambda(D) \end{array}$$

Expansion into eigenfns on $\Gamma\backslash X$ with indefinite metric

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<u>Theorem E (spectral theory)</u>

$^\exists$ measure μ on $\widehat{\mathbb{D}_G(X)} = \{\lambda : \mathbb{D}_G(X) \to \mathbb{C}, \text{alg. hom}\}$ and
$^\exists$ measurable family of maps

$$\mathcal{F}_\lambda : C_c^\infty(\Gamma\backslash X) \to C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda) \ (\subset C^\infty(\Gamma\backslash X))$$

$$f = \int_{\widehat{\mathbb{D}_G(X)}} \mathcal{F}_\lambda f \, d\mu(\lambda) \quad {}^\forall f \in C_c^\infty(\Gamma\backslash X).$$

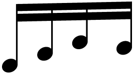
$$\|f\|^2 = \int_{\widehat{\mathbb{D}_G(X)}} \|\mathcal{F}_\lambda f\|^2 \, d\mu(\lambda).$$

* F. Kassel–T. Kobayashi, Spectral Analysis on Standard Locally Homogeneous Spaces, 118 pages, Lect. Notes in Math., Springer-Nature, to appear (cf. arXiv:1912.12601)

Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

Geometry	Analysis

Have prepared 

Let us hear 

A. Construct $\Gamma\backslash G/H$
(existence *vs.* obstruction
of compact forms)

B. Deform $\Gamma \curvearrowright G/H$
(rigidity *vs.* deformation)

C. Universal sound
(analysis on G/H
+ "counting" Γ-orbits)

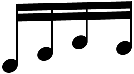
D. Capture all sounds
(branching $G \downarrow G'$
+ nonsymmetric spherical sp)

Plan of talk — $\Gamma\backslash G/H$ beyond Riemannian setting

Geometry

Analysis

Have prepared

Let us hear 

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Thank you very much!

References for Topic C

- T. Kobayashi,
  Intrinsic sound of anti-de Sitter manifolds, PROMS, 191 (2016), pp. 83–99.

- F. Kassel and T. Kobayashi,
  Poincaré series for non-Riemannian locally symmetric spaces.
  Adv. Math. 287, (2016), pp.123–236.

- K. Kannaka, 2021 (Ph. D. thesis, U. Tokyo)
  (3-dimensional AdS manifold: Selecta Math. 2024)

References for Topic D

T. Kobayashi,

- Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups I, Invent. Math. (1994); II Annals of Math. (1998); III Invent. Math. (1998).

- Global analysis by hidden symmetry, Progr. Math. vol. 323, pp. 359–397, 2017. (in honor of Roger Howe)

F. Kassel and T. Kobayashi

- Invariant differential operators on spherical spaces with overgroups, J. Lie Theory **29** (2019), pp. 663–754.

- Spectral analysis on standard locally homogeneous spaces, Proc. Japan Acad. 2020;
full paper to appear in Lecture Notes in Math., Springer-Nature (available also at arXiv:1912.12601).