Hidden Symmetry and Spectral Analysis on Locally Pseudo-Riemannian Symmetry Spaces

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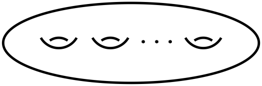
Introduction: Spectral analysis on $\Gamma \backslash G/H$

Spectral analysis of the Laplace–Beltrami operator Δ on a closed Riemann surface Σ_g



$$= \begin{cases} \mathbb{R}^2/\mathbb{Z}^2 & (g = 1) \\ \pi_1(\Sigma_g)\backslash SL(2,\mathbb{R})/SO(2) & (g \geq 2) \end{cases}$$

$\rightsquigarrow$ more generally

What if beyond the classical Riemannian setting?

Discrete isometry groups beyond the Riemannian setting

X: Riemannian manifold
Γ: torsion-free, discrete group of isometries of X

$\rightsquigarrow$ The Γ-action on X is properly discontinuous, and the quotient space $X_\Gamma := \Gamma\backslash X$ becomes a Riemannian manifold:

$$X \to X_\Gamma \qquad \text{(isometric covering)}.$$

However, the positivity is crucial in this result:

X: indefinite Riemannian manifold
Γ: torsion-free, discrete group of isometries of X

$\rightsquigarrow$ The Γ-action on X is not necessarily properly discontinuous, and the quotient space $X_\Gamma = \Gamma\backslash X$ is not always Hausdorff.

Discrete group of isometries of Lorentzian manifolds

Lorentzian manifolds with constant sectional curvature κ:

(1) $(\kappa > 0)$ $\quad \mathrm{dS}^n = SO(n+1,1)/SO(n,1)$ $\quad \cdots$ de Sitter space,

(2) $(\kappa < 0)$ $\quad \mathrm{AdS}^n = SO(n,2)/SO(n,1)$ $\quad \cdots$ anti-de Sitter space

Example 1 (Calabi–Markus phenomenon) There does not exist an infinite discrete group of isometries that acts properly discontinuously on the de Sitter space dS^n.

Example 2 (anti-de Sitter manifold) There exists a discrete group Γ of isometries that acts properly discontinuously and cocompactly on the anti-de Sitter space AdS^n if and only if n is odd.

Reminder · · · Proper Action (Topology)

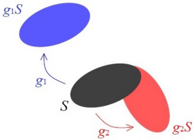
Suppose that G' acts on a manifold X continuously.

Definition We say the G'-action is proper if
the map $G' \times X \to X \times X$, $(g, x) \mapsto (x, gx)$ is proper.

This means that for every compact subset $S \subset X$,

$$\{g \in G' : gS \cap S \neq \emptyset\}$$

is compact.



The G'-action on X is called properly discontinuous
if G' is discrete and the action is proper.

Formulation in terms of group languages $\Gamma \subset G \supset H$

Let $G \supset H$ be a pair of Lie groups.

Consider an action of a subgroup Γ on $X = G/H$.

We remark, for non-compact H:

- Γ is discrete in G.

$$\not\Downarrow \quad \Uparrow$$

- The Γ-action on $X = G/H$ is properly discontinuous.

Quotient space $X_\Gamma = \Gamma\backslash G/H$ in the general setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}$$

$$\begin{array}{ccccc} & & G & & \\ & \swarrow & & \searrow & \\ \Gamma\backslash G & & & & G/H \\ & \searrow & & \swarrow & \\ & & \Gamma\backslash G/H & & \end{array}$$

Suppose that the Γ-action on X is properly discontinuous and free.

$$\underbrace{G/H}_{\text{local geometric structure}} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\text{global}} \backslash G/H$$

Quotient space $X_\Gamma = \Gamma\backslash G/H$ in the general setting

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Suppose that the Γ-action on X is properly discontinuous and free.

$$\underbrace{G/H}_{\text{local geometric structure}} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\text{global}}\backslash G/H$$

A G-invariant differential operator on $X = G/H$ induces an "intrinsic differential operator" on $X_\Gamma = \Gamma\backslash G/H$.

Standard locally homogeneous spaces $\Gamma\backslash G/H$

$X = G/H$ with non-compact subgroup $H \subset G$.

Question How can we find a discrete subgroup Γ acting properly discontinuously on $X = G/H$?

Observation

- Any discrete subgroup Γ will do if H is compact.
- Any lattice Γ will never work if H is non-compact.

Idea (discrete $\leftrightarrow$ continuous)

Definition Take any subgroup G' of G acting on X properly. Then any discrete subgroup Γ of G' acts properly discontinuously on X.
Such a quotient space $\Gamma\backslash X = \Gamma\backslash G/H$ is referred to as a standard quotient of X, when G' is reductive.

Example of compact standard quotients X_Γ

Example 4 (compact anti-de Sitter manifold) Let

$$\mathrm{AdS}^{2m+1} = SO(2m,2)/SO(2m,1).$$

Take $G' := U(m,1)$. Then, for any torsion-free, cocompact, discrete subgroup Γ of G', $\Gamma\backslash \mathrm{AdS}^{2m+1}$ is a compact Lorentzian manifold with negative constant sectional curvature.

Example 5 (3-dimensional indefinite-Kähler manifold) Let

$$X := \{z \in \mathbb{C}^4 : |z_1|^2 + |z_2|^2 > |z_3|^2 + |z_4|^2\}/\mathbb{C}^\times$$
$$\simeq SU(2,2)/U(2,1) \ (=: G/H).$$

Take $G' := Spin(4,1)$ $(\subset G)$. Then, for any torsion-free cocompact discrete subgroup Γ of G',
$X_\Gamma = \Gamma\backslash G/H$ is a compact indefinite-Kähler manifold.

Locally pseudo-Riemannian symmetric space

Let G be a real reductive Lie group,
σ an involutive automorphism of G, and
H an open subgroup of G^σ.
Then $X = G/H$ is called a symmetric space.

Let Γ be a subgroup of G, which acts properly discontinuously and freely on $X = G/H$. Then the quotient space

$$X_\Gamma = \Gamma \backslash G/H$$

is a locally symmetric space.

Example 3 $X = G/K$: Riemannian symmetric space,
$\rightsquigarrow X_\Gamma = \Gamma \backslash G/K$ is a locally Riemannian symmetric space.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H, \quad X_\Gamma := \Gamma\backslash G/H.$$

$\mathbb{D}_G(X)$:= ring of G-invariant differential operators on X

Problem Find spectral decomposition of $C_c^\infty(X_\Gamma)$ and $L^2(X_\Gamma)$ for "intrinsic differential operators" on X_Γ induced from $\mathbb{D}_G(X)$.

We explore Problem when G/H is a reductive symmetric space. Then $\mathbb{D}_G(X)$ is a commutative ring, which contains the pseudo-Riemannian Laplace-Beltrami operator.

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Problem Find spectral decomposition of $C_c^\infty(X_\Gamma)$ and $L^2(X_\Gamma)$ for "intrinsic differential operators" on X_Γ induced from $\mathbb{D}_G(X)$.

Special cases (classical cases) are already deep and rich.

- $\Gamma = \{e\}$
- H compact
- $G = \mathbb{R}^{p,q}$

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

Problem Find spectral decomposition of $C_c^\infty(X_\Gamma)$ and $L^2(X_\Gamma)$ for "intrinsic differential operators" on X_Γ induced from $\mathbb{D}_G(X)$.

Special cases (classical cases) are already deep and rich.

- $\Gamma = \{e\} \cdots$ non-commutative harmonic analysis on $L^2(G/H)$

  Gelfand, Harish-Chandra, S. Helgason, Flensted-Jensen, T. Oshima, Delorme, ...
- H compact, Γ arithmetic $\cdots$ automorphic forms (local theory)

  Siegel, Selberg, Piateski-Shapiro, Langlands, Arthur, Sarnak, Müller, ...
- $G = \mathbb{R}^{p,q}$ (abelian, but non-Riemannian), $\Gamma = \mathbb{Z}^{p+q}$, $H = \{e\}$

  Oppenheim conjecture, Dani, Margulis, Ratner, Eskin, Mozes, ...

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H, \quad X_\Gamma := \Gamma\backslash G/H.$$

<u>Challenge</u>: Spectral analysis on X_Γ by $\mathbb{D}_G(X)$
for non-trivial Γ, non-abelian G and non-compact H.

New difficulties arise

- (<u>geometry</u>)

- (<u>analysis</u>)

- (<u>representation theory</u>)

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

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New difficulties arise

- (geometry) existence of good geometry $\Gamma\backslash X$?
  $\cdots$ "local to global" beyond the Riemannian setting
- (analysis)
- (representation theory)

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H, \quad X_\Gamma := \Gamma\backslash G/H.$$

Challenge: Spectral analysis on X_Γ by $\mathbb{D}_G(X)$
for non-trivial Γ, non-abelian G and non-compact H.

New difficulties arise

- (geometry) existence of good geometry $\Gamma\backslash X$?
  $\cdots$ "local to global" beyond the Riemannian setting
- (analysis) The Laplacian $\square$ is no more elliptic.
  Not obvious whether $\square$ is essentially self-adjoint on $L^2(\Gamma\backslash X)$.
- (representation theory) $\mathrm{vol}(\Gamma\backslash G) = \infty$ even when $\Gamma\backslash X$ is compact

Spectral analysis on $\Gamma\backslash G/H$ beyond Riemannian setting

$$\underset{\text{discrete subgp}}{\Gamma} \subset \underset{\text{Lie group}}{G} \supset \underset{\text{subgroup}}{H}, \quad X := G/H, \quad X_\Gamma := \Gamma\backslash G/H.$$

Challenge: Spectral analysis on X_Γ by $\mathbb{D}_G(X)$
for non-trivial Γ, non-abelian G and non-compact H.

New difficulties arise
$\rightsquigarrow$ need to change methods for the study!

Plan

- Spectral analysis on $\Gamma\backslash G/H$ beyond the Riemannian setting.
- Admissible restriction to reductive subgroups.
- Bounded/finite multiplicities in the branching laws.
- Restriction of H-distinguished representations.
- Three rings of invariant differential operators.
- Main theorem: Spectral Analysis on $\Gamma\backslash G/H$.

Towards Spectral Analysis for $X_\Gamma = \Gamma\backslash G/H$

Ideas and Methods

(• generalized Poincaré series)

- Branching of the restriction $G \downarrow G'$

$$H \underset{\text{induction}}{\nearrow} G \underset{\text{restriction}}{\searrow} G'.$$

  - Discrete decomposability
  - Uniformly bounded Multiplicities.

- Transfer of spectrum
  - Structure of the rings of invariant differential operators.

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G'-admissible restriction — for non-compact subgroup G'

$G \supset G'$ real reductive Lie groups,

$\Pi \in \widehat{G}$ irred unitary rep of G.

Definition The restriction $\Pi|_{G'}$ is said to be G'-admissible if

$$\Pi|_{G'} \simeq \sum_{\pi \in \widehat{G'}}{}^{\oplus} m_\pi \pi \quad (\text{discrete sum})$$

with $m_\pi := [\Pi|_{G'} : \pi] < \infty$ for all $\pi \in \widehat{G'}$.

- Condition: No continuous spectrum & finite multiplicity
- ($G' = K$ case) Any $\Pi \in \widehat{G}$ is K-admissible in our terminology, if K is a max compact subgroup of G (Harish-Chandra's admissibility theorem).

Restriction $G \downarrow G'$

$G \supset G'$ reductive groups $/\mathbb{R}$

Consider the restriction $\Pi|_{G'}$ for $\Pi \in \widehat{G}$.

Idea: We derive useful information for the restriction $G \downarrow G'$ from $G \downarrow K'$. There are two paths to reach K' from G:

$$\begin{array}{ccc} G & \supset & G' \\ \cup & & \cup \\ K & \supset & K' \end{array}$$

reductive groups $/\mathbb{R}$

max compact subgroups

Criterion for admissible restrictions $G \downarrow G'$

Theorem 1 (K– 1998, 2021)* Let $G \supset G'$ be real reductive groups and $\Pi \in \widehat{G}$. If

$$\mathrm{AS}_K(\Pi) \cap C_K(K') = \{0\},$$

then the restriction $\Pi|_{G'}$ is admissible.

Two closed cones in the dual $\sqrt{-1}\mathfrak{t}^*$ of a Cartan subalg $\mathfrak{t} \subset \mathfrak{k}$:

$\mathrm{AS}_K(\Pi)$: asymptotic K-support of Π,

$C_K(K')$: momentum set of $T^*(K/K') \to \sqrt{-1}\mathfrak{t}^*$.

Remark. When $G' = K$, the assumption $C_K(K') = \{0\}$ is obvious, and the conclusion corresponds to Harish-Chandra's admissibility.

* Kobayashi, Ann Math 1998; see also Kobayashi, Proc. ICM 2002; PAMQ 2021 (Kostant memorial issue).

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Restriction under good control

$$G \supset G' \quad \text{and} \quad \Pi \in \widehat{G} \quad (\subset \mathrm{Irr}(G)).$$

A. Admissible restriction $\Pi|_{G'}$ (Theorem 1)
(discretely decomposable with finite multiplicity).

$\rightsquigarrow$ Allow continuous spectrum & non-unitary reps

B. Finite multiplicity restriction $\Pi|_{G'}$

$$[\Pi|_{G'} : \pi] < \infty \qquad {}^\forall \pi \in \mathrm{Irr}(G').$$

C. Bounded multiplicity restriction $\Pi|_{G'}$

$$\sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

Multiplicity of the restriction $\Pi|_{G'}$ including non-unitary case

G : real reductive Lie group

$\mathcal{M}(G)$: smooth admissible reps of G of finite length
with moderate growth (defined on Fréchet spaces)
$\mathrm{Irr}(G)$: irreducible objects

$$\underset{\text{unitary dual}}{\widehat{G}} \hookrightarrow \mathrm{Irr}(G), \quad \Pi \mapsto \Pi^{\infty}.$$

$G \supset G'$: real reductive groups

Definition (multiplicity) For $\Pi \in \mathrm{Irr}(G)$ and $\pi \in \mathrm{Irr}(G')$, we set

$$\mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) := \{\text{symmetry breaking operators}\}$$

$$[\Pi|_{G'} : \pi] := \dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi) \in \mathbb{N} \cup \{\infty\}$$

Comparison: GL(n,R) ↓ O(n) *vs* GL(n,R) ↓ O(p,n-p)

Harish-Chandra's admissibility theorem concerns the restriction with respect to a maximal compact subgroup

$$G \supset K, \qquad \text{e.g., } GL(n, \mathbb{R}) \supset O(n)$$

and asserts

$$[\Pi|_K : \pi] < \infty \qquad {}^\forall \Pi \in \mathrm{Irr}(G) \text{ and } {}^\forall \pi \in \mathrm{Irr}(K).$$

In contrast,

For a reductive symmetric pair

$$G \supset G', \qquad \text{e.g., } GL(n, \mathbb{R}) \supset O(p, n-p)$$

it may well happen that

$$[\Pi|_{G'} : \pi] = \infty \qquad \text{for some } \Pi \in \mathrm{Irr}(G) \text{ and } \pi \in \mathrm{Irr}(G').$$

Spherical Space

$G_{\mathbb{C}}$ complex reductive $\curvearrowright X_{\mathbb{C}}$ complex manifold (connected)

Definition $X_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Example Grassmannian manifolds, flag manifolds, symmetric spaces are spherical spaces.

Restriction $G \downarrow G'$ with uniformly bounded multiplicity property

Theorem 2 (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (also (ii)′ or (ii)″).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \mathrm{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(ii)″ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

- The equivalence (i) ⇔ (ii) is proved in [TK–T. Oshima]*.
- A stronger estimate for (ii) ⇒ (i), namely, multiplicity-free theorem holds for most of (not all of) the cases (Sun–Zhu)**.
- Classification for (ii): $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}'_{\mathbb{C}})$ is $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{gl}(n-1,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$, or up to direct product, abelian factors, or automorphisms (Cooper, Kostant, Krämer).

* T. Kobayashi–T. Oshima, "Finite multiplicity theorems for induction and restriction", Adv. Math., (2013), 921–943.

** Sun–Zhu, "Multiplicity one theorems: the Archimedean case", Ann. of Math., (2012), 23–44.

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Restriction of H-distinguished rep $H \nearrow G$

Definition Suppose H is a closed subgroup of G.

We say $\Pi \in \mathrm{Irr}(G)$ is an H-distinguished rep of G, if $(\Pi^{-\infty})^H \neq \{0\}$, or equivalently, if

$$\mathrm{Hom}_G(\Pi, C^\infty(G/H)) \neq \{0\}.$$

We set

$\mathrm{Irr}(G)_H$:= $\{H$-distinguished irreducible admissible reps$\} \subset \mathrm{Irr}(G)$.

Borel subgroup $B_{G/H}$ for a symmetric space G/H

Let G/H be a reductive symmetric space defined by an involution σ of G.

Definition* (Borel subalg $\mathfrak{b}_{G/H}$)

A Borel subalgebra $\mathfrak{b}_{G/H}$ for G/H is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ defined by a generic semisimple element in $\mathfrak{g}_{\mathbb{C}}^{-\sigma}$ or its conjugate.

Remark • Our "Borel subalgebra" $\mathfrak{b}_{G/H}$ is not necessarily solvable.

• $\mathfrak{b}_{G/H}$ is determined by the complexified symmetric pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

* T. Kobayashi, Multiplicity in restricting small representations, Proc. Acad. Japan (2022).

Bounded multiplicity theorem for H-distinguished reps

(G, H) reductive symmetric pair

G' reductive subgroup of G.

Theorem 3 (K– 22)* (bounded multiplicity criterion) (i) $\Longleftrightarrow$ (ii).

(i) (Rep) The triple $H \subset G \supset G'$ satisfies

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$$

(ii) (Complex Geometry) $G_{\mathbb{C}} / B_{G/H}$ is $G'_{\mathbb{C}}$-spherical.

* T. Kobayashi, Bounded multiplicity for induction and restriction, J. Lie Theory, (2022); Proc. Japan Academy., (2022).

Special case of Theorem 3: $\mathrm{diag}(G) \nearrow G \times G \searrow G' \times G'$

Observe that

$$\mathrm{Irr}(G) \simeq \mathrm{Irr}(G \times G)_{\mathrm{diag}\,G}$$

$$\pi \quad \leftrightarrow \quad \pi \boxtimes \pi^{\vee}$$

Theorem 2 is a special case of Theorem 3.

$$\mathrm{Irr}(G) \qquad\qquad \mathrm{Irr}(G)_H$$

$$\| $$

$$\mathrm{Irr}(G \times G)_{\mathrm{diag}\,G}$$

Theorem 2 (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\,\mathrm{diag}(G'_{\mathbb{C}})$ is spherical.

Theorem 3 (K– 22)* (bounded multiplicity criterion) (i) $\Longleftrightarrow$ (ii).

(i) (Rep) The triple $H \subset G \supset G'$ satisfies $\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

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T. Kobayashi–T. Oshima, "Finite multiplicity theorems for induction and restriction", Adv. Math., (2013), 921–943.

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Bounded multiplicity triple $H \nearrow G \searrow G'$ — Classification

Classification: All the triples $H \subset G \supset G'$ having the bounded multiplicity property:

$$\sup_{\Pi \in \mathrm{Irr}(G)_H} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty \qquad (*)$$

was classified in [K–22]*. This extends the classification of Cooper, Krämer, Kostant in the case where G/H is a group manifold.

Example Let $p_1 + p_2 = p$, $q_1 + q_2 = q$, and
$(H, G, G') = (O(p-1, q), O(p, q), O(p_1, q_1) \times O(p_2, q_2))$.
Then one has the bounded multiplicity property $(*)$.

* Kobayashi, Adv. Math., (2021), Bounded multiplicity theorem for induction and restriction, J. Lie Theory (2022) 197–238.

Question: $H \nearrow G \searrow G'$

Induction: $H \nearrow G$:

For $H \subset G$, we set $X = G/H$.

$$G \curvearrowright C^\infty(X) = C^\infty(G/H).$$

Restriction: $G \searrow G'$

For $G \supset G'$, we consider the restriction of actions.

$$G' \ (\subset G) \ \curvearrowright \Pi \in \mathrm{Irr}(G).$$

$H \nearrow G \searrow G'$: Consider the restriction $\Pi|_{G'}$ when $\Pi \in \mathrm{Irr}(G)_H$, that is, when Π occurs in $C^\infty(G/H)$.

Question What if the G'-action on $X = G/H$ is proper?

Proper action and Admissible restriction $H \nearrow G \searrow G'$

Setting: $G' \subset G \supset H$ real reductive, $X := G/H$.

reductive symmetric space

Theorem 4* Suppose $G' \curvearrowright X$ proper and $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.
If $\Pi \in \widehat{G}$ is H-distinguished, then the restriction $\Pi|_{G'}$ is G'-admissible and the multiplicities are uniformly bounded.

Cf. Theorem 1 (admissibility criterion) is formulated purely by representation theory. The proof of Theorem 4 is geometric and interacts with a structure of three rings of invariant differential operators.

* T. Kobayashi, Global analysis by hidden symmetry, Progr. Math., **323** (2017), 359–397; Kassel–TK, Lecture Notes in Math. (2015).

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Good Control of Restriction $G \downarrow G'$

Theorem 2 (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (also (ii)′ or (ii)″).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty.$

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \mathrm{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

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Geometry $\qquad$ Representation

$G_{\mathbb{C}} \times G'_{\mathbb{C}} / \mathrm{diag}(G'_{\mathbb{C}}) \quad \rightsquigarrow \quad \Pi|_{G'}$

$U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$

Algebra

* T. Kobayashi, T. Oshima, Adv. Math., **248** (2013), 921–944 for (i) ⇔ (ii).

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Ring structure of $\mathbb{D}_{G'_{\mathbb{C}}}(X_{\mathbb{C}})$ for $H \overset{\text{ind}}{\nearrow} G \overset{\text{rest}}{\searrow} G'$

We focus on two rings of differential operators on $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$.

(1) $\mathcal{P} := \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$: $G_{\mathbb{C}}$-invariant differential operators;

(2) $\mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))$: induced from the center $Z(\mathfrak{g}'_{\mathbb{C}})$ of $U(\mathfrak{g}'_{\mathbb{C}})$.

$$\mathcal{P} \rightsquigarrow \Pi \in \mathrm{Irr}(G)_H,$$
$$\mathcal{R} \rightsquigarrow \pi \in \mathrm{Irr}(G').$$

Understanding the relation between $\mathcal{P}$ and $\mathcal{R}$ will help us to understand branching $G \downarrow G'$.

Hidden Symmetries and Invariant Differential Operators

We recall $G'_{\mathbb{C}} \subset G_{\mathbb{C}} \curvearrowright X_{\mathbb{C}}$.

We shall see a "hidden symmetry" of the algebra $Z(\mathfrak{g}'_{\mathbb{C}})$ in the space of joint eigenfunctions for the algebra $\mathbb{D}_G(X_{\mathbb{C}})$:

hidden symmetry

$$Z(\mathfrak{g}'_{\mathbb{C}}) \quad \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \curvearrowright C^{\infty}(X;\mathcal{M}_{\lambda}).$$

under the assumption is that $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

Geometry for $H \overset{\text{ind}}{\nearrow} G \overset{\text{rest}}{\searrow} G'$

Let $H \subset G \supset G'$ be real reductive Lie groups.
Assume that $G' \curvearrowright X = G/H$ proper and that $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

$\exists$ G'-fibration	F	$\to$	X	$\to$	Y
Signature	$(0, q)$		(p, q)		$(p, 0)$

Example $(G, G') = (SO(2m, 2), U(m, 1))$

$S^1 \to$ AdS^{2m+1} (anti de Sitter space) $\to$ $\{z \in \mathbb{C}^m : |z| < 1\}$ (Hermitian ball).

* TK, Invent. Math., 1994.

Invariant differential operator for overgroups

G' $\overset{\text{proper, spherical}}{\curvearrowright}$ $X = G/H$

$\rightsquigarrow$ ${}^{\exists}$ G'-fibration $F \to X \to Y$.

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Two subalgebras of $\mathbb{D}_{G'}(X)$

$\mathcal{P} := \mathbb{D}_G(X)$, $\quad Q := \iota(\mathbb{D}_{K'}(F))$, $\quad \mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))$

Invariant differential operator for overgroups

$$F = K'/H' \hookrightarrow X = G/H \to Y = G'/K' \text{ (fibration)}$$

$$\rightsquigarrow {}^{\exists} G'\text{-fibration } F \to X \to Y.$$

Three ~~Two~~ subalgebras of $\mathbb{D}_{G'}(X)$

$$\mathcal{P} := \mathbb{D}_G(X), \quad \mathcal{Q} := \iota(\mathbb{D}_{K'}(F)), \quad \mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))$$

Theorem 5 (Kassel–TK,19)* Assume $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

(1) The commutative algebra $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{P}$ and $\mathcal{R}$.

* F. Kassel–K, Invariant differential operators on spherical homogeneous spaces ⋯, JLT (2019).

Invariant differential operator for overgroups

$$\boxed{F = K'/H'} \hookrightarrow \boxed{X = G/H} \to \boxed{Y = G'/K'} \text{ (fibration)}$$

$\rightsquigarrow$ $^{\exists}$ G' -fibration $F \to X \to Y$.

Three ~~Two~~ subalgebras of $\mathbb{D}_{G'}(X)$

$$\mathcal{P} := \mathbb{D}_G(X), \quad \mathcal{Q} := \iota(\mathbb{D}_{K'}(F)), \quad \mathcal{R} := dl(Z(\mathfrak{g}'_{\mathbb{C}}))$$

<u>Theorem 5</u> (Kassel–TK,19)* Assume $X_{\mathbb{C}}$ is $G'_{\mathbb{C}}$-spherical.

(1) The commutative algebra $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{P}$ and $\mathcal{R}$.

Let K' be a maximal subgroup of G' containing H'.

(2) $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{P}$ and $\mathcal{Q}$, too.

(3) $\mathbb{D}_{G'}(X)$ is generated by $\mathcal{Q}$ and $\mathcal{R}$, in the quotient field.

* F. Kassel–K, Invariant differential operators on spherical homogeneous spaces ⋯, JLT (2019).

Plan

- Spectral analysis on $\Gamma\backslash G/H$ beyond the Riemannian setting.
- Admissible restriction to reductive subgroups.
- Bounded/finite multiplicities in the branching laws.
- Restriction of H-distinguished representations. $H \nearrow G \searrow G'$
- Three rings of invariant differential operators. $H \nearrow G \searrow G'$
- Main theorem: Spectral Analysis on $\Gamma\backslash G/H$.

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Application of branching problem $G \downarrow G'$

We apply these results to the spectral analysis of

standard, pseudo-Riemannian locally symmetric space $\Gamma \backslash G/H$

beyond the Riemannian setting:

- (G, H) is a reductive symmetric pair with H non-compact.
- G' acts properly on $X = G/H$.
- Γ is a torsion-free discrete subgroup of G'.

Spectral analysis of standard locally symmetric space $\Gamma\backslash G/H$

Let $X = G/H$ be a reductive symmetric space. We set

$$\mathbb{D}_G(X)^\wedge := \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{D}_G(X), \mathbb{C}) \ni \lambda$$

$$\rightsquigarrow \mathcal{M}_\lambda : Df = \lambda(D)f \quad {}^\forall D \in \mathbb{D}_G(X).$$

$$X = \underbrace{G/H}_{\text{local geometric structure}} \xrightarrow{\text{covering}} \underbrace{\Gamma}_{\text{global}} \backslash G/H = X_\Gamma$$

<u>Problem</u> Find spectral decomposition of $C_c^\infty(X_\Gamma)$ and $L^2(X_\Gamma)$ for "intrinsic differential operators" on X_Γ induced from $\mathbb{D}_G(X)$.

Spectral analysis of standard locally symmetric space $\Gamma\backslash G/H$

Let $X = G/H$ be a reductive symmetric space. We set

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$$\rightsquigarrow \mathcal{M}_\lambda : Df = \lambda(D)f \quad {}^\forall D \in \mathbb{D}_G(X).$$

Suppose that a reductive subgroup G' acts on X properly such that $X_\mathbb{C}$ is $G'_\mathbb{C}$-spherical. Take any discrete subgroup Γ of G'.

Main Theorem (expansion into eigenfunctions, Kassel–TK*, 2025)
There exist measure μ on $\mathbb{D}_G(X)^\wedge$ and a measurable family of maps

$$\mathcal{F}_\lambda \colon C_c^\infty(\Gamma\backslash X) \to C^\infty(\Gamma\backslash X; \mathcal{M}_\lambda)$$

s.t. any $f \in C_c^\infty(\Gamma\backslash X)$ is expanded into joint eigenfunctions on $\Gamma\backslash X$:

$$f = \int_{\mathbb{D}_G(X)^\wedge} \mathcal{F}_\lambda f \, d\mu(\lambda),$$

$$\|f\|^2_{L^2(\Gamma\backslash X)} = \int_{\mathbb{D}_G(X)^\wedge} \|\mathcal{F}_\lambda f\|^2_{L^2(\Gamma\backslash X)} d\mu(\lambda).$$

* F. Kassel–K, Spectral analysis on standard locally homogeneous spaces. Lecture Notes in Math. 2025, 126 pages. (in press).

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Strategy for Spectral Analysis on $\Gamma\backslash G/H$

1. Standard quotient

proper

$$\Gamma \subset G' \subset G \curvearrowright X \quad \rightsquigarrow \quad \Gamma\backslash X = \Gamma\backslash G/H$$

2. (Hidden symmetry) If $G'_{\mathbb{C}} \curvearrowright X_{\mathbb{C}}$ is spherical, one has (Theorems 2, 3, 5)

hidden symmetry

$$Z(\mathfrak{g}'_{\mathbb{C}}) \quad \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \curvearrowright C^{\infty}(X; \mathcal{M}_{\lambda}).$$

3. (Branching law $G \downarrow G'$) If $G' \curvearrowright X$ proper, then any $\pi \in \mathrm{Irr}(G)$ realized in $C^{\infty}(X)$ is G'-admissible (Theorem 4).

Thank you very much!

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