Basic Questions
in Group-Theoretic Analysis on Manifolds

Basic Questions in Group-Theoretic Analysis on Manifolds

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Basic Questions in Group-Theoretic Analysis on Manifolds

$$G \curvearrowright X \qquad \rightsquigarrow \qquad G \curvearrowright C^\infty(X),\ L^2(X), \cdots$$

Geometry

Functions

- Plan
  1. Is representation theory useful to the global analysis on X?
  2. What can we say about the “spectrum” on $L^2(X)$?

IHP Minicourses: More details on two “young” areas

The forthcoming two mini-courses, designed for students and non-experts, will address two “young” research themes related to these questions, to be held in January and February at IHP, Paris.

1. Branching Problems and Symmetry Breaking

   IHP Mini-Courses by B. Ørsted, M. Pevzner, B. Speh, and TK

   January 13–17, Paris,

2. “Tempered Spaces”
   — “Geometry” for Tempered Representations

   IHP Mini-Courses by TK

   February 17–21, Paris.

Regular Representation 1

$G \curvearrowright X$ (manifold) $\rightsquigarrow$ $G \curvearrowright C^\infty(X), L^2(X), \cdots$

Geometry

Functions

$G \curvearrowright C^\infty(X)$

One deduces a rep of G on $C^\infty(X)$ by $f(x) \mapsto f(g^{-1}x)$.

$G \overset{\lambda_X}{\curvearrowright} L^2(X)$: the canonical unitary representation of G.

$L^2(X) := L^2(X, \nu_X)$ if ${}^\exists G$-invariant Radon measure ν_X.

More generally, define $L^2(X)$ by using the half-density bundle of X or by a multiplier rep built on the cocycle $c(g,x)$, where

$g_*\nu_X = c(g,x)\nu_X$ (Radon–Nykodim derivative).

Regular Representation 2

$$\boxed{G \curvearrowright X \text{ (manifold)} \ \rightsquigarrow\ G \overset{\lambda_X}{\curvearrowright} L^2(X) \quad \text{(Hilbert space)}}$$

$\widehat{G}$:= {irreducible unitary representations} (unitary dual).

<u>Mautner</u>: Any unitary rep Π of G is disintegrated into irreducibles:

$$\boxed{\Pi \simeq \int_{\widehat{G}}^{\oplus} \underline{m_\pi} \pi \, d\mu(\pi) \qquad \text{(direct integral)}}$$

$$m \colon \widehat{G} \to \mathbb{N} \cup \{\infty\}, \quad \pi \mapsto m_\pi \quad \text{(multiplicity)}.$$

$$\underline{m_\pi} \pi = \underbrace{\pi \oplus \cdots \oplus \pi}_{m_\pi}$$

Simple Lie groups, Reductive Lie groups

Viewpoints from “Analysis and Synthesis”

– The “ smallest units ” of (unitary) representations are irreducible (unitary) representations.

– The “ smallest units ” of Lie groups are one-dimensional abelian groups and simple Lie groups.

- Simple Lie groups :
  A Lie group G of dimension N (> 1) is a simple Lie group

  e.g. $SL(n, \mathbb{R})$, $SL(n, \mathbb{C})$, $SO(p, q)$ $(p + q \neq 2, 4)$,

- Reductive Lie group $\underset{\text{local}}{\fallingdotseq}$ abelian × simple Lie groups

  e.g. $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$, $SO(p, q)$ $(p, q$: any$)$, $\cdots$

Basic Questions in Group-Theoretic Analysis on Manifolds

- General questions on regular representations

  1. Does the group sufficiently control the space of functions?
  2. What can we say about the “spectrum” on $L^2(X)$?

Global analysis via representation theory

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X),\ L^2(X), \cdots$$

Geometry

Functions

Basic Question 1

Is representation theory useful to the global analysis on X?

Connection of the two viewpoints

X : (pseudo-)Riemannian manifold

Spectral analysis of Δ_X: $L^2(X) \simeq \int \mathcal{H}_\lambda d\tau(\lambda)$.

“generalize” $\overset{\wr}{\downarrow}$ $\underset{\wr}{\uparrow}$ if $m_\pi = 1$

Representation Theory: $L^2(X) \simeq \int_{\widehat{G}}^{\oplus} m_\pi \pi d\mu(\pi)$.

Group action: $G \curvearrowright X$

Example

$O(n+1)$	$\curvearrowright$	$\mathbb{S}^n$,	
$O(n,1)$	$\curvearrowright$	$\mathbb{H}^n$	(hyperbolic space),
$O(p,q)$	$\curvearrowright$	Space form	(pseudo-Riemannian).

Multiplicities in regular representations

$$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^{\infty}(X),\ L^2(X), \cdots$$

Geometry | Functions

Basic Question 1

Is representation theory useful to the global analysis on X?

Hint for rigorous formulation. In group representations:

–strong point: Can distinguish inequivalent irreducible reps even they are infinite-dimensional.

–weak point: Multiplicity .

(cannot distinguish a multiple of the same irreducible reps)

Multiplicities in regular representations

$G \curvearrowright X$ $\rightsquigarrow$ $G \curvearrowright C^{\infty}(X)$ (regular rep)

Geometry   Functions

Basic Question 1

Does the group G " control well " the function space $C^{\infty}(X)$?

Formulation Consider the multiplicity, *i.e.*,

the dimension of $\mathrm{Hom}_G(\pi, C^{\infty}(X))$ for $\pi \in \mathrm{Irr}(G)$.

infinite, finite, bounded, 0 or 1

control better

Spherical manifold

$G_{\mathbb{C}}$: a complex reductive Lie group.

Definition Borel subgroup B of $G_{\mathbb{C}}$

$\underset{\text{def}}{=}$ maximal connected solvable subgp of $G_{\mathbb{C}}$

e.g. $$B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \subset GL(3, \mathbb{C}) = G_{\mathbb{C}}$$

$G_{\mathbb{C}}$ complex reductive $\curvearrowright X_{\mathbb{C}}$ complex manifold (connected)

Definition $X_{\mathbb{C}}$ is spherical if a Borel subgroup B of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$.

Example Grassmannian varieties, flag varieties, symmetric spaces, are typical examples of spherical spaces.

Multiplicities in regular representations

$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X)$ (regular rep)

Geometry Functions

Basic Question 1

Does the group G “control well” the function space $C^\infty(X)$?

Formulation Find a geometric estimate of the **multiplicity**

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\pi, C^\infty(X)) \quad \text{for } \pi \in \operatorname{Irr}(G).$$

infinite, finite, bounded, 0 or 1

control better

When does the group “control” well the function space?

For a pair of reductive Lie groups $G \supset H$, consider $X = G/H$.

Theorem A* The following 4 conditions are equivalent:

(i) (Global analysis & rep theory) There exists $C > 0$ s.t. $\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq C$ for all $\pi \in \mathrm{Irr}(G)$.

(ii) (Complex geometry) $X_{\mathbb{C}}$ is spherical.

(ii)′ (Algebra) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is commutative.

(ii)′′ (Algebra) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is a polynomial ring.

- Remarkably, (i) uniform boundedness of the multiplicity is detected soley by the complexification $X_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ in (ii)-(ii)′′.
- The equiv (ii) ⇔ (ii)′ ⇔ (ii)′′ was proven by Vinberg, Knop, $\cdots$.
- The equivalence (i) ⇔ (ii) gives a strong tie between

Global analysis ⟺ Algebra, Geometry,

which was proven in TK–T. Oshima*.

* T. Kobayashi, T. Oshima, Adv. Math., **248** (2013), 921–944 for (i) ⇔ (ii).

When does the group “control” well the function space?

Theorem A* Let $X = G/H$, where $G \supset H$ are reductive Lie groups.

The following four conditions (i), (ii), (ii)′ and (ii)″ are equivalent:

(i) (Global analysis & rep theory) There exists $C > 0$ such that

$$\dim \mathrm{Hom}_G(\pi, C^\infty(X)) \leq C \text{ for all } \pi \in \mathrm{Irr}(G).$$

(ii) (Complex geometry) $X_{\mathbb{C}}$ is spherical.

(ii)′ (Ring) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is commutative.

(ii)″ (Ring) The ring $\mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}})$ is a polynomial ring.

$$\begin{tikzcd}
\text{Geometry} & & \text{Analysis} \\
G_{\mathbb{C}} \curvearrowright X_{\mathbb{C}} \arrow[rr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & G \curvearrowright C^\infty(X) \\
& \mathbb{D}_{G_{\mathbb{C}}}(X_{\mathbb{C}}) \arrow[ur, rightsquigarrow] & \\
& \text{Algebra} &
\end{tikzcd}$$

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* T. Kobayashi–T. Oshima, “Finite multiplicity theorems for induction and restriction”, Adv. Math., (2013), for (i)⇔(ii).

Sketch of proof

(i) Global Analysis & Rep Theory $\Longleftrightarrow$ (ii) Complex Geometry

$\dim \mathrm{Hom}_{\mathbb{C}}(\pi, C^{\infty}(X)) \leq C$ $\qquad$ $G_{\mathbb{C}} \curvearrowright X_{\mathbb{C}}$ spherical

The original proof in [KO]* uses PDEs and integration.

Methods of proof*: Interpret (i) by means of PDEs

(i) $\Leftarrow$ (ii) (Differential equations)

Determine "solutions to PDEs" by "boundary values"

$\rightsquigarrow$ Reduction to geometry of boundaries :

Equivariant compactification

+ hyperfunction-valued boundary maps for a system of PDEs.

(i) $\Rightarrow$ (ii) (Integral operator)

Construct "solutions to PDEs" from "data on boundaries"

$\rightsquigarrow$ Find integral operators from functions on boundaries

(a generalization of the Poisson integral).

* T. Kobayashi, T. Oshima, Adv. Math., **248** (2013), 921–944.

Induction *vs.* Restriction

$H \qquad \subset \qquad G$ groups

$$\begin{tikzcd} \{H\text{-modules}\} \arrow[r, rightsquigarrow, shift left, "\text{Induction}"] & \{G\text{-modules}\} \arrow[l, rightsquigarrow, shift left, "\text{Restriction}"] \end{tikzcd}$$

<u>Example</u>(Induction) $\operatorname{Ind}_H^G(\mathbf{1}) \simeq C^\infty(G/H), L^2(G/H), \cdots ,$
(depending on the class of "Induction").

We now consider the H-module by Restriction :

$$\Pi|_H \equiv \operatorname{Rest}_H^G(\Pi) \quad \text{for} \quad \Pi \in \operatorname{Irr}(G).$$

Branching problems in the general setting

$$\begin{tikzcd}
G \arrow[r, "\Pi", "\text{irreducible}"'] & GL(V) \\
\cup & \\
G' \arrow[uur, dashed, "\Pi|_{G'}"'] &
\end{tikzcd}$$

Branching problems in the general setting

$$\begin{tikzcd}
G \arrow[r, "\Pi", "\text{irreducible}"'] & GL(V) \\
\cup & \\
G' \arrow[uur, dashed, "\Pi|_{G'}"'] &
\end{tikzcd}$$

Example (tensor product of two representations)

$$\begin{tikzcd}
G_1 \times G_1 \arrow[r, "\pi' \boxtimes \pi''", "\text{outer tensor product}"'] & GL(V' \otimes V'') \\
\cup & \\
\operatorname{diag} G_1 \arrow[uur, dashed, "\pi' \otimes \pi''"'] &
\end{tikzcd}$$

Branching problems in the general setting

$$\begin{tikzcd}
G \arrow[r, "\Pi", "\text{irreducible}"'] & GL(V) \\
\cup & \\
G' \arrow[uur, dashed, "\Pi|_{G'}"'] &
\end{tikzcd}$$

Branching problem (in a broader sense than the usual)

$\cdots$ wish to understand

how the restriction $\Pi|_{G'}$ behaves as a G'-module.

- For $\Pi \in \mathrm{Irr}(G)$, $\pi \in \mathrm{Irr}(G')$,

$$[\Pi|_{G'} : \pi] := \dim \mathrm{Hom}_{G'}(\Pi|_{G'}, \pi).$$

Good Control of Restriction $G \downarrow G'$

Theorem B (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (⇔ (ii)′ ⇔ (ii)″).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \operatorname{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(ii)″ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

- The equivalence (i) ⇔ (ii) is proved in (T. Kobayashi–T. Oshima)*.
- A stronger estimate for (ii) ⇒ (i), namely, multiplicity-free theorem holds for most of (not all of) the cases (Sun–Zhu)**.
- Classification for (ii): If G is simple, $(\mathfrak{g}, \mathfrak{g}')$ is $(\mathfrak{sl}(n,\mathbb{C}), \mathfrak{gl}(n-1,\mathbb{C}))$, $(\mathfrak{so}(n,\mathbb{C}), \mathfrak{so}(n-1,\mathbb{C}))$, or their real forms up to automorphisms.

[*] T. Kobayashi–T. Oshima, “Finite multiplicity theorems for induction and restriction”, Adv. Math., (2013), 921–943.

[**] Sun–Zhu, “Multiplicity one theorems: the Archimedean case”, Ann. of Math., (2012), 23–44.

Good Control of Restriction $G \downarrow G'$

Theorem B (Uniformly bounded multiplicity criterion)

For a pair $G \supset G'$ of real reductive groups, (i)⇔(ii) (also (ii)′ or (ii)″).

(i) (Rep) $\sup_{\Pi \in \mathrm{Irr}(G)} \sup_{\pi \in \mathrm{Irr}(G')} [\Pi|_{G'} : \pi] < \infty$.

(ii) (Geometry) $(G_{\mathbb{C}} \times G'_{\mathbb{C}}) / \operatorname{diag}(G'_{\mathbb{C}})$ is spherical.

(ii)′ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is commutative.

(ii)″ (Ring) The ring $U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}}$ is a polynomial ring.

$$\begin{tikzcd}
\text{Geometry} & & \text{Representation} \\
G_{\mathbb{C}} \times G'_{\mathbb{C}} / \operatorname{diag}(G'_{\mathbb{C}}) \arrow[rr, rightsquigarrow] \arrow[dr, rightsquigarrow] & & \Pi|_{G'} \\
& U(\mathfrak{g}_{\mathbb{C}})^{G'_{\mathbb{C}}} \arrow[ur, rightsquigarrow] & \\
& \text{Algebra} &
\end{tikzcd}$$

Basic Questions
in Group-Theoretic Analysis on Manifolds

Plan

- General questions on regular representations

$$G \curvearrowright X \text{ (manifold)} \rightsquigarrow G \curvearrowright C^\infty(X), L^2(X), \cdots$$

Geometry

Functions

1. Does the group sufficiently control the space of functions on X?

2. What can we say about the "spectrum" on $L^2(X)$?

Second theme of this talk

$G \curvearrowright X$ $\rightsquigarrow$ $G \curvearrowright L^2(X)$ (regular rep)

Geometry — Function Space

Basic Question 2 What can we say about the "spectrum" on $L^2(X)$?

Tempered representations

Let G be a locally compact group.

Def A unitary rep π of G is called tempered if $\pi \prec L^2(G)$.

$\prec$ $\cdots$ weakly contained

i.e., every matrix coefficient of π is a uniform limit on every compacta of G by a sequence of sum of coefficients of $L^2(G)$.

- $G \curvearrowright L^2(G)$ (regular rep)

$$L^2(G) \ni f(x) \mapsto f(g^{-1}x) \in L^2(G).$$

- For a unitary rep π of G on a Hilbert space $\mathcal{H}$, matrix coefficients are functions on G defined by

$$\varphi_{u,v}(g) := (\pi(g)u, v)_{\mathcal{H}} \in C(G)$$

for $u, v \in \mathcal{H}$.

When is $L^2(X)$ tempered?

$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright L^2(X)$ regular rep

Geometry — Function Space

<u>Basic Question 2</u> When is $L^2(X)$ tempered?

i.e., for which G-space X, does one have

$$L^2(X) \prec L^2(G)?$$

Question: When is $L^2(X) \prec L^2(G)$?

“Young” research topics that have been recently explored from various disciplines such as

algebra	(polyhedral combinatorics, $\cdots$);
analysis	(functional analysis, L^p-matrix coefficients, $\cdots$);
geometry	(dynamical system, geometric quantization, $\cdots$);
topology	(limit algebras, quantification of proper actions, $\cdots$).

More details:

“Tempered Spaces”

— “Geometry” for Tempered Representations

IHP Mini-Courses, February 17–21, Paris.

References Y. Benoist–TK, “Tempered Homogeneous Spaces”

I (2015), II (2022), III (2021), IV (2023), $+\varepsilon$.

Temperedness under disintegration

Mautner: Any unitary rep Π can be decomposed into irreducibles:

$$\Pi \simeq \int_{\widehat{G}}^{\oplus} m_\pi \, \pi \, d\mu(\pi) \qquad \text{(direct integral).}$$

<u>Fact</u> Π is tempered $\Leftrightarrow$ irreducible reps π are tempered for μ-a.e.

$$\begin{array}{ll} \widehat{G} & = \{\text{irreducible unitary reps}\} \\ \cup & \\ \widehat{G}_{\text{temp}} \equiv \widehat{G_r} & := \{\text{irreducible tempered reps}\}. \end{array}$$

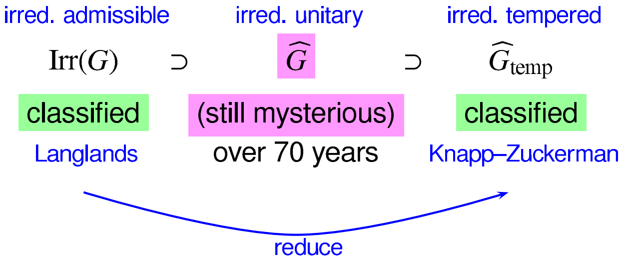
That is,

$$\Pi \text{ is tempered} \iff \int_{\widehat{G}_{\text{temp}}}^{\oplus} m_\pi \, \pi \, d\mu(\pi).$$

Classification theory of the unitary dual $\widehat{G}$

Fact (Duflo) Classification problem of the unitary dual $\widehat{G}$ for real algebraic groups G is reduced to that for real reductive Lie groups.

Suppose G is a real reductive Lie group (*e.g.*, $GL(n,\mathbb{R})$, $O(p,q)$).



irred. admissible		irred. unitary		irred. tempered
$\mathrm{Irr}(G)$	$\supset$	$\widehat{G}$	$\supset$	$\widehat{G}_{\mathrm{temp}}$
classified		(still mysterious)		classified
Langlands		over 70 years		Knapp–Zuckerman

reduce

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$= \{\,\mathbf{1}\,\} \amalg \{$ principal series $\} \amalg \{$ complementary series $\}$

$\amalg \{$ discrete series $\} \amalg \{$ limit of discrete series $\}$

Tempered representations (warming up)

V. Bargmann (1947): Irreducible unitary reps of $SL(2,\mathbb{R})$

$$= \{\, \mathbf{1} \,\} \amalg \{\, \text{principal series} \,\} \amalg \{\, \text{complementary series} \,\}$$

$$\amalg \{\, \text{discrete series} \,\} \amalg \{\, \text{limit of discrete series} \,\}$$

$-\frac{1}{2}$ Casimir operator acts on them as scalars

$$\{0\}, \qquad [\tfrac{1}{4}, \infty), \qquad (0, \tfrac{1}{4}), \qquad \{\tfrac{1}{4}(n^2 - 1) : n \in \mathbb{N}_+\}, \qquad \{0\}$$

Γ: congruence subgroup of $G = SL(2,\mathbb{R})$

Selberg's $\frac{1}{4}$ eigenvalue conjecture *:

All eigenvalues of Δ on Maas wave forms for $\Gamma \geq \frac{1}{4}$.

$\Longleftrightarrow$ The unitary rep of $G \curvearrowright L^2_{\mathrm{cusp}}(\Gamma\backslash G)$ is tempered.

Just one irred non-tempered rep would disprove the conjecture.

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* A. Selberg, On the estimate of Fourier coefficients of modular forms, Proc. Symp. Pure Math. 1965.

When is $L^2(X)$ tempered?

$G \curvearrowright X$ $\rightsquigarrow$ $G \curvearrowright L^2(X)$ regular rep

Geometry Function Space

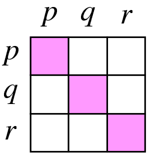
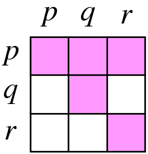
Basic Question 2 When is $L^2(X)$ tempered, that is, when is $L^2(X)$ weakly contained in $L^2(G)$?

- Even when $X = G/H$ is a reductive symmetric space, this question involves a hard problem regarding vanishing conditions of cohomological parabolic inductions with singular parameters.
- How about more general (non-symmetric) space $X = G/H$?

$\rightsquigarrow$ A new machinery?

Examples of temperedness criterion

Example(2022*) $G = GL(p+q+r,\mathbb{R})$

H	$L^2(G/H)$ is tempered
	$\begin{cases} p \leq q+r+1 \\ q \leq p+r+1 \\ r \leq p+q+1 \end{cases}$
	$\begin{cases} p = 1 \\ q \leq r+1 \\ r \leq q+1 \end{cases}$

Why?

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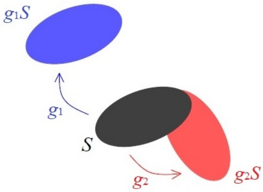
* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces II, In; Festschrift of Margulis, Chicago Univ. Press, (2022).

Topology (proper actions) $\rightsquigarrow$ Quantification

Definition A continuous action $G \curvearrowright X$ is called proper if the subset

$$G_S := \{g \in G : S \cap gS \neq \emptyset\}$$

is compact for any compact subset $S \subset X$.



$$\Longrightarrow \quad g_1 \notin G_S, \quad g_2 \in G_S$$

Two “quantifications” of properness of the action on $X = G/H$:

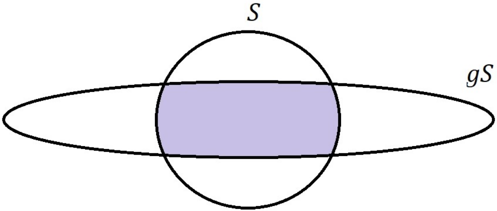
“asymptotic” volume (Benoist–K, ’15) $\xrightarrow[\text{“more proper”}]{\text{proper}}$ “sharpness condition” (Kassel–K, ’15)

$\rightsquigarrow$ spectrum of $G \curvearrowright L^2(X)$

$\rightsquigarrow$ deformation theory of $\Gamma\backslash G/H$

First key idea for temperedness criterion

- Study the asymptotic decay of $\underline{\mathrm{vol}(S \cap gS)}$ as $G \ni g$ tends to "infinity" when S is a compact subset in X.



Function $\rho_{\mathfrak{h}} : \mathfrak{h} \to \mathbb{R}_{\geq 0}$

Setting

$\mathfrak{h}$: Lie algebra

$\mathrm{ad} : \mathfrak{h} \to \mathrm{End}_{\mathbb{R}}(\mathfrak{h})$: adjoint representation

We set

$$\rho_V : \mathfrak{h} \to \mathbb{R}_{\geq 0}, \quad Y \mapsto \frac{1}{2} \sum |\mathrm{Re}\, \lambda|$$

where the sum is taken over all eigenvalues λ of the complex linear extension $\tau(Y)_{\mathbb{C}} \in \mathrm{End}(\mathfrak{h}_{\mathbb{C}})$.

- Coincide with the usual ρ on the dominant chamber.

Tempered criterion for G/H

Theorem C (2015*, 2022)*

Let H be a connected subgp of a real reductive Lie group G.

Then (i) $\Leftrightarrow$ (ii).

(i) (Global analysis & rep theory) $L^2(G/H)$ is tempered.

(ii) (Combinatorial geometry) $2\rho_{\mathfrak{h}}(Y) \le \rho_{\mathfrak{g}}(Y), \quad {}^{\forall}Y \in \mathfrak{h}$.

$\rho_{\mathfrak{h}}$ is for $\mathrm{ad}\colon \mathfrak{h} \to \mathrm{End}(\mathfrak{h})$,

$\rho_{\mathfrak{g}}$ is for $\mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathrm{End}(\mathfrak{g})$.

Remark The criterion can be used to detect whether $L^2(X)$ is tempered or not for any real algebraic variety X with algebraic G-action, even when the G-action on X is not transitive.

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* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces I, II, Euro. J. Math. **17** (2015), pp. 3015–3036; Univ. Chicago

$G/H = GL(p+q+r)/GL(p) \times GL(q) \times GL(r)$

Example Equivalent (i) $\Longleftrightarrow$ (ii).

(i) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$ (Temperedness Criterion).

(ii) $2\max(p,q,r) \leq p+q+r+1$.

Proof The condition $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$, amounts to:

$$\sum_{1 \leq i<j \leq p} |x_i - x_j| + \sum_{1 \leq i<j \leq q} |y_i - y_j| + \sum_{1 \leq i<j \leq r} |z_i - z_j|$$
$$\leq \sum_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} |x_i - y_j| + \sum_{\substack{1 \leq j \leq q \\ 1 \leq k \leq r}} |y_j - z_k| + \sum_{\substack{1 \leq k \leq r \\ 1 \leq i \leq p}} |z_k - x_i|$$

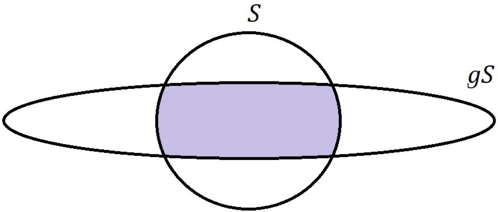
for all $(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_r) \in \mathbb{R}^{p+q+r}$ with $\sum x_i = 0$, $\sum y_j = 0$, $\sum z_k = 0$.

By some combinatorics on convex polyhedral cones, one sees

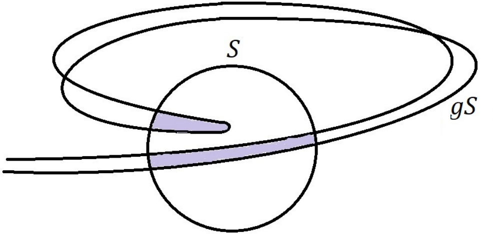
$$2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}} \Longleftrightarrow 2\max(p,q,r) \leq p+q+r+1.$$

First key idea for temperedness criterion

- Study the asymptotic decay of $\mathrm{vol}(S \cap gS)$ as $G \ni g$ tends to "infinity" when S is a compact subset in $X = G/H$.



- Global picture



+ some further ideas for nonreductive H.

Collapsing Lie algebras

Definition (limit algebra) $\mathfrak{h} \subset \mathfrak{g}$ Lie algebras

We say $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$ if

${}^{\exists}g_j \in G$ such that $\lim\limits_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is a solvable Lie algebra.

Example Let $n \geq 3$. $\mathfrak{h} = \mathfrak{so}(n)$ is a semisimple subalgebra of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R})$.

Take $g_j = \begin{pmatrix} n & & & \\ & n-1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}^j \qquad (j = 1, 2, \cdots)$.

Then $\lim\limits_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{so}(n) = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ \mathbf{0} & & 0 \end{pmatrix} \right\} \cdots$ solvable Lie algebra.

Example: $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h} \subset \mathfrak{g}$

$$G = GL(p+q+r)$$

$$\cup$$

$$H = GL(p) \times GL(q) \times GL(r)$$

The Lie algebra $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$,

i.e. $\exists$ a sequence $g_j \in G$ such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is solvable.

$\iff$ $2\max(p,q,r) \le p+q+r+1.$

Example $G/H = GL(p+q+r)/GL(p)\times GL(q)\times GL(r)$

Theorem The following conditions on (p, q, r) are equivalent:

(i) (Rep Theory) $L^2(G/H)$ is a tempered representation of G.

(ii) (Combinatorics: $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$)

$$\sum_{1\leq i<j\leq p} |x_i - x_j| + \sum_{1\leq i<j\leq q} |y_i - y_j| + \sum_{1\leq i<j\leq r} |z_i - z_j|$$

$$\leq \sum_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} |x_i - y_j| + \sum_{\substack{1\leq j\leq q\\ 1\leq k\leq r}} |y_j - z_k| + \sum_{\substack{1\leq k\leq r\\ 1\leq i\leq p}} |z_k - x_i|$$

for all $(x_1, \cdots, x_p, y_1, \cdots, y_q, z_1, \cdots, z_r) \in \mathbb{R}^{p+q+r}$ with $\sum x_i = 0$, $\sum y_j = 0$, $\sum z_k = 0$.

(iii) (Collapsing Lie algebra) $\exists$ a sequence g_j such that $\lim_{j\to\infty} \mathrm{Ad}(g_j)\mathfrak{h}$ is a solvable Lie algebra.

(iv) (Classification) $2\max(p, q, r) \leq p + q + r + 1$.

Geometric quantization and temperedness

$\mathrm{Ad}\colon G \to GL_{\mathbb{R}}(\mathfrak{g})$ adjoint representation.

Coadjoint orbit $\mathcal{O}_\lambda := \mathrm{Ad}^*(G)\lambda$ for $\lambda \in \mathfrak{g}^*$.

Every $\mathcal{O}_\lambda$ carries a symplectic structure (Kostant–Kirillov–Souriau).

“Geometric quantization”: $\underset{\text{symplectic mfd}}{\mathfrak{g}^* \supset \mathcal{O}_\lambda = \mathrm{Ad}^*(G)\lambda} \overset{?}{\rightsquigarrow} \underset{\text{unitary rep}}{\pi_\lambda \in \widehat{G}}$

<u>Theorem</u> (2023*)

Suppose G is a complex reductive Lie group,

and H a connected closed subgroup. Then (i) ⇔ (ii).

(i) $G \curvearrowright L^2(G/H)$ is tempered.

(ii) $\mathfrak{g}^*_{\mathrm{reg}} \cap \mathfrak{h}^\perp \neq \emptyset$.

$$\mathfrak{g}^*_{\mathrm{reg}} := \{\lambda \in \mathfrak{g}^* : \mathrm{Ad}^*(G)\cdot\lambda \text{ is of maximal dimension}\}$$

$$\mathfrak{h}^\perp := \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{h}} \equiv 0\}$$

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[*] Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, **22** (2023), 2879–2906.

Further interactions for “tempered spaces”

Theorem D (2023)* Let $\mathfrak{g}$ be a complex reductive Lie algebra.
The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

(i) (unitary rep) $L^2(G/H)$ is tempered.

(ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.

(iii) (limit algebra) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.

(iv) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.

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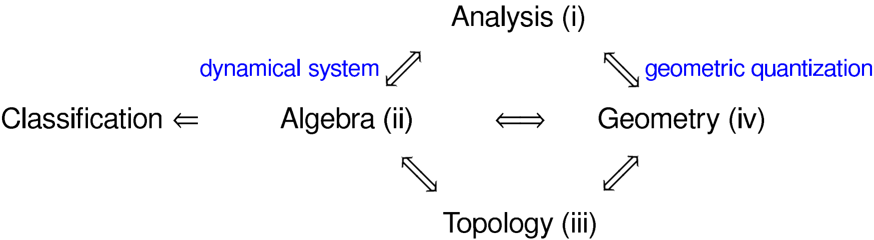
* Y. Benoist–T. Kobayashi, Tempered homogeneous spaces IV, J. Inst. Math. Jussieu, **22** (2023), 2879–2906.

Equivalent characterization: Tempered spaces

Thm (2023) Let $\mathfrak{g}$ be a complex reductive Lie algebra.

The following 4 conditions on a Lie subalgebra $\mathfrak{h}$ are equivalent.

- (i) (unitary rep) $L^2(G/H)$ is tempered .
- (ii) (combinatorics) $2\rho_{\mathfrak{h}} \leq \rho_{\mathfrak{g}}$.
- (iii) (limit algebra) $\mathfrak{h}$ has a solvable limit in $\mathfrak{g}$.
- (iv) (orbit method) $\mathfrak{h}^{\perp} \cap \mathfrak{g}^*_{\mathrm{reg}} \neq \emptyset$ in $\mathfrak{g}^*$.



Analysis (i)

dynamical system ⤢ ⤡ geometric quantization

Classification ⇐ Algebra (ii) ⟺ Geometry (iv)

⤡ ⤢

Topology (iii)

Basic Questions
in Group-Theoretic Analysis on Manifolds

$G \curvearrowright X \quad \rightsquigarrow \quad G \curvearrowright C^\infty(X),\ L^2(X),\ \cdots$

Geometry Function Space

Basic Question 1 (Multiplicity)

Does the group sufficiently control the space of functions?

Basic Question 2 (Tempered homogeneous spaces)

Is $G \curvearrowright L^2(X)$ a tempered representation?

Thank you very much!

References

The second topic is joint with Yves Benoist. For more details of the talk today, we discuss in IHP, February 17–21.

Tempered Homogeneous Spaces:

- ——— I. (J. Euro Math., 2015)
  Method (Dynamical System)
- ——— II. (Margulis Festschrift, 2022, Chicago Univ. Press)
  Representation Theory
- ——— III. (J. Lie Theory, 2021)
  Classification Theory (Combinatorics)
- ——— IV. (J. Inst. Math. Jussieu, 2023)
  Limit algebra, geometric quantization

Tensor product of GL_n (J. Algebra, 2023)