Lie Group and Representation Theory Seminar, Kyoto 2006

Date: **October 23 (Mon), 2006, 10:00–12:00**
October 24 (Tue) , 2006, 16:30–18:00
October 25 (Wed), 2006, 9:00–10:30
October 27 (Fri), 2006, 13:00–15:00

Place: **RIMS, Kyoto University : Room 402**

Speaker: **Oksana Yakimova (Humboldt fellow, Cologne)**

Title: **Gelfand pair, definitions, main properties, and generalisations**

Abstract: The concept of a Gelfand pair is a natural generalisation of a symmetric Riemannian homogeneous space. It plays an important rôle in representation theory, differential geometry, symplectic geometry, and functional analysis.

Let $X = G/K$ be a connected Riemannian homogenous space of a real Lie group G. Then (G, K) is called a *Gelfand pair* and the homogeneous space X is said to be *commutative* if the algebra $D(X)^G$ of G-invariant differential operators on X is commutative; or, equivalently, if the representation of G on $L^2(X)$ is multiplicity free. In this lectures we will consider other characterisations of Gelfand pairs. For example, X is commutative if and only if the action of G on the cotangent bundle T^*X is coisotropic with respect to the standard G-invariant symplectic structure.

In 1956, Gelfand and Selberg independently introduced two sufficient conditions for commutativity. These conditions turned out to be equivalent and can be formulated as follows:

(GS) there is a group antiautomorphism σ of G such that each double coset of K is σ-stable.

If condition (GS) is satisfied, then X is said to be *weakly symmmetric*. **This condition is not necessary for commutativity. In 1998, Lauret constructed the first example of a commutative but not weakly symmetric homogeneous space.**

Suppose that G is reductive. Then, by a result of Akhiezer and Vinberg, (G, K) is a Gelfand pair if and only if X is weakly symmetric. Moreover, (G, K) is a Gelfand pair if and only if the complexification $X(\mathbb{C})$ of X is a *spherical* **$G(\mathbb{C})$-variety. Spherical variety is a well-studied object of algebraic geometry. If $X(\mathbb{C})$ is homogeneous and affine, then the complete classification was obtained by Krämer, Brion, and Mikityuk. Classification of Gelfand pairs with reductive G easily follows from their results.**

In general, if (G, K) is commutative, then, up to a local isomorphism, G has a factorisation $G = N \rtimes L$, where L is reductive, N is commutative or two-step nilpotent, and $K \subset L$. We will present an effective commutativity criterion in terms of representations of L and K on $\mathfrak{n} = \mathrm{Lie}\, N$; and discuss main ideas of the classification of Gelfand pairs.

The notion of a Gelfand pair can be generalised in different ways. If K is not compact, then one can give various reasonable definitions that are not equivalent. In this way, we will obtain different objects, which belong to several areas of mathematics, like spherical varieties in case G is reductive (invariant theory, algebraic geometry); generalised Gelfand pairs (harmonic analysis); coisotropic actions (symplectic geometry, integrable Hamiltonian systems).

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