The Heat equation, the Segal-Bargmann transform and generalizations

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Let $\Delta = \sum \partial^2/\partial x_i^2$ be the Laplace operator on $\mathbb{R}^n$. The *heat equation* is given by

$$\begin{aligned} \Delta u(x,t) &= \frac{\partial}{\partial t} u(x,t) \\ \lim_{t\to 0^+} u(x,t) &= f(x) \end{aligned}$$

where f can be a L^2-function, a distribution or an element in some other natural class of objects. The solution $u(x,t) = e^{t\Delta} f(x) = H_t f(x)$ is given by

$$H_t f(x) = \int_{\mathbb{R}^n} f(y) h_t(x-y)\, dy = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-(x-y)\cdot(x-y)/4t}\, dy \tag{1}$$

where $h_t(x) = (4\pi t)^{-n/2} e^{-x\cdot x/4t}$ is the heat kernel, i.e. the solution corresponding to $f = \delta_0$. The map $H_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is smoothing, in fact one reads of from the explicit formula (1), that $\mathbb{R}^n \ni x \mapsto H_t f(x) \in \mathbb{C}$ extends to an entire function on $\mathbb{C}^n$. The transform $L^2(\mathbb{R}^n) \ni f \mapsto H_t f \in \mathcal{O}(\mathbb{C}^n)$ is the *Segal-Bargmann transform.* Its image is the space of holomorphic functions $F : \mathbb{C}^n \to \mathbb{C}$, such that

$$\|F\|_t^2 := (2\pi t)^{-n/2} \int |F(x+iy)|^2 e^{-\|y\|^2/2t}\, dxdy < \infty$$

and $\|f\| = \|H_t f\|_t$.

The Heat equation has a natural generalization to all Riemannian manifolds. The solution is again given by the Heat transform

$$u(x,t) = H_t f(x) = \int f(y) h_t(x,y)\, dy$$

where h_t is the *heat kernel*, but – as there is no *natural* complexification in general – it is not at all clear how to realize the image in a Hilbert space of holomorphic functions. One exception is the class of semisimple Riemannian symmetric spaces. The special case of K-invariant functions leads to generalizations of the Segal-Bargmann transform for the heat equation related to arbitrary multiplicity functions.

In those two talks, we will discuss the following:

(1) The Heat equation and Segal-Bargmann transform on $\mathbb{R}^n$. Our proof of the unitarity and surjectivity will involve the restriction principle [2].
(2) Basic structure of symmetric spaces of noncompact type and the crown, [1].
(3) The image of the Segal Bargmann transform for Riemannian symmetric spaces of the noncompact type, [1].
(4) The case of K-invariant functions and the Heckmann-Opdam hypergeometric transform, [3].
(5) (Only if there is time enough) the restriction principle for semisimple symmetric spaces, [2]

All the following references, except [2, 11], are available on arxiv. The articles [2, 11] can be downloaded from `www.math.lsu.edu/∼preprint`.

References

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