The 7th Takagi Lectures

ABSTRACT

Michael Harris: *Arithmetic applications of the Langlands program*

The functoriality conjecture is at the heart of the Langlands program and will undoubtedly remain as a challenge to number theorists for many decades to come. Shortly after formulating his program, however, Langlands proposed to test it in two interdependent settings. The first was the framework of *Shimura varieties*, already understood by Shimura as a natural setting for a non-abelian generalization of the Shimura–Taniyama theory of complex multiplication. The second was the phenomenon of endoscopy, which can be seen alternatively as a classification of the *obstacles* to the stabilization of the trace formula or as an *opportunity* to prove the functoriality conjecture in some of the most interesting cases. After three decades of research, much of it by Langlands and his associates, these two closely related experiments are coming to a successful close, at least for classical groups, thanks in large part to the recent proof of the so-called *Fundamental Lemma* by Waldspurger, Laumon, and especially Ngô.

My primary interest in these lectures is to give an account of these developments insofar as they are relevant to the Galois groups of number fields. Algebraic number theorists are just beginning to take stock of the new information provided by the successful resolution of the problem of endoscopy and the analysis of the most important classes of Shimura varieties. I will devote special attention to the application in this setting of the methods pioneered by Wiles, and developed further by Taylor, Kisin, and others, in his proof of statements in non-abelian class field theory that imply Fermat's Last Theorem.

Michael Hopkins: *The Kervaire invariant problem*

The existence of framed manifolds of Kervaire invariant one is one of the oldest unresolved problems in algebraic topology. Important questions about smooth structures on spheres and on the homotopy groups of spheres depend on its solution. In these talks I will describe joint work with Mike Hill and Doug Ravenel which solves this problem in all dimensions except 126.

Uwe Jannsen: ***Weights in arithmetic geometry***

The concept of weights on the cohomology of algebraic varieties was initiated by fundamental ideas and work of A. Grothendieck and Pierre Deligne. It is deeply connected with the concept of motives and appeared first on the singular cohomology as the weights of (possibly mixed) Hodge structures and on the etale cohomology as the weights of eigenvalues of Frobenius. But weights also appear on algebraic fundamental groups and in p-adic Hodge theory, where they become only visible after applying the comparison functors of Fontaine. After rehearsing various versions of weights, the talk will present some more recent applications of weights, e.g., to Hasse principles and the computation of motivic cohomology, and will discuss some open questions and speculations.

Chandrashekhar Khare: ***Serre's conjecture and its consequences***

I will give a historically motivated account of Serre's conjecture about mod p representations of the absolute Galois group of the rationals. This was proved by J-P. Wintenberger and myself, together with a certain input of Kisin.

The context in which Serre made his conjecture was the work of Serre and Swinnerton–Dyer which explained the congruences Ramanujan had found for the Ramanujan τ-function. The explanation was via the study of images of Galois representations Deligne attached to the Ramanujan Δ-function (again conjectured by Serre). Here $\Delta(z) = q\Pi(1-q^n)^{24} = \sum_n \tau(n)q^n$ with $q = e^{2\pi i z}$.

I will also explain some of the consequences of Serre's conjecture. For instance it implies Artin's conjecture for 2-dimensional, complex, odd representations of the absolute Galois group of the rationals.

I will also talk about the ideas of the proof and some questions they lead to.

James McKernan: ***Mori dream spaces***

A fundamental result of Hilbert says that if R is a finitely generated $\mathbb{C}$-algebra then the ring of invariants R^G is finitely generated, provided G is a reductive algebraic group (for example products of the multiplicative group $(\mathbb{C}^*)^k$). Nagata gave examples where R is a polynomial ring and $G = \mathbb{C}^k$ is a product of the additive group and yet R^G is not finitely generated. In fact R^G is the total coordinate ring of a blow up of projective space $\mathbb{P}^n$.

If X is a projective variety and the Cox ring is finitely generated then X is called a Mori dream space; every toric variety and every Fano variety is a Mori dream space. As the name might suggest, Mori dream spaces have very many nice properties; every section ring is finitely generated; flips always terminate; there is a natural combinatorial structure to the set of minimal models, and all of this controlled by the geometric invariant theory of some Thaddeus master space. We will explore this circle of ideas in the talk.