Mackey’s theory of \( \tau \)-conjugate representations for finite groups

Tullio Ceccherini-Silberstein · Fabio Scarabotti · Filippo Tolli

Received: 25 November 2013 / Revised: 10 July 2014 / Accepted: 11 July 2014
Published online: 18 December 2014
© The Mathematical Society of Japan and Springer Japan 2014

Communicated by: Yasuyuki Kawahigashi

Dedicated to our mentors and friends
Toni Machi on his 75th birthday and Pierre de la Harpe on his 70th birthday

Abstract. The aim of the present paper is to expose two contributions of Mackey, together with a more recent result of Kawanaka and Matsuyama, generalized by Bump and Ginzburg, on the representation theory of a finite group equipped with an involutory anti-automorphism (e.g. the anti-automorphism \( g \mapsto g^{-1} \)). Mackey’s first contribution is a detailed version of the so-called Gelfand criterion for weakly symmetric Gelfand pairs. Mackey’s second contribution is a characterization of simply reducible groups (a notion introduced by Wigner). The other result is a twisted version of the Frobenius–Schur theorem, where “twisted” refers to the above-mentioned involutory anti-automorphism.

Keywords and phrases: representation theory of finite groups, Gelfand pair, Kronecker product, simply reducible group, Clifford groups, Frobenius–Schur theorem

Mathematics Subject Classification (2010): 20C15, 43A90, 20G40
Contents

1. Introduction ............................................................................................................... 44
2. Preliminaries and notation ...................................................................................... 48
   2.1. Linear algebra ............................................................................................... 48
   2.2. Representation theory of finite groups .......................................................... 51
3. The $\tau$-Frobenius–Schur number ................................................................. 52
4.Multiplicity-free permutation representations: the Mackey–Gelfand criterion .... 58
5. Simply reducible groups I: Mackey’s criterion .................................................... 63
6. Simply reducible groups II: Mackey’s generalizations of Wigner’s criterion .... 65
7. An example: the Clifford groups ........................................................................ 72
8. The twisted Frobenius–Schur theorem ............................................................... 76
9. The twisted Frobenius–Schur theorem for a Gelfand pair .................................. 81
10. Examples ............................................................................................................. 85
11. Open problems and further comments ................................................................ 88