Real Harmonic Analysis
for Geometric Automorphic Forms

Takayuki Oda

Introduction

This is a survey of the results in the last decade on spherical functions over real semisimple Lie groups, and their applications to the theory of automorphic forms of many variables, mainly due to the author and his collaborators. The origin of our project is the former article [40] of the two decades ago by the author. So this article might be also considered as its continuation.

It is difficult to have substantial and general results in the theory of automorphic forms of many variables. Therefore our results in this article are classified into two categories: when the representations generated by automorphic forms are (a) either small representations on rather general semisimple Lie groups, (b) or rather general representations on small Lie groups. The largeness of representations is measured by the Gelfand-Kirillov dimension, and the largeness of groups by the $\mathbb{R}$-split rank. The aspect (a) is shown in the subsection 1.5, which discuss the vanishing of the middle Hodge component of certain modular symbol, and in the subsection 6.2 to discuss the Green functions associated with modular algebraic cycles. These are also the geometric aspect in our results. The other aspect (b) is shown in the section 5 to discuss the generalized spherical functions on the "small group" $Sp(2, \mathbb{R})$ related to the various Fourier expansions of automorphic forms. The remaining part is for the preparation to give the backgrounds for these results.

But before to proceed to the outline of this paper, let me recall some personal memory about $Sp(2, \mathbb{R})$ and its representations.

Sometime around 1980, in the Hongo Campus of the University of Tokyo on a slope west of the Yasuda Hall, Ihara who had been my former teacher asked my opinion about the non-holomorphic discrete series representations of $Sp(2, \mathbb{R})$: "What should be the role of such representations in the arithmetic of automorphic forms on $Sp(2, \mathbb{R})". A part of the reason of this question was because he noticed a relation between the characters of finite-dimensional representations of the compact form $Sp(2)$ of $Sp(2, \mathbb{R})$ and the characters of the discrete series representations of $Sp(2, \mathbb{R})$, which became available by a result of Hirai [8] at that time. Also he investigated automorphic forms on the compact group $Sp(2)$ (cf. [16]) sometime ago.

The specific problem, which might probably be in his mind, was developed by Ibukiyama [13] and Hashimoto-Ibukiyama [14] (or by Ibukiyama-Ihara [15] ?), but the general question has been lingered in my mind. This was one of the motivations for me to write [40].

For the lack of the knowledge of real harmonic analysis on real semisimple Lie
groups, I have no much progress for long time. But around early 90’s, I noticed the series of papers by Yamashita ([52] and related papers). This gives me the method to compute various spherical functions on $Sp(2, \mathbb{R})$ and on $SU(2, 2)$. And fortunately I had many bright students to work out. These are the results presented in §5 basically.

Meanwhile there were development in cohomological representations in 80’s and 90’s. Obviously these are related to the modular symbols on higher dimensional modular varieties, which have been my interest from the time of the doctoral thesis on Hilbert modular surfaces. Modular symbols on the higher dimensional arithmetic quotients are the theme of the subsections 1.5 and 6.2 mentioned above.

Let me explain the contents of this paper. The first section is mainly devoted to a short review of the “classical” results on the cohomology groups of the (cocompact) discrete subgroups $\Gamma$ of real semisimple Lie groups $G$ (§§1.1 ~ 1.4, 1.6). We recall here the Matsushima isomorphism and the relative Lie algebra cohomology. The exception is the subsection 1.5, here I review shortly the content of the joint paper with Toshiyuki Kobayashi [23]. The subsection 1.6 contains the important example, i.e., the case $G = Sp(2, \mathbb{R})$. The section 2 is for non-cocompact $\Gamma$ in $G = Sp(2, \mathbb{R})$. Here we “review” the analogue of Eichler-Shimura isomorphism for the third cohomology of $\Gamma$, which is a variant of the Matsushima isomorphism, in the context of $(\mathfrak{g}, K)$-modules (Theorem 2.1). Though we do not claim the result here is new, but the statement related to the Dirac-Schmid operator does not seem to be found in the literature. In §3, we recall the “classical” results by Harish-Chandra on the discrete series representations, but mainly limited to the case $G = Sp(2, \mathbb{R})$ (§§3.1~ 3.3). And we also recall the important results of W. Schmid on the realization of the discrete series and the characterization of the minimal $K$-types (the sections 3.5 and 3.4, respectively), which are the deep and crucial results in the discrete series. The section 4 is the preparation for the next section 5. We review the general frame work of the Fourier expansion of automorphic forms with respect to relatively “large” closed subgroup $R$, and the related problem of spherical functions (the subsections 4.1 and 4.2). The section 5 is the review of the various spherical functions related to the Fourier expansions of automorphic forms with respect to the different type parabolic subgroups. In the subsection 6.1, we comment about spherical functions with respect to symmetric pairs $(G, R)$. And in the subsection 6.2, we see how the spherical functions are utilized to have a global object in the theory of automorphic forms taking the Green function as an example.

The reference list of this paper is already rather long. I am forced to skip even the papers which are historically important, if they are not directly related to this article. They are probably found in the references in the quoted papers.

The contents of this article grew in the course of a few occasions to talk at some workshops. I had a chance to give a series of talks at Tsuda College in September 2000 under the title “Geometry of arithmetic quotients of the classical domains”. Also the English version of this talk is given at a Japan-Germany seminar at 2001 under the title “Modular cycles on arithmetic quotients of classical domains”. These correspond to the section 1 and the subsection 6.1 of this article. The contents of §5 is presented at the workshop: “Motifs and cycles of abelian varieties”, January 2004 at Tohoku University, under the title “The known and unknown on Fourier expansions of automorphic forms”. The author
thanks to the organizer of these workshops. He also thanks to the former students who obtained excellent results spending much time of concentration for difficult computations. Finally he thanks to the organizers of the this workshop at Texel Island, for giving me the chance to write this.

1 The Matsushima isomorphism for cohomology of discrete subgroups

In this section we recall some basic facts around the Matsushima isomorphism on cohomology groups of discrete subgroups in real semisimple Lie groups.

A good reference on this theme is a survey article by A. Borel [1]. The book [2] of Borel-Wallach also has been a very important reference. The original paper about this theme is Matsushima’s [27]. In the case when $G/K$ is Hermitian, there is a paper by the author which is more specialized to the Hodge theoretic aspect of the problem [40].

1.1 Shift to the relative Lie algebra cohomology groups

We start with a given cocompact discrete subgroup $\Gamma$ in a real semisimple Lie group $G$ with finite center. For simplicity we may assume that $G$ is connected.

We consider its Eilenberg-Maclane cohomology group $H^i(\Gamma, \mathbb{C})$. Or more generally, if a finite dimensional rational representation $r : G \to GL(E)$ of $G$ is given, we may regard it as $\Gamma$-module by restriction, and can form cohomology group $H^i(\Gamma, E)$.

Fix a maximal compact subgroup $K$ of $G$ to get a Riemannian symmetric space $X = G/K$. For simplicity assume that we have no elements of finite order in $\Gamma$, then $\Gamma$ acts on $X$ from the left side without fixed point, and the quotient $\Gamma \backslash X$ becomes a manifold. In this case $\Gamma$ is isomorphic to the fundamental group of this quotient manifold ($X$ is contractible to a point), and the $\Gamma$-modules $E$ defines a local system $\tilde{E}$ on this quotient manifold. Then we have a canonical isomorphism of cohomology groups: $H^\ast(\Gamma, E) \cong H^\ast(\Gamma \backslash X, \tilde{E})$.

Let $\sigma : X \to \Gamma \backslash X$ be the canonical map, then by pulling back differential forms with respect to $\sigma$ we have a monomorphism of de Rham complexes $\sigma^0 : \Omega^\ast_{\Gamma \backslash X} \to \Omega^\ast_X$. Moreover on the target complex, $\Gamma$ acts naturally. Let $(\Omega^\ast_X)^\Gamma$ be the invariant subcomplex. Then it coincides with the image of $\sigma^0$, and by the de Rham theorem, we have an isomorphism of cohomology groups:

$$H^\ast(\Gamma, E) \cong H^\ast(\Gamma \backslash X, \tilde{E}) = H^\ast(\Omega^\ast_X(E)^\Gamma).$$

The last complex in the above isomorphisms is identified with the complex of differential forms on $G$ as follows.

Let $\pi^0$ be the pull-back homomorphism of differential forms with respect to the canonical map $\pi : \Gamma \backslash G \to \Gamma \backslash X$. Then a form $\omega \in \Omega^\ast_X(E)^\Gamma$ defines a differential form $\omega^0$ on $G$ by

$$x \in G \mapsto r(x)^{-1} \pi^0(\omega)(x),$$

and denote by $A^0_\ast(G, \Gamma, E)$ the image of this homomorphism. Then since $\sigma^0$ is a monomorphism, we have an isomorphism of complexes $\Omega^\ast_X(E)^\Gamma \cong A^0_\ast(G, \Gamma, E)$. 
Here the last complex is identified with the right $K$-invariant subcomplex of the de Rham complex $\Omega^*_{\Gamma\backslash G}(E) = \Omega^*_{\Gamma\backslash G} \otimes E$, which is defined over $\Gamma\backslash G$ and takes values in $E$.

Since the tangent space at each point of $G/K$ is identified with the orthogonal complement $p$ of $k$ in $g$ with respect to the Killing form, the module of the $i$-th cochains becomes $\text{Hom}_K(\wedge^i p, C^\infty(\Gamma\backslash G) \otimes E)$. Therefore, the cochain complex defined in this manner gives the relative Lie algebra cohomology groups. When $G$ is connected we have an isomorphism:

$$H^*(\Gamma, E) \cong H^*(g, K; C^\infty(\Gamma\backslash G), E).$$

### 1.2 Matsushima isomorphism

Let $G$ be a connected semisimple real Lie group with finite center. Assume that the discrete subgroup $\Gamma$ is cocompact, i.e. the quotient $\Gamma\backslash G$ is compact.

Let $L^2(\Gamma\backslash G)$ be the space of $L^2$-functions on $G$ with respect the Haar measure on $G$, on which $G$ acts unitarily by the right action. By assumption the space $C^\infty(\Gamma\backslash G)$ is a subspace of this space.

**Proposition 1.1** (Gelfand, Graev, and Piatetski-Shapiro) If $\Gamma$ is cocompact, we have the direct sum decomposition of the unitary representation $L^2(\Gamma\backslash G)$ into irreducible components:

$$L^2(\Gamma\backslash G) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma)M_{\pi},$$

with finite multiplicities $m(\pi, \Gamma)$. Here $\hat{G}$ is the unitary dual of $G$, i.e. the unitary equivalence classes of irreducible unitary representations of $G$, and $M_{\pi}$ denotes the representation space of $\pi$. The symbol $\bigoplus$ means the Hilbert space direct sum.

By the above proposition, one has a decomposition

$$C^\infty(\Gamma\backslash G) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma)M^\infty_{\pi}$$

of topological linear spaces. Here $M^\infty_{\pi}$ is the subspace consisting of $C^\infty$-vectors in the representation space $M_{\pi}$.

**Theorem 1.1** For a finite dimensional rational $G$-module $E$ over the complex number field, we have an isomorphism:

$$H^*(\Gamma, E) = \bigoplus_{\pi \in \hat{G}} m(\pi, \Gamma)H^*(g, K, M^\infty_{\pi} \otimes E)$$

$$= \bigoplus_{\pi \in \hat{G}} \{\text{Hom}_G(M_{\pi}, L^2(\Gamma\backslash G) \otimes C^\infty(g, K, M^\infty_{\pi} \otimes E))\}$$

The key point of the proof here is the complete direct sum $\bigoplus$ is replaced by a simple algebraic direct sum by passing to the cohomology (cf. Borel [1]).

We can equipp a $K$-invariant inner product on $E$. By using this, we may regard $H^m(g, K, M^\infty_{\pi} \otimes E)$ as a space of a kind of harmonic forms, i.e. the totality of cochains vanishing by the Laplace operator. Thus we have the following.

**Proposition 1.2** Let $(\pi, E)$ be an irreducible $G$-module of finite dimension. For any $\pi \in \hat{G}$ we have the following.

(i) If $\chi_{\pi}(C) = \chi_{\pi}(C)$, there is an isomorphism

$$\text{Hom}_K(\wedge^m p, M_{\pi} \otimes C^\infty \otimes E) = C^m(g, K, M^\infty_{\pi} \otimes E) = H^m(g, K, M^\infty_{\pi} \otimes E).$$
Here $C^m(\mathfrak{g}, K, M_\pi^\infty \otimes E)$ denotes the $m$-th module of the standard complex of $(\mathfrak{g}, K)$ cohomology.

(ii) If $\chi_r(C) \neq \chi_\pi(C)$, there is an isomorphism

$$H^m(\mathfrak{g}, K, M_\pi^\infty \otimes E) = \{0\}.$$

Here $C$ denotes the Casimir operator.

In particular, when $E$ is the trivial $G$-module $C$, the above theorem is no other than the original formula of Betti numbers of $\Gamma \backslash X$ by Matsushima [27].

Also there is a variant of this type vanishing theorem shown by D. Wigner. But it is omitted here.

We recall here that the relative Lie algebra cohomology group $H^m(\mathfrak{g}, K, M_\pi^\infty \otimes E)$ is isomorphic to the continuous cohomology group $H^m_{ct}(G, M_\pi^\infty \otimes E)$ if $G$ is connected. This is shown by using differential cohomology and van Est spectral sequence.

### 1.3 "Classical" vanishing theorems

A number of vanishing theorems were found in ’60’s: Calabi-Vesentini, Weil etc. In their proof, the same type of computation of “curvature forms” is done, which is similar to a proof of Kodaira vanishing theorem.

Firstly, Matsushima’s vanishing theorem of the first Betti number of $\Gamma \backslash X$ was also proved by such method ([28]).

This type of vanishing theorem is vastly improved by representation theoretic method. Probably the best result of this category is the following result by Zuckerman ([54]) (see also [2], Chapter V, §2 - §3 (p.150–155)).

**Theorem 1.2** Let $G$ be a simple real algebraic group, $(\pi, M_\pi)$ a nontrivial irreducible unitary representation of $G$, and $(r, E)$ a finite dimensional representation of $G$. Then for $k < \text{rank}_RG$,

$$H^k_{ct}(G, M_\pi^\infty \otimes E) = \{0\}.$$

**Corollary 1.3** Given a cocompact discrete subgroup $\Gamma$ of $G$, for $k < \text{rank}_RG$, the restriction homomorphism

$$H^k_{ct}(G, E) \to H^k(\Gamma, E)$$

is an isomorphism.

### 1.4 Enumeration and construction of unitary cohomological representations

We shortly review the state of arts on the cohomological representations defined below.

**Definition 1.1** An irreducible (unitary) representation $(\pi, M_\pi) \in \hat{\mathcal{G}}$ is called cohomological, if there is a finite dimensional $G$-module $F$ such that

$$H^i(\mathfrak{g}, K; M_\pi^\infty \otimes_C F) \neq \{0\}$$

for some $i \in \mathbb{N}$. The set of equivalent classes of cohomological representations is denoted by $\hat{\mathcal{G}}_{coh}$. 

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Enumeration of such cohomological representations was done for the case trivial $F = \mathbb{C}$ by Kumaresan [25], and for general case by Vogan-Zuckermann [50]. This was originally described by using the cohomological induction functor $\mathcal{A}_\lambda(g)$ first. And later a global realization of this Zuckermann module was obtained by H.-W. Wong [51].

1.5 A new vanishing theorem

Recently T. Kobayashi is developing a theory of branching rule for such cohomological representations when they are restricted to a large reductive subgroup $H$ of $G$ ([20, 21, 22]). We have an application of this result to show the vanishing of certain Hodge components of some modular symbols (cf. Kobayashi-Oda [23]). This is a bit deeper result of vanishing than the classical ones.

1.5.1 Formulation of problem

Firstly we recall the construction of generalized modular symbols associated with reductive subgroups of $G$. Given a double coset space $V = \Gamma \backslash G$, which is compact. Let $\iota : H \subset G$ be a reductive subgroup of $G$, such that

(a) $H \cap K$ is maximally compact in $H$;
(b) $H \cap \Gamma$ is cocompact discrete subgroup of $H$.

Then we have a natural map of double cosets:

$$\tilde{\iota} : (H \cap \Gamma) \backslash G / (H \cap K) \to V.$$ 

Set $d = \dim \mathbb{R} W = \dim \mathbb{R} H / (H \cap K)$, $N = \dim \mathbb{R} V = \dim \mathbb{R} G / K$. Then the fundamental class $[W] \in H_d(W, \mathbb{C})$ mapped naturally by $\iota$ to

$$\iota_*([W]) \in \mathbb{H}_d(V, \mathbb{C}) \cong \mathbb{H}^{N-d}(V, \mathbb{C}) \cong \mathbb{H}^{N-d}(\Gamma, \mathbb{C})$$

$$\cong \oplus_{\pi \in \hat{G}, \chi_{\infty}(\pi) = \chi_{\infty}(1)} \{ \text{Hom}_G(H_{\pi}, L^2(\Gamma \backslash G)) \otimes \mathbb{H}^{N-d}(\mathfrak{g}, K, H_{\pi}^\infty) \}.$$ 

Here the first isomorphism denoted by $P$ is the Poincaré duality. According to the last decomposition in the above formula, we have a decomposition

$$P \cdot \iota_*([W]) = \sum_{\pi \in \hat{G}, \chi_{\infty}(\pi) = \chi_{\infty}(1)} \mathcal{M}(\pi)(W),$$

where $\mathcal{M}(\pi)(W)$ denotes the $\pi$-th component of the Poincaré dual of $[W]$.

**Problem** Describe $\mathcal{M}(\pi)(W)$, using the special values of automorphic $L$-functions etc, \ldots

Dually speaking, it is to consider the restriction map:

$$H^d(V, \mathbb{C}) = \oplus_{\pi \in \hat{G}} \{ \text{Hom}_G(M_{\pi}, L^2(\Gamma \backslash G)) \oplus \mathbb{C} H^d(\mathfrak{g}, K; M_{\pi}) \}.$$ 

$$\to H^d(W, \mathbb{C}) = \mathbb{C}.$$ 

This is done by investigation of the period integrals $\int_W \omega$ for elements $\omega$ in the $\pi$-component

$$H^d(V, \mathbb{C})(\pi) = \{ \text{Hom}_G(M_{\pi}, L^2(\Gamma \backslash G)) \oplus \mathbb{C} H^d(\mathfrak{g}, K; M_{\pi}) \}.$$

of $H^d(V, \mathbb{C})$. 

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A nice situation is when the restriction $\pi|H$ of $\pi \in \hat{G}_{coh}$ is admissible, i.e.
(a) $\pi|H$ decomposes discretely as $H$-modules;
(b) $\pi|H = \bigoplus_{\sigma \in \hat{H}} m(\pi, \sigma)\sigma$ has finite multiplicity $m(\pi, \sigma) < \infty$.

For cohomological representations $\pi = A_q(q)$, Koyabayashi obtained certain sufficient condition for the admissibility of $\pi|H$, which is described geometrically.

As an application, under the same condition as the above criterion of admissibility, the restriction to
\[ H_d(W, C) \times H^d_dR(V, C) \rightarrow C \]
vanishes. Here $d = \dim W$.

1.5.2 Example
Let $G = SO(2n, 2) \times \text{compact factor}$ such that there is an algebraic group $G$ over $Q$ with $G = G(R)$. Then we may form an algebraic subgroup $H$ of $G$ such that $H = H(R)$ is isomorphic to $SO(2n, 1) \times \text{compact factor}$ and there is a maximal compact subgroup $K$ of $G$ with $H \cap K$ being maximal compact in $H$.

Set $X = G/K$ and $X_H = H/(H \cap K)$. Then $X_H$ is a totally real submanifold of $X$. This means that for some holomorphic local coordinates $(z_1, z_2, \cdots, z_{2n})$ at each point of $X_H$, it is defined locally by the equalities
\[ \text{Im}(z_1) = 0, \cdots, \text{Im}(z_{2n}) = 0. \]

Now for a cocompact arithmetic discrete subgroup $\Gamma$ of $G$, we can define a generalized modular symbol:
\[ \iota : W = (H \cap \Gamma) \backslash X_H \rightarrow V = \Gamma \backslash X. \]
In this case the fundamental class $[W] \in H_{2n}(W, C)$ mapped to
\[ H_{2n}(V, C) \cong H^d_dR(V, C) = \bigoplus_{p+q=2n} H^{p,q} \]
by $\iota_*$. Here the last isomorphism is the Hodge decomposition. Hence we have the Hodge decomposition of $M(W)$:
\[ M(W) = \sum_{p+q=2n} M^{(p,q)}(W). \]
Its $(n, n)$-type component $M^{(n,n)}(W)$ has further decomposition:
\[ M^{(n,n)}(W) = \sum_{\pi \in \hat{G}, H^{n,n}(g, K; H^\infty) \neq \{0\}} M^{(\pi)}(W). \]
Here $H^{n,n}(g, K; M^\infty)$ is the $(n, n)$-type component of $H^{2n}(g, K; M^\infty)$. In this case we have the following.
Proposition 1.3 If \( \pi \neq C \), \( M^{(\pi)}(W) = 0 \) for \( \pi \) satisfying \( H^{n,n}(g, K; M_C^\infty) \neq \{0\} \). This means that \( M(n, n)(W) \) is a constant multiple of a universal cohomology class \( \eta \in H^{(n,n)}_{dR}(V, C) \), which is one of two natural generators of

\[
H^{n,n}(g, K; C) = C \wedge^n \kappa \oplus C\eta,
\]

where \( \kappa \) is the Kaehler class. Moreover the constant is \( \text{vol}(W) / \text{vol}(V) \).

Remark In place of the pair \((SO(2n, 2), SO(2n, 1))\), we can consider the pair \((SU(2n, 2), Sp(n, 1))\) for example.

1.6 The Matsushima isomorphism in the case of Hermitean \( G/K \)

Recall the part (i) of Proposition 1.2:

"If \( \chi r(C) = \chi r(C) \), there is an isomorphism

\[
H^m(g, K, M^\infty \otimes E) = C^m(g, K, M^\infty \otimes E) = \text{Hom}_K(\wedge^m p_C, M^\infty \otimes E) \text{.}"
\]

When the quotient \( X = G/K \) is Hermitean, the tangent space \( p \) at the identity \( [e] = eK \in G/K \) has the canonical complex structure to give the associated decomposition: \( p_C = p_+ \oplus p_- \). Then we have the induced decomposition of \( K \)-module:

\[
\wedge^m p_C = \oplus_{a+b=m} \wedge^a p_+ \otimes \wedge^b p_-.
\]

Thus the last space of the above isomorphism is a direct sum:

\[
\text{Hom}_K(\wedge^m p_C, M^\infty \otimes E) \cong \oplus_{a+b=m} \text{Hom}_K(\wedge^a p_+ \otimes \wedge^b p_-, M^\infty \otimes E).
\]

1.6.1 Example: the case \( G = SL(2, R) \)

Let \( E \) be the symmetric tensor representation of degree \( k \) of the standard representation of \( G = SL(2, R) \). It is the unique irreducible finite dimensional rational representation of \( G \) of degree \( k + 1 \). For a cocompact \( \Gamma \) the Matsushima isomorphism is the Eichler isomorphism:

\[
H^1(\Gamma \backslash G/SO(2), \text{Sym}^k) = \oplus_{\pi \in G : \chi r(\pi) = \chi r(\pi)} \text{Hom}_{SO(2)}(p, M_{\pi} \otimes \text{Sym}^k) \otimes I_\Gamma(\pi) = \text{Hom}_{SO(2)}(p, M_{\pi} \otimes \text{Sym}^k) \otimes I_\Gamma(D^+_{k+2}) = \text{Hom}_{SO(2)}(p, M_{\pi} \otimes \text{Sym}^k) \otimes I_\Gamma(D^-_{k+2}) = \text{Hom}_G(M_{\pi} \otimes L^2(\Gamma \backslash G)) \oplus \text{Hom}_G(M_{\pi} \otimes L^2(\Gamma \backslash G)) = \text{M}_{k+2}(\Gamma) \oplus \text{M}_{k+2}(\Gamma).
\]

Here

\[
\text{Hom}_{SO(2)}(p, M_{D^+_{k+2}} \otimes \text{Sym}^k) \cong C \quad \text{and} \quad \text{Hom}_{SO(2)}(p, M_{D^-_{k+2}} \otimes \text{Sym}^k) \cong C,
\]

and \( \text{M}_{k+2}(\Gamma) \) the space of holomorphic automorphic forms of weight \( k + 2 \) on the upper half plane \( SL(2, R) / SO(2) \) and \( \text{M}_{k+2}(\Gamma) \) the space of antiholomorphic automorphic forms of weight \( k + 2 \).
1.6.2 The case $G = Sp(2, \mathbb{R})$

We put

$$G = Sp(2, \mathbb{R}) := \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$ 

Let $(\rho_{(l_1,t_2)}, E_{(l_1,t_2)})$ be the irreducible finite-dimensional representation of $G$ with highest weight $l = (l_1, t_2)$. Let $\tilde{E}_l$ be the constructible sheaf on $\Gamma \backslash G/K$ associated with the restriction of $E_l$ to a cocompact discrete subgroup $\Gamma$ of $G$.

There are exactly 4 representations $\pi \in \hat{G}$ contributing to the non-vanishing cohomology group

$$H^3(g, K; M_{\pi,K} \otimes_C E_l) \neq \{0\}.$$ 

Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = (2, 1)$ be the half-sum of positive roots

$$\{2e_1 = (2, 0), e_1 + e_2 = (1, 1), 2e_2 = (0, 2), e_1 - e_2 = (1, -1)\}.$$ 

Then these 4 discrete series representations of $G$ have Harish-Chandra parameters (cf. §3 for definition):

$$(l_1 + 2, l_2 + 1) = l + \rho, \quad (l_1 + 2, -(l_2 + 1)) = w_1(l + \rho),$$

$$(l_2 + 1, -(l_1 + 2)) = w_2(l + \rho), \quad (-l_2 + 1, -(l_1 + 2)) = w_3(l + \rho),$$

where $w_i$ ($i = 1, 2, 3$) are elements of the Weyl group of $Sp(2, \mathbb{R})$. We denote them by

$$\{\pi_{l+\rho}^{++}, \pi_{w_1(l+\rho)}^{+-}, \pi_{w_2(l+\rho)}^{-+}, \pi_{w_3(l+\rho)}^{--}\},$$

though the superscripts $\pm, \pm$ are redundant symbols. We have

$$\text{Hom}_K(\wedge^3 p_+, M_{\pi^{++}_{l+\rho}} \otimes E_l) \cong C$$

and

$$\text{Hom}_K(\wedge^3 p_+ \otimes \wedge^3 p_-, M_{\pi^{++}_{l+\rho}} \otimes E_l) = \{0\}$$

for the remaining $(a,b)$. We may refer to this fact by saying that $\pi^{++}_{l+\rho}$ has type $(3,0)$. Similarly the remaining representations $\pi_{w_1(l+\rho)}^{+-}$, $\pi_{w_2(l+\rho)}^{-+}$ and $\pi_{w_3(l+\rho)}^{--}$ have the types $(2,1)$, $(1,2)$, and $(0,3)$ respectively. Thus the Matsushima isomorphism gives an isomorphism:

$$H^3(g, K; M_{\pi,K} \otimes_C E_l) = \text{Hom}_G(\pi_{l+\rho}^{++}) \oplus \text{Hom}_G(\pi_{w_1(l+\rho)}^{+-}) \oplus \text{Hom}_G(\pi_{w_2(l+\rho)}^{-+}) \oplus \text{Hom}_G(\pi_{w_3(l+\rho)}^{--}).$$

For a dominant integral weight $\lambda = (\lambda_1, \lambda_2)$ of the fixed compact Cartan subgroup $T$ of $K \cong U(2)$, let $(\tau_\lambda, W_\lambda)$ be the associated finite-dimensional irreducible representation of $K$. 

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1.6.3 The evaluation map

For example, when we consider \( \pi_{l+1}^{+,+} \), the non-zero element of \( \text{Hom}_K(\wedge^3 p_+, M_{\pi_{l+1}}^{+,+} \otimes E_l|_K) \) is realized as follows.

Choose the minimal \( K \)-type \((\tau_{\text{min}}, W_{\text{min}})\) of \((\pi_{l+1}^{+,+}, M_{\pi_{l+1}}^{+,+})\). Then there is the unique non-zero \( K \)-homomorphism

\[
\iota : \wedge^3 p_+ \to \tau_{\text{min}} \otimes E_l|_K
\]

which is equivalent to give a non-zero element in the \( K \)-invariant space

\[
\{\tau_{\text{min}} \otimes E_l|_K \otimes \wedge^3 p_+^*\}
\]

In our case we have \((\tau_{\text{min}}, W_{\text{min}}) \cong (\tau_{(l_1+3,l_2+3)}^{(l_1+3,l_2+3)}, W_{(l_1+3,l_2+3)})\). On the other hand we have the unique \( K \)-homomorphism \( \iota : (\tau_{\text{min}}, W_{\text{min}}) \to (\pi_{l+1}^{+,+}, M_{\pi_{l+1}}^{+,+})\).

The composition of \( \iota \) with \( \text{Hom}(id_{\wedge^3 p_+}, \iota \otimes id_{E_l}) \) gives the generator of the space \( \text{Hom}_K(\wedge^3 p_+, M_{\pi_{l+1}}^{+,+} \otimes E_l|_K) \). On the other hand, for

\[
I_{\iota}(\pi_{l+1}^{+,+}) = \text{Hom}_{(g,K)}(\pi_{l+1}^{+,+}, L^2(\Gamma\backslash G)),
\]

we can consider the evaluation map:

\[
\text{Hom}_{(g,K)}(\pi_{l+1}^{+,+}, L^2(\Gamma\backslash G)) \otimes M_{\pi_{l+1}}^{+,+} \to L^2(\Gamma\backslash G),
\]

which is a \( G \)-homomorphism.

On one part we consider the restriction this homomorphism to the image of \( \iota : \tau_{\text{min}} \to H_{\pi_{l+1}}^{+,+} \), which is identified with an element in

\[
\{f : \Gamma\backslash G \to W_{\tau_{\text{min}}}^*, f(xk) = \tau_{\text{min}}^*(k)f(x), \forall X \in p_-, \}
\]

because \( \pi_{l+1}^{+,+} \) is a highest weight module and the elements in \( \text{Im}(\iota) \) is annihilated by \( p_- \). This is the space \( M_{\iota_{l+1}}(\Gamma) \) of vector-valued holomorphic Sieg modular forms of weight \((l_1 + 3, l_2 + 3)\) belonging to \( \Gamma \).

**Remark 1.** On the hand the images of the evaluation map are given by elements in the space \( A^3(\Gamma\backslash G/K, E_l) \) of \( C^\infty \) \( 3 \)-forms with values in \( E_l \), which is identified with \( \{C^\infty(\Gamma\backslash G) \otimes E_l|_K \otimes \wedge^3 p_C^*\}^K \).

We can consider the remaining 3 cases similarly to have analogous spaces:

\[
M_{w_1(l+1)-\rho}^{+,+}(\Gamma) := \{f : \Gamma\backslash G \to W_{(l_1+1,-l_1-3)}, C^\infty \text{ function} | \begin{align*}
(i) & \ f(xk) = \tau_{(l_1+1,-l_1-3)}(k)f(x), \forall x \in G, \forall k \in K \\
(ii) & \ D^+ f = 0, \text{ Here } D^+ \text{ is the Dirac-Schmid operator}
\end{align*}\]

\[
M_{w_2(l+1)-\rho}^{+,+}(\Gamma) := \{f : \Gamma\backslash G \to W_{(l_1+3,-l_2-1)}, C^\infty \text{ function} | \begin{align*}
(i) & \ f(xk) = \tau_{(l_1+3,-l_2-1)}(k)f(x), \forall x \in G, \forall k \in K \\
(ii) & \ D^- f = 0, \text{ Here } D^- \text{ is the Dirac-Schmid operator}
\end{align*}\]

\[
M_{w_3(l+1)-\rho}^{+,+}(\Gamma) := \{f : \Gamma\backslash G \to W_{(l_1+3,l_2+3)}, C^\infty \text{ function} | \begin{align*}
(i) & \ f(xk) = \tau_{(l_1+3,l_2+3)}(k)f(x), \forall x \in G, \forall k \in K \\
(ii) & \ D^- f = 0, \text{ Here } D^- \text{ is the Dirac-Schmid operator}
\end{align*}\]

The precise definition of the Dirac-Schmid operator is not given here. We shall define that in later subsections 3.4 and 3.5 to discuss the realization of the discrete series representations of \( Sp(2,\mathbb{R}) \).
2 The case of non-cocompact $\Gamma$

We again consider the general $G$. When $\Gamma$ is not cocompact, it becomes difficult to describe the cohomology groups $H^i(\Gamma, E)$ in terms of automorphic forms. Because the space $C^\infty(\Gamma \backslash G)$ contains those functions which increase quite rapidly "at infinity", i.e., when they approach to the cusps, it is quite inconvenient to develop natural analysis. So we have to impose some conditions at infinity. There are two standard ways to do so.

2.1 Automorphic forms

Before to discuss the Fourier expansion of automorphic forms, we recall their definition. Firstly we have to define the norm

$$\|g\| := \text{tr}(g \cdot ^t g).$$

of an element $g \in G = \text{Sp}(2, \mathbb{R})$. From now on we let $\Gamma = \text{Sp}(2, \mathbb{Z})$ for simplicity.

**Definition** An element $f \in C^\infty(\Gamma \backslash G)$ is an automorphic form, if it satisfies the following three conditions:

(i) $f$ is (right) $K$-finite, i.e., the linear span of right translations $R_k(f)(x) = f(xk)$ ($k \in K$) of $f$ by $K$ is of finite dimension;

(ii) $f$ is $Z(g)$-finite, i.e., there is an ideal $I$ of finite codimension in the center $Z(g)$ of the universal enveloping algebra $U(g)$, which annihilate $f$;

(iii) There are a natural number $N$ and a positive constant $C$ such that

$$|f(x)| \leq C\|x\|^N \text{ for any } x \in G.$$

We denote by $\mathcal{A}(\Gamma \backslash G)$ the space of automorphic forms defined above. It is a $G$-module by the right action of $G$ which is smooth under an adequate topology, and a $(g, K)$-module algebraically. If necessary, it is better to replace the second condition of the above definition by a more restrictive condition:

(ii)' $f$ generates an irreducible $(g, K)$-module $V_f$ by right translation.

**Remark.** Moreover, in the following we discuss the explicit formulae of Fourier expansion or spherical functions, mainly we consider only such $f$ that has a scalar $K$-type, i.e., there is a continuous unitary character

$$\chi_m : k = \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in K \mapsto \det(A + \sqrt{-1}B)^m \in \mathbb{C}^*$$

such that $f(xk) = \chi_m(k)f(x)$.

2.2 $L^2$-space and spectral decomposition

The other standard way to control the automorphic forms at infinity is to consider the $L^2$-space on $\Gamma \backslash G$. Let us recall a fundamental result on the spectral decomposition of the $L^2$-space $L^2(\Gamma \backslash G)$, which is a unitary $G$-module by the right quasi-regular action: $\varphi(x) \mapsto \varphi(xg)$ $(x, g \in G)$. Since an algebraic real Lie group $G$ is of type I in the sense of $C^*$-algebra, the space $L^2(\Gamma \backslash G)$ is a direct integral of irreducible unitary representations of $G$ uniquely. The discrete part of
this decomposition $L^2_{\text{dis}}(\Gamma \backslash G)$ is defined as the closure of the sum of irreducible $G$-subspaces. The continuous part $L^2_{\text{con}}(\Gamma \backslash G)$ is the orthogonal complement of this discrete part. Hence we have the direct sum decomposition:

$$L^2(\Gamma \backslash G) = L^2_{\text{dis}}(\Gamma \backslash G) \oplus L^2_{\text{con}}(\Gamma \backslash G).$$

The generalized Plancherel measure on $\hat{G}$ to describe $L^2_{\text{con}}(\Gamma \backslash G)$ is given as a sum of contribution from various Eisenstein series. An the cuspidal subspace $L^2_{\text{cusp}}(\Gamma \backslash G)$ is a closed subspace of $L^2_{\text{dis}}(\Gamma \backslash G)$. The fundamental theorem of (Selberg-)Langlands in [26] is to describe the orthogonal complement of $L^2_{\text{cusp}}(\Gamma \backslash G)$ in $L^2_{\text{dis}}(\Gamma \backslash G)$ as the residual part coming from the residues of the poles of Eisenstein series.

### 2.3 Various cohomology groups

We introduce some variations of cohomology of $\Gamma$ which are mutually correlated:

(i) The cuspidal cohomology group:

$$H^i_{\text{cusp}}(\Gamma, E) := H^i(\mathfrak{g}, K; E \otimes C L^2_{\text{cusp}}(\Gamma \backslash G)),$$

where $L^2_{\text{cusp}}(\Gamma \backslash G)$ is the closed subspace of $L^2(\Gamma \backslash G)$ consisting of cusp forms, which decomposes discretely into irreducible unitary $G$-modules with finite multiplicities.

(ii) The square-integrable cohomology group:

$$H^i_{(2)}(\Gamma, E) := H^i(\mathcal{A}^*(2)(\Gamma \backslash X, E)),$$

where $(\mathcal{A}^*(\Gamma \backslash X, E), d)$ is the complex consisting square-integrable forms. As shown by A. Borel, this space is isomorphic to

$$H^i(\mathfrak{g}, K; E \otimes C L^2_{\text{dis}}(\Gamma \backslash G)),$$

where $L^2_{\text{dis}}(\Gamma \backslash G)$ is the discrete part of $L^2(\Gamma \backslash G)$, that is the closure of the sum of the closed irreducible subspaces of $L^2(\Gamma \backslash G)$.

(iii) The interior cohomology group:

$$H^i(\Gamma, E) := \text{Im}\{H^i_{\text{c}}(\Gamma \backslash X, \tilde{E}) \rightarrow H^i(\Gamma \backslash X, \tilde{E})\},$$

where $H^i_{\text{c}}(\ast, \ast)$ means the cohomology group with compact supports.

Fortunately for $G = \text{Sp}(2, \mathbb{R})$ and $i = 3$, all these cohomology groups are isomorphic and have a similar decomposition as the cocompact case, replacing $L^3(\Gamma \backslash G)$ by $L^3_{\text{dis}}(\Gamma \backslash G)$ or by a smaller space $L^3_{\text{cusp}}(\Gamma \backslash G)$. This was shown by Oda-Schwermer ([42], Formula (4) in p. 488) for the constant coefficient case, and for the twisted case coefficient cases it is handled easier. Since the $K$-finite part of $L^3_{\text{cusp}}(\Gamma \backslash G)$ is identical with the space of cusp forms $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$, we have

$$H^3(\Gamma, E_i) = H^3_{\text{cusp}}(\Gamma, E_i) = H^3_{(2)}(\Gamma, E_i).$$

Moreover these space have a similar decomposition as the case of cocompact subgroups, replacing $C^\infty(\Gamma \backslash G)$ by the space of cusps forms $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$ to form the intertwining spaces

$$\text{Hom}_{(\mathfrak{g}, K)}(\pi^{\pm \rho}_{i+\rho}, \mathcal{A}_{\text{cusp}}(\Gamma \backslash G) \otimes E_i).$$
Similarly as the cocompact case, we introduce 4 spaces:

\[
S^+_{l^+}(\Gamma) := \{ f \in \mathcal{A}_{cusp}(\Gamma) \otimes W_{(-l_2-3,-l_1-3)} \mid \\
(\text{i}) f(xk) = \tau_{(-l_2-3,-l_1-3)}(k)f(x), \forall x \in G, \forall k \in K \\
(\text{ii}) D^{++}f = 0 \}
\]

\[
S^-_{w_1(l+\rho)-\rho}(\Gamma) := \{ f \in \mathcal{A}_{cusp}(\Gamma) \otimes W_{(l_1+3,-l_2-1)} \mid \\
(\text{i}) f(xk) = \tau_{(l_1+3,l_2-1)}(k)f(x), \forall x \in G, \forall k \in K \\
(\text{ii}) D^{++}f = 0 \}
\]

\[
S^+_{w_2(l+\rho)-\rho}(\Gamma) := \{ f \in \mathcal{A}_{cusp}(\Gamma) \otimes W_{(l_1+3,l_2+3)} \mid \\
(\text{i}) f(xk) = \tau_{(l_1+3,l_2+3)}(k)f(x), \forall x \in G, \forall k \in K \\
(\text{ii}) D^{++}f = 0 \}
\]

\[
S^-_{w_3(l+\rho)-\rho}(\Gamma) := \{ f \in \mathcal{A}_{cusp}(\Gamma) \otimes W_{(l_1+3,l_2+3)} \mid \\
(\text{i}) f(xk) = \tau_{(l_1+3,l_2+3)}(k)f(x), \forall x \in G, \forall k \in K \\
(\text{ii}) D^{++}f = 0 \}
\]

Here \(D^{\pm,\pm}\) are the Dirac-Schmid operators.

As we see in the subsection 3.4, the definition of the Dirac-Schmid operator is independent of \(\Gamma\). Therefore it is the same as the case of cocompact \(\Gamma\).

Now we have the analogue of Eichler-Shimura isomorphism:

**Theorem 2.1**

\[
H^3_{\mathcal{C}}(\Gamma, E_\ell) = \bigoplus_{\pm,\pm} \text{Hom}_{(G,K)}(s_{l_1+\rho'}^\pm, \mathcal{A}_{cusp}(\Gamma) \otimes E_\ell) = \\
S^+_{l^+}(\Gamma) \oplus S^-_{w_1(l+\rho)-\rho}(\Gamma) \oplus S^+_{w_2(l+\rho)-\rho}(\Gamma) \oplus S^-_{w_3(l+\rho)-\rho}(\Gamma).
\]

Here the proof of the second equality of the above theorem is the immediate consequence of the characterization of the discrete series representations by their minimal \(K\)-types (cf. the subsection 3.3).

One important achievement in 80’s on the \(L^2\)-cohomology is the proof of Zucker’s conjecture. We may refer the article Zucker [53] for the survey of this history. We should mention the project of (Harder-)Schwermer to describe the “difference” between \(H^3_{\mathcal{C}}(\Gamma, G, E)\) and \(H^3_\mathcal{C}(G, \mathcal{A}(\Gamma \backslash G; E))\) by using Eisenstein series ([47]).
3 The representations of the discrete series of $Sp(2, \mathbb{R})$

A representation of discrete series of a real semisimple Lie group $G$ is a both tempered and cohomological representation. The investigation of this very important class of unitary representations are initiated by Harish-Chandra in later '50's. The fundamental results are all obtained by him: the equivalence with square-integrability, the criterion of existence, the character formula on compact Cartan subgroups, etc. Realization (Okamoto-Narashiman, Schmid), the character formula (Hirai, Schmid) $K$-type theorem (the Blattner conjecture, (Hecht)-Schmid). These deep results are not yet used for geometry and arithmetic of automorphic forms.

In general, let $G$ be a locally compact topological group which satisfies the second axiom of countability; let $\hat{G}$ be the unitary dual of $G$, i.e., the set of unitary equivalence classes of irreducible unitary representations of $G$. As described in Dixmier [3] by the $C^*$-algebra of $G$, $\hat{G}$ under the hull-kernel topology is a locally compact Baire space.

For a semisimple Lie group $G$, which is postliminaire, the abstract Plancherel theorem is void (due to Segal): fix a Haar measure $d_G$ on $G$, then there exists unique positive measure $\mu$ on $\hat{G}$ such that

$$\int_G |f(x)|^2 d_G(x) = \int_{\hat{G}} \text{tr}(\pi(f)\pi(f)^*) d\mu(\pi)$$

for all $f \in L^1(G) \cap L^2(G)$.

The support of the Plancherel measure $\mu$ on $\hat{G}$ is a closed subset $\hat{G}_{\text{temp}}$, and those unitary representations belonging to this are called tempered representations.

3.1 Definition

For a general semisimple Lie group $G$ with finite center, the unitary representations of $G$ belonging to the discrete series are defined as follows:

Definition

(i) Let $\hat{G}$ be the unitary dual of $G$ with the hull-kernel or the Fell topology. Then $\pi$ is of discrete series, if and only if $\{\pi\}$ is isolated in $\hat{G}$.

(ii) The following was shown by Harish-Chandra:

Theorem 3.1 $\pi$ is of discrete series, if and only if $\pi$ is square-integrable, i.e.,

$$\forall v, w \in H_e, \text{ the matrix coefficient } c_{v, w}(g) = (\pi(g)v, w)_{H_e} \in L^2(G).$$

(iii) Let us consider $L^2(G)$ as a $G \times G$-bimodule via the left and right regular representations. Then we have the Plancherel formula:

$$L^2(G) = \int_{\hat{G}_{\text{temp}}} \pi_\lambda \boxtimes \pi_\lambda^* d\mu(\lambda).$$

Here $\mu$ is the Plancherel measure which has support only on the tempered spectre $\hat{G}_{\text{temp}}$ of $G$. Then for discrete series $\pi_\lambda$, we have $\mu(\{\pi_\lambda\}) > 0$ and this value is proportional to the formal degree of $\pi_\lambda$.

The following fundamental result is also due to Harish-Chandra:
Theorem 3.2 (Harish-Chandra) The discrete spectrum $\hat{G}_{DS}$ is non-empty, if and only if $\text{rank-}K = \text{rank-}G$, i.e., $G$ has a compact Cartan subgroup $T$.

3.2 Parametrization of the discrete series

There is also the parametrization of the discrete series by Harish-Chandra, by using the set of dominant integral weights on $T$.

We discuss on the special case $G = \text{Sp}(2, \mathbb{R})$. A compact Cartan subgroup $T$ of $G$ in $K$ is given by

$$ T = \left\{ \begin{pmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{pmatrix} \mid \theta_1, \theta_2 \in \mathbb{R} \right\}. $$

The set $L_T$ of unitary characters of $T$ is parametrized by pairs of integers $(l_1, l_2) \in \mathbb{Z}^2$ and the set of dominant weights is given by

$$ L^+_T := \{(l_1, l_2) \in L \mid l_1 \geq l_2\}. $$

The irreducible finite-dimensional representation of $K$ with highest weight $(l_1, l_2) \in L^+_T$ is denoted by $(\tau_{\pi^{l_1, l_2}}, W_{\pi^{l_1, l_2}})$. Let

$$ \Xi := \{(l_1, l_2) \in \mathbb{Z}^2 \mid l \text{ regular, i.e., } l_1 \neq 0, l_2 \neq 0, l_1 \neq \pm l_2\}. $$

We note here that $\rho$ the half-sum of positive roots for $(g, t)$ is given by

$$ \rho = \frac{1}{2} \{2e_1 + (e_1 + e_2) + 2e_2 + (e_1 - e_2)\} = \frac{1}{2} \{(2, 0) + (1, 1) + (0, 2) + (1, -1)\} = (2, 1). $$

Theorem 3.3 (Harish-Chandra) For $G = \text{Sp}(2, \mathbb{R})$, there is a bijection $\pi_{\lambda} \in \hat{G}_{DS} \leftrightarrow \lambda \in \Xi$. Moreover the set $\Xi$ is written as the sum of 4 Weyl chambers:

$$ \Xi = \Xi^{++} \cup \Xi^{+-} \cup \Xi^{-+} \cup \Xi^{--} $$

with

$$ \Xi^{++} := \{(l_1, l_2) \in \Xi \mid l_1 > l_2 > 0\} $$
$$ \Xi^{+-} := \{(l_1, l_2) \in \Xi \mid l_1 > 0 > l_2, l_1 + l_2 > 0\} $$
$$ \Xi^{-+} := \{(l_1, l_2) \in \Xi \mid l_1 > 0 > l_2, l_1 + l_2 < 0\} $$
$$ \Xi^{--} := \{(l_1, l_2) \in \Xi \mid 0 > l_1 > l_2\}. $$

For a given $\lambda = (l_1, l_2) \in \Xi$ satisfying $l_1 > l_2 > 0$, there are 4 parameters:

$$ \lambda^{++} = (l_1, l_2) \in \Xi^{++}, \quad \lambda^{+-} = (l_1, -l_2) \in \Xi^{+-}, $$
$$ \lambda^{-+} = (l_2, -l_1) \in \Xi^{-+}, \quad \lambda^{--} = (-l_2, -l_1) \in \Xi^{--} $$

with the same infinitesimal characters $\chi_{\pi_{\lambda}, \lambda} : Z(g) \to \mathbb{C}$. 

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3.3 The minimal $K$-types

From some time, there was the Blattner conjecture to describe the multiplicities of $K$-types a given discrete series. This was finally proved by Hecht-Schmid [7]. Some what earlier the minimal $K$-type $\tau$ whose highest weight is nearest to the origin that occurs with multiplicity one is known to exist. We recall this minimal $K$-type in the case $G = \text{Sp}(2, \mathbb{R})$. For a given Harish-Chandra parameter $\lambda \in \Xi$, we choose the positive root system $\Phi^+(\lambda)$ which is positive with respect to $\lambda$. Let $\rho_c$ be the half-sum of the compact positive roots which is independent of the choice of $\lambda$, and let $\rho_n(\lambda)$ be the half-sum of non-compact positive roots in $\Phi^+_{nc}(\lambda)$. Then we have the following (cf. Schmid [46]).

Proposition 3.1 Set $\mu = \lambda - \rho_c + \rho_n(\lambda)$. Then $\tau_\mu$ is the minimal $K$-type of the discrete series $\pi_\lambda$, and for any $\beta \in \Phi^+_{nc}(\lambda)$, the $K$-type $\tau_{\mu - \beta}$ does not occur in $\pi_\lambda$. Moreover the condition that such $K$-type exists, which is far enough from the walls of the Weyl chambers, characterises the discrete series $\pi_\lambda$ infinitesimally. Namely, if $\tau_\mu$ is far enough from the walls of Weyl chambers, and $\tau_\mu$ occurs with multiplicity one in some irreducible $(\mathfrak{g}, K)$ module $\pi$, and if and for any $\beta \in \Phi^+_{nc}(\mu)$, the $K$-type $\tau_{\mu - \beta}$ does not occur in $\pi$. Then $\pi$ is isomorphic to the $(\mathfrak{g}, K)$-modules of the discrete series $\pi_{\mu + \rho_c - \rho_n(\mu)}$.

Here is the table of the minimal $K$-types $\mu$ of the discrete series for $\text{Sp}(2, \mathbb{R})$. Here $\lambda = (\lambda_1, \lambda_2)$ are Harish-Chandra parameters.

<table>
<thead>
<tr>
<th>type of parameter $\lambda$</th>
<th>$\lambda \in \Xi^{++}$</th>
<th>$\lambda \in \Xi^{+-}$</th>
<th>$\lambda \in \Xi^{-+}$</th>
<th>$\lambda \in \Xi^{--}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>the minimal $K$-type $\mu$</td>
<td>$\lambda + (1, 2)$</td>
<td>$\lambda + (1, 0)$</td>
<td>$\lambda - (0, 1)$</td>
<td>$\lambda - (2, 1)$</td>
</tr>
</tbody>
</table>

Remark 1 The condition "if $\tau_\mu$ is far enought from the walls of Wely chambers" is not necessary for $G = \text{Sp}(2, \mathbb{R})$ case. We know the infinitesimal character of the discrete series representation in question, and also the composition series of the principal series representation with this infinitesimal character explicitly.

3.4 Realization of discrete series represenations

There are realizations of the discrete series represenations in the spaces of square-integrable $\partial$-cohomology groups over $G/K$ or over $G/T$ with values in holomorphic vector bundles associated with irreducible finite-dimensional representaions of $K$ or with unitary characters of $T$ (cf. [37], [45]), which are employed to prove a conjecture of Langlands. And this smooth realization was later generalized for smooth realization of cohomological representasions by Wong [51].

But for explicit calculation of the $A$-radial part of the functions belonging to the minimal $K$-type, the following realization of the discrete series representations, utilizing the Dirac-Schmid operator is quite essential.

For a finite dimensional continuous representation $(\tau, W)$ of $K$, we set

$$C^\infty_K(G; K) := \{C^\infty(G) \otimes W\}_K.$$

Here $K$-invariant part $\{\ast\}_K$ is considered with respect ot the right quasiregular action of $K$ on $C^\infty(G)$ and $\tau$ on $W$. Let $(\text{Ad}, p_C)$ and $(\text{Ad}, p_C^*)$ be the adjoint representation of $K$ and its contragradient representation, which are isomorphic
by the non-degenerateness of the restriction of the Killing form to $p$. Then we have an isomorphism of $K$-modules

$$(Ad, p_C) \cong (Ad, p_C^*) \cong \oplus_{\beta \in \Phi_{nc}(l)} (\tau_\beta, W_\beta)$$

with $\Phi_{nc}$ the set of non-compact roots in $\Phi(g, t)$. Choose a basis $\{x_i\}$ in $p$ and its dual basis $\{\xi_i\}$ in $p^*$, then we can define the gradient operator

$$\nabla : C^\infty_T(G; K) \to C^\infty_{T \otimes (Ad^*, p^*)}(G; K)$$

by

$$\nabla f := \sum_i R_{xi} f \otimes \xi_i$$

with $R_{xi}$ the right action of $p \subset g$ on $C^\infty(G)$.

Now choose a dominant integral weight $l$ of $T$ to have the associated irreducible representation $(\tau_l, W_l)$ of $K$. We assume that $l$ satisfies the regularity condition such that there is unique positive system of non-compact roots $\Phi_{nc}^+(l)$, which is positive with respect to $l$ (and compatible with the given system of compact positive roots $\Phi^+_l$). Then we have the associated decomposition of $p_C = p^*_+(l) \oplus p^*_-(l)$ with

$$p^*_+(l) := \oplus_{\beta \in \Phi_{nc}^+(l)} \tau_\beta, \text{ and } p^*_-(l) := \oplus_{\beta \in \Phi_{nc}^+(l)} \tau_{-\beta}.$$

Then from the tensor product

$$(\tau_l \otimes Ad^*, W_l \otimes p_C^*) \cong \oplus_{\beta \in \Phi_{nc}^+} (\tau_{l+\beta}, W_{l+\beta}),$$

there is a projector of $K$-modules

$$pr_- : W_l \otimes p^* \to W_l \otimes p^*_-(l) \cong \oplus_{\beta \in \Phi_{nc}^+} (\tau_{l-\beta}, W_{l+\beta}).$$

Lastly we consider the composition of the gradient operator $\nabla$ with the projector

$$C^\infty_{\tau_l \otimes Ad^*}(G; K) \to C^\infty_{\tau_l \otimes Ad^*}(G; K),$$

to get an operator $D_l$ between the smooth left $G$-modules:

$$C^\infty_{\tau_l}(G; K) \to C^\infty_{\tau_l \otimes Ad^*}(G; K).$$

Here $Ad^*$ means the $K$-module $(Ad^*_+, p^*_+(l))$. This operator, which we call the Dirac-Schmid operator, is an elliptic operator under some condition on $l$, which is satisfied when $l$ is the minimal $K$-type of a discrete series with paramater $\lambda$ positive with respect to $\Phi^+(l)$.

**Theorem 3.4** Let $\tau_l$ be the contragradient representation of the minimal $K$-type $\mu = \lambda - \rho_c + \rho_n$ of the discrete series with Harish-Chandra parameter $\lambda$. Then the kernel space $\text{Ker}(D_l)$ of the operator is non-zero and by the left quasi-regular action, it gives the contragradient discrete series representation $\pi^*_\lambda$.

**Remark 1.** It is easy to understand the "meaning" of the above theorem. For each discrete series $\pi_\lambda$, it occurs as the outer tensor product $\pi^*_\lambda \otimes \pi_\lambda$ in the $(G \times G)$-birepresentation $L^2(G)$ uniquely. Consider the intertwining space

$$\text{Hom}_G(\pi_\lambda, L^2(G))$$
to absorb the right $G$-action and to realize $\pi_\lambda^*$ as this intertwining space. And consider further the evaluation map

$$\text{Hom}_G(\pi_\lambda, L^2(G)) \otimes \pi_\lambda \to L^2(G).$$

The restriction of this evaluation map to the minimal $K$-type $\tau_\mu$ gives canonically

$$ev_K : \text{Hom}_G(\pi_\lambda, L^2(G)) \to \{L^2(K) \otimes \tau_\mu^*\}^K.$$  

Here the superscript $\{\ast\}^K$ means the $K$-invariant part. Then the condition of the annihilation $p_-(\lambda) \cdot \tau_\mu = \{0\}$ implies that the image of the induced evaluation map $ev_K$ is contained in the kernel of the Dirac-Schmid operator. The surjectivity follows from the latter part of Proposition 3.1. (cf. Hotta-Parthasarathy [12]).

**Remark 2** Original proof was done under stronger regularity condition, to assure the characterization of the minimal $K$-type of the discrete series to assure the ellipticity of $D_l$. This is unnecessary here for $G = \text{Sp}(2,\mathbb{R})$.

### 3.5 The Dirac-Schmid operators on automorphic forms

Let $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$ be the space of automorphic cusp forms on $G$ with respect to $\Gamma$. Then for a given discrete series representation $\pi_\lambda$, we consider the evaluation map

$$ev_\Gamma : \text{Hom}_{(g, K)}(\pi_\lambda, \mathcal{A}_{\text{cusp}}(\Gamma \backslash G)) \otimes \pi_\lambda \to \mathcal{A}_{\text{cusp}}(\Gamma \backslash G).$$

The image of this is the $\pi_\lambda$-isotypic component in $\mathcal{A}_{\text{cusp}}(\Gamma \backslash G)$. Take the minimal $K$-type $i : W_\mu \mapsto \pi_\lambda$ to have the induced $K$-homomorphism by restriction:

$$ev_\Gamma : \text{Hom}_{(g, K)}(\pi_\lambda, \mathcal{A}_{\text{cusp}}(\Gamma \backslash G)) \otimes W_\mu \to \mathcal{A}_{\text{cusp}}(\Gamma \backslash G),$$

or passing to the adjoint as $K$-modules, we have

$$ev_{\Gamma, K} : \text{Hom}_{(g, K)}(\pi_\lambda, \mathcal{A}_{\text{cusp}}(\Gamma \backslash G)) \to \mathcal{A}_{\text{cusp}}(\Gamma \backslash G) \otimes W_\mu^* \to \mathcal{A}_{\text{cusp}}(\Gamma \backslash G) \otimes \{W_\mu^* \otimes p_-(\lambda)\}^K.$$  

Discussing similarly as the previous remark 1, we can show that the image of this belongs to the kernel of the Dirac-Schmid operator

$$pr_\cdot \cdot \nabla : \{\mathcal{A}_{\text{cusp}}(\Gamma \backslash G) \otimes W_\mu^*\}^K \to \mathcal{A}_{\text{cusp}}(\Gamma \backslash G) \otimes \{W_\mu^* \otimes p_-(\lambda)\}^K,$$

which we also denote by $\mathcal{D}^\pm$. The surjectivity is obtained similarly as the last remark 1.
4 Some general aspect on Fourier expansion of automorphic forms

At first sight it seems to be a rather trivial problem to have Fourier expansions of automorphic forms. But after a serious study, one find that it is a difficult problem to find some reasonable form of Fourier expansion for general automorphic forms in the sense of Harish-Chandra [5], with respect rational parabolic subgroups. Even for the rather small group $SU(2,1)$ it was worked out rather recently by Ishikawa [19]. And for holomorphic automorphic forms on a symmetric domain $G/K$, though there were Fourier-Jacobi expansions with respect to various rational maximal parabolic subgroups $P$, the Fourier expansion with respect to general parabolic subgroups including the minimal parabolic subgroup has settled quite recently in a satisfactory manner by Narita [38] for $G/K$ being tube domains. The well-known theory of Fourier expansion on $GL(n)$ by H. Jacquet, I. Piatetski-Shapiro, and J. Shalika is basically valid only for the maximal parabolic subgroup of type $(1, n-1)$ utilizing the semi-global information of Whittaker functions, and moreover Whittaker functions are defined with respect to the minimal parabolic subgroup of $GL(n)$ and linear characters of its unipotent radical.

4.1 General formulation of the problem

Before discussing the Fourier expansion of automorphic forms with respect specific parabolic subgroups, we consider the general setting of Fourier expansion with respect to a "good" closed subgroup $R$ of $G$ defined over $\mathbb{Q}$. In the beginning we assume that

$$(H1) : \quad (R \cap \Gamma) \setminus R \text{ is compact},$$

to avoid the trouble of the continuous spectrum. For $r \in R$ and $g \in G$, we consider a function $f(rg)$ of two variables $(r, g) \in R \times G$ for a vector $f$ in the image of the evaluation map

$$ev_{\pi} : \text{Hom}_{(g, K)}(M_{\pi}, A(\Gamma \setminus G)) \otimes M_{\pi} \to A(\Gamma \setminus G)$$

for some fixed $(g, K)$ module $M_{\pi}$.

Fix the second variable for a while to regard $f(rg)$ as a function in $r$. Then it is real analytic in $r$ which is left $(R \cap \Gamma)$-invariant and right $R \cap (g^{-1}Kg)$-invariant. Since it is not of $Z(r)$-finite in general, it is an infinite sum

$$f(rg) = \sum_{\eta \in \hat{R}} f(rg)[\eta]$$

of automorphic forms on $R$ via the spectral decomposition

$$L^2((R \cap \Gamma) \setminus R) = \bigoplus_{\eta \in \hat{R}} I(\eta)$$

with the $\eta$-isotypic component

$$I(\eta) \cong \text{Hom}_R(V_\eta, L^2((R \cap \Gamma) \setminus R)) \otimes V_\eta.$$

Here $V_\eta$ is the representation space of $\eta$. 

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For each fixed \( \eta \), let \( B_\eta = \{ \beta_{\eta,i} \}_{1 \leq i \leq m(\eta)} \) be a basis of the finite-dimensional vector space \( \text{Hom}_R(\eta, L^2((R \cap \Gamma)\setminus R)) \) of dimension \( m(\eta) \). Then

\[
f(rg) = \sum_{i=1}^{m(\eta)} ev_\eta(\beta_{\eta,i} \otimes v_{\eta,i})(r)
\]

with some vectors \( v_{\eta,i} = v_{\eta,i}(f; g) \in V_\eta \) depending on \( f \) and \( g \). Here \( ev_\eta \) is the evaluation isomorphism:

\[
ev_\eta : \text{Hom}_R(\eta, L^2((R \cap \Gamma)\setminus R)) \otimes V_\eta \cong L^2((R \cap \Gamma)\setminus R)[\eta].\]

Computing the function \( f(rr'g) \) \((r, r' \in R, g \in G)\) of 3 variables in two ways, we have an intertwining property:

\[
v_{\eta,i}(f; rg) = \eta(r)v_{\eta,i}(f; g) \quad (r \in R, g \in G).
\]

Therefore for fixed \( f \), the function

\[
g \in G \mapsto v_{\eta,i}(f; g) \in V_\eta
\]
defines an element of the representation space of \( \text{Ind}_R^G(\eta) \). Moreover the association

\[
f \in M_\pi \subset \mathcal{A}(\Gamma\setminus G) \mapsto v_{\eta,i}(f; g) \in \text{Ind}_R^G(\eta)
\]
is a \( G \)-homomorphism. Thus each \( v_{\eta,i} \) \((i = 1, \cdots, m(\eta))\) is considered as an element of the intertwining space \( \text{Hom}_{(g, K)}(M_\pi, \text{Ind}_R^G(\eta)) \).

If we have "the multiplicity-free theorem" \( \text{Hom}_{(g, K)}(M_\pi, \text{Ind}_R^G(\eta)) \leq 1 \), then there is the unique normalized element \( v_\eta \) in this space, and each \( v_{\eta,i} \) is written in the form

\[
v_{\eta,i}(f; g) = c_{\eta,i}(f)v_\eta(f; g)
\]
with a constant (i.e., the Fourier coefficient) \( c_{\eta,i}(f) \) which is independent of \( g \) and \( i \).

When \( R \) is the unipotent radical \( N \) of a minimal parabolic subgroup \( P \) of \( G \), and \( \eta \) a unitary character of \( N \), then all the above produre is valid and \( v_\eta \) is called the Whittaker functional. For a infinite-dimensional \( \eta \) of \( M \), or more generally for general parabolic subgroups \( P \), the multiplicity-free property is not valid even for the characters \( \eta \) of the unipotent radical \( R = N \) of \( P \). In this case, we have to take as \( R \) a subgroup between \( P \) and \( N \), which is in many cases (a metaplectic cover of) the semidirect product of \( N \) and the stablizer \( S \) of a represenation \( \xi \) of \( N \) in the Levi-part \( L \) of \( P \).

Anyway we are naturally lead to the following problem of spherical functions.

### 4.2 The local problem: spherical functions

Let us formulate our problem on generalized spherical functions. Let \( R \) be a closed subgroup, and \( \eta \) a unitary irreducible representation of \( R \). Form the \( C^\infty \)-induction \( C^\infty-\text{Ind}_R^G(\eta) \). Given a smooth representation \( \pi \) of \( G \), or a \((g, K)\)-module \((\pi, H_\pi)\). Then we can consider the intertwinning space

\[
I_{(R, \eta)}(\pi) := \text{Hom}_{(g, K)}(\pi, C^\infty-\text{Ind}_R^G(\eta)).
\]

We hope the situation when \( R \) is "large enough" and \( \eta \) is "small enough", to satisfy the following conditions:
(i) For the split Cartan subgroup $A$, we have a double coset decomposition $G = RAK$.

(ii) The dimension of $I(R,\eta)(\pi)$ is finite.

In the context of ordinary Fourier expansion, it is enough to consider when $R$ is a subgroup of a parabolic subgroup $P$ containing the unipotent radical $N$ of $R$. But it is also important to consider the case when $R$ is a reductive subgroup of $G$, as we see in the later section 6.

4.3 The $P_J$ principal series representations of $Sp(2, \mathbb{R})$

We should review the results on the spherical functions belonging to the discrete series representations of $Sp(2, \mathbb{R})$. But for holomorphic discrete series, this problem is rather easy, and more or less it is known classically. For the large discrete series one need to prepare some more terminology to specify the basis of irreducible representations $\tau_l$ of $K$ (especially for the minimal $K$-type $\tau$). This requires much more space.

In place of the large discrete series themselves, we will review the results on the spherical functions belonging to the $P_J$ principal series representation, which is closely related to the large discrete series representations of $Sp(2, \mathbb{R})$ in two ways: (i) the large discrete series are embedded in a $P_J$ principal series; (ii) the $K$-types of the large discrete series are translations of the $K$-types of the $P_J$ principal series. As a result, the forms of spherical functions of the large discrete series are quite similar to those of the $P_J$ principal series.

4.3.1 The definition of $P_J$ principal series

Here is the definition of the $P_J$ principal series. Firstly, we consider the standard parabolic subgroup $P_J$ associated with the long root $2e_2 = (0, 2)$ in the set of positive roots

$\Delta = \{e_1 - e_2 = (1, -1), 2e_2 = (0, 2)\}.$

Its unipotent radical $N_J$ is the Heisenberg group of dimension 3 with Lie algebra

$$n_J = \mathfrak{g}_{e_1} + e_2 \oplus \mathfrak{g}_{2e_1} \oplus \mathfrak{g}_{e_1 - e_2}.$$ 

Thus the half sum of positive roots $\rho_J$ is given by $\frac{1}{2}(e_1 + e_2 + 2e_1 + e_1 - e_2)$. The split component $A_J$ is associated with the Lie algebra $a_J = RH_1$ with

$H_1 = \text{diag}(1, 0, -1, 0).$

The group $M_J$ is the product of the centralizer of $A_J$ in $K$ and the connected component $M_J^0$ which corresponds to the Lie algebra

$$m_J = \mathfrak{g}_{2e_2} \oplus \mathfrak{g} - 2e_2 \oplus RH_2.$$ 

Since the group $M_J$ is isomorphic to the product $\{\pm 1\} \times SL(2, \mathbb{R})$, the discrete series representation of $M_J$ is the tensor product $\varepsilon \otimes D^\pm_k$ of a character $\varepsilon$ of $\{\pm 1\}$ and a discrete series representation $D^\pm_k$ or $D^\pm_k$ of $SL(2, \mathbb{R})$ with minimal $K$-type

$\chi_\pm: r(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \in SO(2) \mapsto \exp(k\theta) \in U(1)$

or $\chi_{-\pm}: r(\theta) \mapsto \exp(-k\theta) \in U(1).$
Fix a complex-valued linear form $\nu_j \in \text{Hom}_\mathbb{R}(a_j, \mathbb{C})$ which is identified with the complex number $\nu_j(H_1)$, and the associated quasi-character

$$e^{\nu_j + \rho_j} : a \in A_j \mapsto \exp\{(\nu + \rho_j)(\log a)\} \in \mathbb{C}^*,$$

and form the outer tensor product $(\varepsilon \otimes D_k^\pm) \boxtimes e^{\nu + \rho}$ to get a representation of the product $M_J A_J$. Via the canonical isomorphism $P_J/N_J \cong M_J A_J$, we can regard this as a representation of $P_J$. On the Hilbert space

$$\text{Ind}_J^G((\varepsilon \otimes D_k^\pm) \boxtimes e^{\nu + \rho}) := \left\{ f : G \to H D_k^\pm \mid \begin{array}{c}
 f(n_{\text{max}}) = (\varepsilon \otimes D_k^{pm})(m)e^{\nu + \rho}(a)f(x),
 \text{ for a.e.n} \in N_J, m \in M_J, a \in A_J, x \in G
 \end{array} \right\}$$

The action of $G$ is by the right translation. We denote the obtained Hilbert representation by $\pi_{\varepsilon \otimes D_k^\pm, \nu}$. Since the cases $D_k^+$ and $D_k^-$ are mutually *-conjugate, we consider only the case of $D_k^+$.

When $\varepsilon(-1) \cdot (-1)^k = 1$, we call this representation even $P_J$ principal series representation, and otherwise odd.

### 4.3.2 The $K$-types of the $P_J$ principal series

We consider an even $P_J$ principal series representation $\pi_{\pm} = \pi_{\varepsilon \otimes D_k^\pm, \nu}$. In order to describe the $K$-types of even $\pi_{\pm}$'s, we introduce some notation first.

**Notation** For a dominant integral weight $(l_1, l_2) \in L_k^+$ with parity condition $(-1)^{l_1 + l_2} = +1$, we put

$$(\hat{l}_1, \hat{l}_2) = \begin{cases}
 (l_1, l_2), & \text{if } (-1)^{l_i} = \varepsilon \ (i = 1, 2); \\
 (l_1 - 1, l_2 + 1), & \text{if } (-1)^{l_i} = -\varepsilon \ (i = 1, 2).
\end{cases}$$

**Proposition 4.1** If $(-1)^{l_1 + l_2} = +1$, the multiplicity $[\pi_{\pm} : \tau_{(l_1, l_2)}]$ of $\tau_{(l_1, l_2)} \in \hat{K}$ in $\pi_{\pm}$ (resp. $\pi_{-}$) is given by

$$[\pi_+ : \tau_{(l_1, l_2)}] = \frac{1}{2} (\sup\{k, \hat{l}_1\} - \sup\{k, \hat{l}_2\}) + 1$$

and

$$[\pi_- : \tau_{(l_1, l_2)}] = \frac{1}{2} (\inf\{-k, \hat{l}_1\} - \inf\{-k, \hat{l}_2\}) + 1.$$

If $(-1)^{l_1 + l_2} = -1$, then $[\pi_{\pm} : \tau_{(l_1, l_2)}] = 0$.

### 4.3.3 The annihilator of the corner $K$-type of even $P_J$ principal series in $U(\mathfrak{g})$

We have

$$[\pi_+ : \tau_{(l_1, l_2)}] = 1 \quad \text{and} \quad [\pi_- : \tau_{(l_1, l_2)}] = 1,$$

respectively. Each $K$-type $\tau_{(\pm k, \pm k)}$ in $\pi_{\pm}$ is called the corner $K$-type, respectively. Here imagine the "picture" of the constituents of the $K$-types of each $\pi_{\pm}$.

Let $v_+$ or $v_-$ be the non-zero vector, unique up to scalar multiple, which is in the image of the non-zero $K$-homomorphism $\tau_{(k, k)} \mapsto \pi_+$ or $\tau_{(-k, -k)} \mapsto \pi_-$. 

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respectively. Then we have some element of degree 2 in the universal enveloping algebra $S(p_-) = U(p_-)$ (resp. $S(p_+) = U(p_+)$), which annihilates the image of this homomorphism. The holomorphic part $p_+$ (or the anti-holomorphic part $p_-$) of $p_C$ with respect to the complex structure $Ad(\iota)$ given by an element $\iota$ of order 8 in the center of $K$, has a basis $\{X_{+11}, X_{12}, X_{22}\}$ (or $\{X_{-11}, X_{-12}, X_{-22}\}$, resp.) specified as follows, by using the matrix units $E_{ij} \in M_4(C)$:

\[
\begin{align*}
X_{\pm 11} &:= E_{11} - E_{33} \pm \sqrt{-1}(E_{13} + E_{31}), \\
X_{\pm 22} &:= E_{22} - E_{44} \pm \sqrt{-1}(E_{24} + E_{42}), \\
X_{\pm 12} &:= \frac{1}{2}(E_{12} + E_{21} - E_{34} - E_{43}) \pm \frac{1}{2}\sqrt{-1}(E_{14} + E_{23} + E_{32} + E_{41}).
\end{align*}
\]

**Proposition 4.2** We have

\[(X_{-11}X_{-22} - X_{-12}^2)v(k,k) = 0, \text{ or } (X_{+11}X_{+22} - X_{+12}^2)v(-k,-k) = 0,\]

respectively. Moreover since $\pi_\pm$ are quasi-simple, they have the infinitesimal characters $\chi_{\pi_\pm} : Z(g) \to C$, and in particular we have equations:

\[\{C - \chi_{\pi_\pm}(C)\}v(\pm k, \pm k) = 0\]

for the Casimir operator $C$.

**Remark 1** The second statement is a standard fact. The first statement in the above proposition is not formulated in [31] so explicitly, but this is the core reason to have one of the main differential equations in [31]. The interesting point is the radial part of this operator gives the Euler-Darboux operator with respect to natural coordinates in various realizations or models of our $P_J$ principal series.

**Remark 2** Logically speaking we do not need it, but it would be helpful to understand $P_J$ principal series to know the fact:

**Proposition 4.3** The large discrete series is embeddable in a $P_J$ principal series.

The details and application of the above proposition will be discussed in a forthcoming paper.
5 The known facts on Fourier expansions of automorphic forms on $Sp(2, \mathbb{Z}) \backslash Sp(2, \mathbb{R})$

We discuss some of the known results on the Fourier expansion on $\Gamma \backslash Sp(2, \mathbb{R})$ up to now, where $\Gamma$ is $Sp(2, \mathbb{Z})$ or one of its congruence subgroups.

5.1 The Fourier expansion with respect to $P_0$

The Fourier expansion with respect to the minimal parabolic subgroup $P_0$ is most difficult to handle. We do not have satisfactory results for general automorphic forms $f$ and for general infinite-dimensional representations $\eta$ of the unipotent radical $N_0$. Usually only the case of linear characters $\eta$ are discussed.

The case of holomorphic discrete series

In this case, there are no terms terms corresponding to the characters $\eta$ of $N$. This fact is expressed as the vanishing of the Whittaker models in the representation-theoretic formulation of the problem. Heuristically speaking, the smallness of the holomorphic discrete series representations (i.e., their Gelfand-Kirillov dimension is 3) strikes out the “enough room” for the existence of the associated Whittaker functions.

In this case we have to consider the terms for infinite-dimensional $\eta \in \hat{N}_0$. We have some partial result. As we have alreay seen in the subsection 2.3, the $(+,+)$-component of $H^2(\Gamma \backslash G/K, E_i)$ is realized by the space of holomorphic automorphic cuspidal forms belonging to the weight $(l_1 + 3, l_2 + 3)$, i.e., vector-valued automorphic cuspidal forms belonging to the automorphy factor $\det^{l_2+3} \otimes \text{sym}^{l_1-l_2}(CZ + D)$ for $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$. For holomorphic modular forms, the Fourier expansion along the minimal parabolic subgroup $P_0$ was discussed by Narita [38]. Similarly the $(-,-)$-component corresponds to the anti-holomorphic automorphic forms.

The case of the large discrete series representations

For the $(+,-)$ or $(-,+)$-component, we have to do something new. The elements in this component generate large discrete series representations. The first problem is to find the Whittaker functions belonging to the large discrete series. This was done by Oda [41] for the large discrete series and by Miyazaki-Oda [31] for $P_J$-principal series.

We see the explicit formula of the $A$-radial part $h_W$ of the spherical function $\varphi = I(v_{(k,k)})$ with the corner $K$-type in the Whittaker model $\text{Im}(I)$ of the $P_J$ principal series $\pi_\pm$ for a Whittaker functional $I \in \text{Hom}_{(\mathfrak{g}, K)}(\pi_\pm, \text{Ind}_{N_0}^G(\eta))$.

5.1.1 The holonomic system of $A$-radial part

Proposition 5.1 ([31]) Let $h_W(a_1, a_2) := h(\text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}))$ be the $A$-radial part of the Whittaker function $\varphi(g)$ with the corner $K$-type, belonging to the $P_J$-principal series $\pi_{\nu_J}$. Then this satisfies a system of equations:

$[\partial_1 \partial_2 + (4\pi c_0 a_2^2 - k) \partial_1 - (k+1) \partial_2 - 4\pi (k+1)c_0 a_2 + k(k+1) + 4\pi^2 c_0^2 (\frac{a_1}{a_2})^2] h_W(a) = 0$

$[\partial_1^2 + \partial_2^2 - 4\partial_1 - 2\partial_2 - 8\pi^2 c_0^2 (\frac{a_1}{a_2})^2 - 16\pi^2 c_0^2 a_2^4 + 8\pi k c_0 a_2^2] h_W(a) = 0.$

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Here $\partial_i = a_i \frac{\partial}{\partial a_i}$ (i = 1, 2).

5.1.2 Integral expression

**Theorem 5.1** (i) Let $\varphi := I(v_{(-k,-k)})|_A$ be a Whittaker function if $\pi_{P_S \mathbb{Z}_k}$ with $K$-type $(-k,-k)$. Then it has an integral expression

$$
\varphi(a_1, a_2) = a_k^{k+1}a_k^k \exp(-\sqrt{-1} \eta_2 a_k^2) \\
\times \int_0^\infty t^{-k+\frac{1}{2}} W_{0,v}(t) \exp(-\frac{t^2}{4\sqrt{-1} \eta_2 a_k^2} + \frac{8\sqrt{-1} \eta_2 a_k^2 a_k^2}{t^2}) \frac{dt}{t},
$$

(ii) (Moriyama) Up to constant multiple

$$
h_W(a_1, a_2) = (2\pi)^{-2} \int_{\text{Re}(s)=\sigma_1} \int_{\text{Re}(s)=\sigma_2} M_{\nu,j,k} a_1^{-s_1} a_2^{-s_2} ds_1 ds_2
$$

with

$$
M_{\nu,j,k}(s_1, s_2) = \Gamma(\frac{s_1 + s_2 + \nu_j - k + 1}{2}) \Gamma(\frac{s_1 + s_2 - \nu_j - k + 1}{2}) \Gamma(\frac{s_1}{2}) \Gamma(-\frac{s_2}{2})
$$

**Remark.** And this result is utilized by Moriyama [35] to show analytic continuation and the functional equation for spinor $L$-functions of automorphic forms belonging to the large (or generic at $\mathbb{R}$ if you prefer this terminology) discrete series representations.

The Whittaker functions are defined only for (linear) characters of $N_0$. The group $N_0$ has infinite-dimensional representations in general. We know little about the generalized Whittaker functions with respect to $\eta$ of $\dim \eta = \infty$.

**Open Problem** Investigate the generalized Whittaker models or Whittaker functions for infinite-dimensional $\eta \in \hat{N}_0$.

5.2 The Fourier expansion with respect to the Siegel parabolic subgroup $P_S$

5.2.1 The naive formulation of the problem

Let

$$
P_S := \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in G|A, B, D \in M_2(\mathbb{Q}) \right\}
$$

be the rational standard maximal subgroup associated with the short root, which is called Siegel parabolic subgroup.

An element of this group is written as

$$
\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} 1_2 & B' \\ 0 & 1_2 \end{pmatrix}
$$

$(A, D \in GL(2, \mathbb{R}), A^{t}A = 1_2, B' = A^{t}B' \in M_2(\mathbb{R}))$ and any element of the unipotent radical $N_S$ is given as

$$
\begin{pmatrix} 1_2 & B' \\ 0 & 1_2 \end{pmatrix} \quad (B = A^{t}B \in M_2(\mathbb{R})).
$$
In this case $N_S$ is abelian, hence all the irreducible unitary representations
of $N_S$ is abelian, i.e., characters. Therefore the function $f(n_S x)$ in $n_S \in N_S$
has the associated Fourier expansion of the form:

$$f(n_S x) = \sum_{\eta \in \text{N}/(\text{T} \cap \text{N})} \eta(n_S) SW_{\eta}^f(x)$$

The unitary characters of $N_S$ trivial on $N_{S,T} = \Gamma \cap N_S$ can be expressed as

$$\eta_{\Gamma}(n_S) = \exp\{2\pi i \text{tr}(T \cdot B(n_S))\} \quad \text{for } n_S = \begin{pmatrix} 1 & B(n_S) \\ 0 & 1 \end{pmatrix}$$

for some symmetric matrix $T = \begin{bmatrix} T \end{bmatrix} \in M_2(\mathbb{Q})$ with some integrality condition
for $N_{S,T}$. Therefore the naive form of the Fourier expansion of an automorphic
form $f$ on $\Gamma \backslash G$ along $P_S$ is given by

$$f(n_S g) = \sum_{T = \begin{bmatrix} T \end{bmatrix} \in (N_S / N_{S,T})} \exp\{2\pi i \text{tr}(T \cdot B(n_S))\} SW_{\eta_T}^f(g),$$

where

$$SW_{\eta_T}^f(g) = \int_{N_S \cdot R / N_{S,T}} \exp\{-2\pi i \text{tr}(T \cdot B(n_S))\} f(n_S g) d(n_S),$$

with $d(n_S)$ is the normalized Haar measure on $N_S / N_{S,T}$. Since $G = P_S \cdot K$, if we
assign the special $K$-type to $f$ from the right side, to know $f$ it suffices to know the restriction
$f|P_S$. Meanwhile we have the Levi decomposition $P_S = L_S \cdot N_S$. Hence to know

$$SW_{\eta_T}^f(p_S) = SW_{\eta_T}^f(n_S \cdot l_S) = \exp\{2\pi i \text{tr}(T \cdot B(n_S))b\} SW_{\eta_T}^f(l_S),$$

it suffice to know its restriction to $L_S$. But the homogeneous space $L_S / (L_S \cap K)$
still has real dimension 3, and the differential equations coming from the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ are not enough many to give a holonomic system on this
homogenous space in general.

Note that the case of holomorphic automorphic forms $f$ which are classical
Siegel modular forms is exceptional, because in this case the Cauchy-Riemann
condition on many variables make the representation spaces of $\pi_{(+,+)}$, $\pi_{(-,-)}$ quite small (i.e., the Gelfand-Kirillov dimension of the automorphic representations
generated by $f$ is 3).

In general, to have tractable form of Fourier expansion, we need further expansion
of $SW_{\eta_T}^f(l_S)$.

Assume that $T \neq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and put

$$St(\eta_T)^0 = SO(T) = \{l_S \in GL(2, \mathbb{R}) = L_S(\mathbb{R}) \mid l_S T l_S = T\}^0.$$ 

Then we have $L_S = \sum_{b \in SO(T)} bAb^{-1}$ for

$$A = \{\text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \mid a_i \in \mathbb{R}_{>0} \ (i = 1, 2)\}.$$

If $T$ is definite, then $SO(T)$ is isomorphic to $SO(2)$, a compact abelian group,
whose characters are given by

$$\chi_m : r_\theta \in SO(2) \mapsto \exp(i m \theta) \in U(1).$$
Here \( r_\theta \in SO(2) \) is the rotation of the angle \( \theta \) on the Euclid plane \( \mathbb{R}^2 \). Then we can expand \( SW_T^f(l_S) \) further as
\[
SW_T^f(\sigma l_S \sigma^{-1}) = \sum_{m \in \mathbb{Z}} \chi_m(\sigma) SW_{T,m}^f(l_S) \quad (\sigma \in SO(T)).
\]

Then finally we have to ask whether we have enough data to specify \( SW_{T,m}^f | A \) for each pair \( \{T, m\} \).

### 5.2.2 The Siegel-Whittaker functions

Apart from the naive formulation of the problem, we discuss the actual formulation of the problem to specify the functions \( SW_{T,m}^f | A \) in the context of representation theory.

For \( T = \tau T \in M_2(\mathbb{R}) \), we can associate the subgroup
\[
SO(T) = St(\eta_T)^0
\]
similarly as in the previous subsection. Assume that \( T \) is definite from now on in this subsection. Then the group \( SO(T) \) is compact and isomorphic to \( SO(2) \).

We can define the unitary characters \( \chi_m \) for each \( m \in \mathbb{Z} \) as above. Then for the semidirect product
\[
R_\eta = N_S \cdot SO(T),
\]
we can associate the represenation \( \eta \boxtimes \chi_m \), the twisted outer tensor product.

Then the local problem at the real place is to investigate the intertwining space
\[
\text{Hom}(\mathfrak{g},K)(\pi, \text{Ind}_{R_\eta}^G(\eta \boxtimes \chi_m))
\]
for the standard representations or all the irreducible representations \( \pi \) of the pair \( (\mathfrak{g}, K) \). The following is known until now.

**Theorem 5.2** (Niwa, Miyazaki, Ishii) Let \( \pi \) be a representation of the discrete series, or principal series, or \( P_J \)-principal serie (i.e., only the generalized principal series obtained by parabolic induction with respect to \( P_S \) is not handled). Then, the dimension of the intertwining space
\[
\text{Hom}(\mathfrak{g},K)(\pi, \text{Ind}_{R_\eta}^G(\eta \boxtimes \chi_m))
\]
is at most the half of the Bernstein degree of the representation in question (i.e., 2, 4, 2, respectively in this case).

Moreover if we replace the target space \( \text{Ind}_{R_\eta}^G(\eta \boxtimes \chi_m) \) of the intertwining operator by the subspace \( \text{Ind}_{R_\eta}^G(\eta \boxtimes \chi_m)^{\text{rapid}} \) consisting of vectors which decrease rapidly on \( A \) at infinity, then the dimension of the intertwining operator:
\[
\text{Hom}(\mathfrak{g},K)(\pi, \text{Ind}_{R_\eta}^G(\eta \boxtimes \chi_m)^{\text{rapid}})
\]
is at most one (the multiplicity-free theorem); Furthermore, we have explicit integral expression of the \( A \)-radial part of the image of some vector in \( \pi \) with some specified \( K \)-types by this unique intertwining operator.
5.2.3 *Explicit formulae*

In this subsubsection, we show some explicit integral formulae for our ‘Siegel-Whittaker functions. Let $I$ be a non-zero intertwining operator in the space $\text{Hom}_{\mathfrak{g},k} (\pi, \text{Ind}_{K_0}^G (\eta \boxtimes \chi_m))$ and let $\phi_{SW}$ be the $A$-radial part of the image $I(\tau(k,h))$ of the non-zero vector of $\tau(k,h)$-type in the $P_J$ principal series $\pi_+$. Here the elements of $A$ are denoted by $\text{diag}(a_1, a_2, a_1^{-1}, a_2^{-1}) \in A$. First we want to have the holonomic system for $\phi_{SW}(a_1, a_2)$.

**The holonomic system of the $A$-radial part**

**Proposition 5.2** ([29]) Let $T = \text{diag}(h, h)$ ($h > 0$). Set

$$\phi_{SW}(a_1, a_2) = (a_1 a_2)^{k+1} \exp(-2\pi h(a_1^2 + a_2^2)) h_{SW}(a_1, a_2).$$

Then $h_{SW}$ satisfies the holonomic system of rank $4$:

$$[\partial_1 \partial_2 - \frac{a_1^2}{a_1-a_2} \partial_1 + \frac{a_1^2}{a_1-a_2} \partial_2 + m^2 \frac{a_1^2 a_2^2}{(a_1-a_2)^2}] h_{SW}(a) = 0$$

$$[\partial_1 + \partial_2 + k - \nu](\partial_1 + \partial_2 + k - \nu) - 8\pi h a_1^2 (\partial_1 + 1) - 8\pi h a_2^2 (\partial_2 + 1) h_{SW}(a) = 0.$$

with $\partial_i = a_i \frac{\partial}{\partial a_i}$ ($i = 1, 2$).

Here note that the first equation comes from the annihilator $X_{-11} X_{-22} - X_{22}^2$ and the second from the eigen-equation with respect to the Casimir operator.

**Integral expressions**

We have to solve the above holonomic system. There are three different integral expressions. The first two are Eulerian, the last is of Barnes’ type. We show only the first two here.

**Theorem 5.3** (Theorem (7.5) of Miyazaki [29]) Here $T = \text{diag}(h, h)$ with $h > 0$. Then the $A$-radial part of the Siegel-Whittaker function is of the form

$$b'(a_1, a_2) = \text{const.} e^{-m \pi \sqrt{1/2} \frac{k+1}{(2m+1)} \int_0^\infty t^m J_{m} (2\pi \sqrt{1} (h(a_1^2 - a_2^2))) \times 2F_1 (\frac{2-k+2m+1}{2}, \frac{2-k+2m+1}{2}, 2m+1; -t) e^{-2\pi h(a_1^2 + a_2^2)(t+1)} dt$$

Here $J_m$ is the Bessel function, $2F_1$ the Gaussian hypergeometric function. And $k, \nu$ are the parameter of the representation of the even $P_J$-principal series.

Here is another integral expression of the same function $b'(a_1, a_2)$. Basically this is obtained in the same way as Gon [4].

**Theorem 5.4** Here $T = \text{diag}(h_1, h_2)$ with $h_1 = h_2 = h > 0$. Then the $A$-radial part of the Siegel-Whittaker function is of the form

$$b'(a_1, a_2) = \text{const.} (a_1 a_2)^{k+1} e^{-2\pi h(a_1^2 + a_2^2)} c(a_1, a_2)$$

where

$$c(a_1, a_2) = \frac{|a_1^2 - a_2^2|^{m/2}}{\Gamma(1-t)} 1^{m/2-1} (1-t)^{|m|/2-1} G(ta_1^2 + (1-t)a_2^2) dt$$

with

$$G(t) = \Gamma \left( \frac{k+|m|+1}{2} \right) e^t \cdot W_{k-|m|-1,1/2} (2t).$$

Here $W_{*,*}$ is the $W$-Whittaker function under the notation of Whittaker-Watson, which decreases very fast at infinity.
Open Problem When $T$ is indefinite, $\text{St}(\eta T)^0 = SO(1,1) \cong \mathbb{R}_{>0}$ is non-compact. For this we are ignorant about the multiplicity-free property of Siegel-Whittaker models (or functions).

5.2.4 Application to automorphic $L$-functions

The explicit integral formula for Siegel-Whittaker functions has application to have analytic continuation and the functional equation for the spinor $L$-function of automorphic forms, by the method of Andrianov. The original papers of Andrianov treated the cases of holomorphic Siegel modular forms. The result was later extended to the wave forms on the Siegel upper half space of degree 2 by Hori [11] utilizing the explicit integral formula of the class one Whittaker functions by Niwa [39]. Miyazaki further extended this method to modular forms belonging to the large discrete series [29].

5.3 Fourier expansion along $P_J$

In this case, we consider a spherical subgroup $N_J \subset \mathbb{R} \subset P_J$ which is now called Jacobi subgroup. We do not yet have global expansion except for the case of holomorphic Siegel modular forms. The holomorphic case, we have the so-called Fourier-Jacobi expansion.

For the local problem of spherical functions, Hirano ([9], [10]) obtained the explicit formulae of spherical functions and the multiplicity-free theorem for the cases of the $P_J$ principal series and the discrete series representations.

Open Problem 1 Find a reasonable Fourier expansion utilizing the above mentioned result of Hirano. Among others, consider how to settle the problem of "continuous" spectrum adequately.

Open Problem 2 There is a yet-another way to have the spinor $L$-function from automorphic forms on $\Gamma \backslash Sp(2, \mathbb{R})$ by Kohnen-Skoruppa [24]. Obviously Hirano’s result should have application if one try to extend the method of [24] for non-holomorphic automorphic forms.

6 Spherical functions with respect to reductive subgroups

There is one productive heuristic idea in the theory of automorphic forms, that is the analogy between the objects involved in parabolic subgroups and those in "hyperbolic" reductive subgroups. The basic credo here is what was done for parabolic subgroups should be also be done analogously for reductive subgroups.

For example as there are Eisenstein series associated with parabolic subgroups, so there are Poincaré series associated with reductive subgroups. One successful construction by this idea is our result on Green function ([43]). Another "example", which should be cultivated, is the hyperbolic Fourier expansion with respect to reductive spherical subgroups. The classical work by C. L. Siegel on the Fourier expansion along the tori $\mathbb{G}_m^2 \hookrightarrow \text{GL}(2)$ is a special case. We do not yet know even the simplest extension of this result to $\mathbb{G}_m^n \hookrightarrow \text{GL}(n)$.
6.1 Results on spherical functions with respect to the reductive spherical subgroups

There are already some papers on the spherical functions with respect to reductive spherical subgroups of a semisimple algebraic groups.

In their theory of construction of automorphic $L$-functions for automorphic forms on algebraic groups of type A and type BD, Murase-Sugano, [36] considered such problem. For real groups of type A, Tsuzuki [49] has worked out rather completely for the case of the pair $(G, R) = (U(n, 1), U(1) \times U(n - 1, 1))$: in this important paper, he obtained the multiplicity-free theorem and explicit formulae of the radial part of the spherical function ([49], I), the local zeta integral ([49], II), and the $c$-function of the Poisson transformation to have the criterion for non-vanishing of spherical functions ([49], III).

For our group $G = Sp(2, R)$ and $R = SL(2, R) \times SL(2, R)$ or $R = SL(2, C)$, Moriyama [32, 33] obtained explicit formulae of spherical functions by using Meijer’s $G$-functions. Hayata has some results for $(G, R) = (SU(2, 2), Sp(2, R))$ (cf. [6]).

6.2 Construction of Green functions for modular divisors on arithmetic quotients of bounded symmetric domains

This subsection is a short review of the joint work with Masao Tsuzuki [43].

6.2.1 Logarithmic Green functions

Given a (smooth, for simplicity) subvariety $Y$ of codimension $d$ in a smooth quasi-projective complex algebraic variety $X$ of dimension $n$, a current $\delta_Y$, i.e. a differential forms with coefficients in distributions, of type $(d, d)$ on $X$ is defined by associating the values of the integral

$$\int_Y i^*(\omega)$$

for every $(n - d, n - d)$-type $C^\infty$ form $\omega$ on $X$. Here $i : Y \subset X$ is the inclusion map. This is a closed form.

Now, if there is a current $g$ of type $(d - 1, d - 1)$ on $X$ such that

$$dd^c g + \delta_Y$$

is a smooth differential form of type $(d, d)$ on $X$, then $g$ is called a Green current for $Y$. Here $d^c = (\partial - \bar{\partial})/2\pi\sqrt{-1}$ if we write $d = \partial + \bar{\partial}$ as a sum of holomorphic part and antiholomorphic part. The form $g$ is not unique.

In the intersection theory of Arakelov type for higher dimensional cases, Gillet-Soulé [48] defined Green current of logarithmic type. When the codimension $d > 1$, this is defined by using the resolution of singularities of Hironaka, but when $d = 1$, i.e. when $Y$ is a divisor, it is done more directly.

If one defines a Hermitian metric $\| \|$ on the holomorphic line bundle $L = O_X(Y)$ by putting $\| f \| = e^{-g}|f|$ for local section $f$ of $L$, then the Chern form of $L$ with respect to this metric is given by $dd^c g + \delta_Y$. Conversely, if $(L, \| \|)$ a holomorphic line bundle with Hermitian metric, then its Green current is known by the following theorem of Poincaré-Lelong.
Proposition 6.1 For a meromorphic section $s$ of $L$, $-\log \|s\|^2$ is a locally integrable function $X$ and gives a logarithmic Green function for for $Y = \text{div}(s)$. Moreover the right hand side of

$$dd^c(-\log \|s\|^2) + \delta_Y = c_1(L, \| \|)$$

is the Chern form for the metric line bundle $(L, \| \|)$.

Existence of logarithmic Green currents are guaranteed in general, but little is known about their concrete construction.

From now on, when $X$ and $Y$ are arithmetic quotients and $Y$ is a divisor of $X$, we show one way to construct the logarithmic Green current of $Y$.

6.2.2 The secondary spherical functions for affine symmetric pairs $(G, H)$ of rank 1

For our construction, we need a pair $(G, H)$ of real reductive Lie groups satisfying the following "axioms":

(i) $G$, a connected real semisimple (algebraic) Lie group such that the quotient $G/K$ by a maximal compact subgroup $K$ is Hermitian symmetric ($K = G^\theta$ with $\theta$ a Cartan involution);

(ii) $H$, a reductive subgroup of $G$, such that there exists an involution $\sigma : G \rightarrow G$ satisfying $\theta \sigma = \sigma \theta$ and $(G^\sigma)_0 \subset H \subset G^\sigma$. Moreover $H \setminus G$ is a semisimple symmetric space of real rank 1.

Then for the Lie algebras $g = \text{Lie}(G)$, $h = \text{Lie}(H)$ we have

$$g = g^\sigma \cap g^\theta \oplus g^\sigma \cap g^{-\theta} \oplus g^\theta \cap g^{-\sigma} \oplus g^{-\sigma} \cap g^{-\theta}$$

and a maximal abelian subalgebra $a$ in the last factor $g^{-\sigma} \cap g^{-\theta}$ is of dimension 1.

By the classification table of symmetric spaces, we have two cases:

(U-type): $g = su(n,1), h = su(1) \times u(n-1,1))$

(O-type): $g = so(n,2), h = so(n-1,2)$.

We choose a generator $Y_0 \in a$ such that $\lambda(Y_0) = 1$ for the short root $\lambda$ in the root system $\Psi = \Psi(a, g)$. Set $U = \exp(g_{a} + g_{2a})$ and $2\rho_0 = \text{tr}(\text{ad}(Y_0)_{|\text{Lie}(U)})$. Also we normalize the Casimir operator $\Omega$ such that it corresponds to the bilinear form $X, Y = 1_{B(Y_0, Y_0)}B(X, Y)$ ($X, Y \in g$).

Set $A = \exp(a) = \{a_t = \exp(tY_0)| t \in \mathbb{R}\}$, then $G = HAK$.

Now we can consider a left $H$-invariant and right $K$-invariant spherical function $\phi_s^{(1)}(g) \in C^\infty(H \setminus G/K)$ satisfying

$$\phi_s^{(1)}(g) * \Omega = (s^2 - \rho_0^2)\phi_s^{(1)}(g) \quad (s \in \mathbb{C}).$$

This function generates class one principal series representation of $G$ in $C^\infty(H \setminus G)$ by right translation under $G$, which has $H$-invariant. This ordinary spherical function is not necessarily good one to define Poincaré series. In place of this, we consider the secondary spherical functions $\phi_s^{(2)}(g) \in C^\infty(G - H \cdot K)$ which has logarithmic singularities along $H \cdot K$. 31
Proposition 6.2 Let \( s \in \mathbb{C}, \Re(s) > \rho_0 \). Then there exists the unique function satisfying the conditions (a)-(d) below:
(a) \( \phi_s^{(2)} \) is \( C^\infty \) on \( G - H \cdot K \), and left \( H \)-invariant and right \( K \)-invariant;
(b) \( \phi_s^{(2)} \) satisfies the differential equation:
\[
\phi_s^{(2)} * \Omega = (s^2 - \rho_0^2) \phi_s^{(2)} \text{ on } G - HK;
\]
(c) For sufficiently small \( \delta > 0 \),
\[
\phi_s^{(2)}(a_t) - \log(t) \text{ is bounded in } (0, \delta);
\]
(d) \( \phi_s^{(2)}(a_t) \) is rapidly decreasing for \( t \to +\infty \).

For our later purpose, it is better to introduce a vector-valued spherical function \( \psi_s \). Recall the Cartan decomposition and its complexification:
\[
g = k \oplus p, \quad g_C = k_C \oplus p_C \oplus p_C.
\]
Here the subspace \( p_C \) is the \( \pm i \)-eigenspaces with respect to the given complex structure in the complexification \( p_C \).

For a function \( F \in C^\infty(G/K) \) we can define the gradient \( \nabla F = \nabla_+ F + \nabla_- F \in C^\infty(G) \otimes p_C \). Then we have \( \nabla_- \nabla_+ F : G \to p_+ \otimes p_- \), which is \( K \)-equivariant under the right \( K \)-action on \( G \) and the tensor product of the adjoint representation \( \text{Ad}_{p_+} \). The \( K \)-module \( p_+ \otimes p_- \) is a direct sum of the trivial representation and the other irreducible representation \( V_{11} \). We denote by \( p_r \) the projection to \( V_{11} \) from \( p_+ \otimes p_- \).

Definition 6.1 \( \phi_s = p_r \cdot \nabla_- \nabla_+ \phi_s^{(2)} \).

6.2.3 Poincaré series
We can now define Poincaré series.

Definition 6.2 For \( s \in \mathbb{C} \) with \( \Re(s) > \rho_0 \), set
\[
G_s(z) = \sum_{\gamma \in (\Gamma \cap H) \setminus \Gamma} \phi_s^{(2)}(l, \gamma(z)) \quad (g \in G)
\]
and
\[
\Psi_s(z) = \sum_{\gamma \in (\Gamma \cap H) \setminus \Gamma} \psi_s^{(2)}(l, \gamma(z)) \quad (g \in G).
\]
Both converge as currents for \( \Re(s) > \rho_0 \) and real analytic except for on the image of
\[
(\Gamma \cap H) \setminus H/(K \cap H) \to \Gamma \setminus G/K.
\]
We can show
(i) a criterion for \( G_s \in L^p(\Gamma \setminus G) \) in particular \( G_s \in L^2(\Gamma \setminus G) \),
(ii) the analytic continuation, when \( G_s \in L^2(\Gamma \setminus G) \), (this is a kind of functional equation for \( G_s \) and \( G_{-s} \).)
(iii) \( G_s(g) \) has a simple pole at \( s = \rho_0 \) with residue
\[
\frac{\text{vol}((H \cap \Gamma) \setminus H)}{\text{vol}(\Gamma \setminus G)} \cdot \frac{1}{2\rho_0}.
\]
6.2.4 $\partial\bar{\partial}$-formula

Let $d = \partial + \bar{\partial}$ be the decomposition of the exterior derivative on $G/K$ into the holomorphic part and the antiholomorphic part.

**Theorem 6.1** We have

\[
\begin{align*}
(a) & & \partial\bar{\partial}G_s + \pi\delta_{D_0} &= -\frac{\sqrt{-1}}{2n}((c(g)s)^2 - \rho_0^2)G_s \wedge \omega_{\Gamma\backslash G/K} + 4\Psi_s \\
(b) & & \Delta\Psi_s &= -((c(g)s)^2 - \rho_0^2)\left(\Psi_s - \frac{\pi\sqrt{-1}}{4}\delta_{D_0} + \frac{\pi\sqrt{-1}}{4n}\delta_{D_0} \wedge \omega_{\Gamma\backslash G/K}\right)
\end{align*}
\]

Here $\Psi_s$ is the Poincaré series of $(1,1)$-form, $\tilde{\delta}_{D_0}$ the current associated with the divisor \((H \cap \Gamma)\backslash H/(H \cap K) \to \Gamma\backslash G/K\), $c(g)$ a constant given by $c(g) = 1$ (U-type), $= 2$ (O-type), and $\omega_{\Gamma\backslash G/K}$ the Kaehler form on $\Gamma\backslash G/K$.

**Theorem 6.2**

(i) The $(1,1)$-type current $\Psi_s$ is holomorphic at $s = \rho_0$, and $\Psi_{\rho_0}$ is harmonic $C^\infty$-form of $(1,1)$-type on $\Gamma\backslash G/K$.

(ii) Put $\mathcal{G} = \lim_{s \to \rho_0}(G_s - \frac{1}{s - \rho_0}\text{Res}_{s - \rho_0}G_s)$. Then

\[
\sqrt{-1}\partial\bar{\partial}\mathcal{G} - \pi\delta_{D_0} = \kappa\omega_{\Gamma\backslash G/K} + 4\sqrt{-1}\Psi_{\rho_0}.
\]

(cf. [TsO01, Chapter 7, Theorem (7.6.1)] and its application).

Here is the meaning of the last formula. There is the subspace of square-integrable forms $H^2_2(\Gamma\backslash G/K, \mathbb{C})$ in $H^2(\Gamma, \mathbb{C}) = H^2(\Gamma\backslash G/K, \mathbb{C})$. There are two irreducible unitary representation of $G$ which contribute to the $(1,1)$-type Hodge component of $H^2_2(\Gamma\backslash G/K, \mathbb{C})$: one is the trivial representation $\mathbb{C}$ and the other is a certain infinite dimensional representations, which we denote by $\pi_1$. Both representations $\pi$ have $H^{1,1}(\mathfrak{g}, K; H_{1,1}) \cong \mathbb{C}$. Hence

\[
H^{1,1} = \text{Hom}_G(1, L^2(\Gamma\backslash G)) \oplus \text{Hom}_G(\pi_1, L^2(\Gamma\backslash G)).
\]

The first factor of the right hand side is generated by the Kaehler form $\kappa$. The right hand side of the statement (ii) of the last theorem is compatible with this decomposition.

**Postscript**

We have no ability to see the future of our project. But let me comment about some new directions in progress. For the spherical functions on $G = Sp(2, \mathbb{R})$, we are interested in the relation between different type spherical functions. i.e., they are different realization of the same irreducible unitary representation $G$. Therefore there should be an isomorphism of $(\mathfrak{g}, K)$-modules, and it is an interesting problem to find explicit realization of this intertwining isomorphism. About this direction, the author, Miki Hirano and Taku Ishii are obtaining some formulae of “confluence from Siegel-Whittaker functions to Whittaker functions”.

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This is, heuristically speaking, the application of the idea of "contraction or deformation of spherical subgroups". This might be the first step toward the seamless formulae to encompass the various spherical functions.

Another direction is the joint work with Hirano to have explicit formula for the radial part of Whittaker functions of the $P_f$ principal series representation of $Sp(3, \mathbb{R})$, whose Bernstein degree is the half of the order of the Weyl group, i.e., $24 = \frac{1}{2}(3!2^3)$. We have already some explicit formulae for the secondary Whittaker functions, which are 24 independent power series solutions of the holonomic system consisting of the radial parts of the annihilators in $U(\mathfrak{g})$ of the corner $K$-type of $P_f$ series.

In the above two attempts, the secondary spherical functions, which is the analogue of Harish-Chandra’s hypergeometric series in his classical theory of spherical functions on semisimple Lie groups, play a crucial role.

Yet another direction is to have Green currents of higher-codimensional modular algebraic cycles. Here we have to consider slightly more general spherical functions proposed by Masao Tsuzuki. But here also the "secondary" functions, which have natural singularities inside $G$, is the seed of the whole construction.

At somewhere Jean-Pierre Serre said that Representation Theory and Algebraic Geometry are the most difficult fields in mathematics. In the arithmetic theory of automorphic forms the One from which everything comes out is the discrete subgroup $\Gamma$, which is the unit group of a non-commutative arithmetic algebra. The emanated objects are written in these two sophisticated languages. But if one wants to have arithmetic results, the direct application of the known results in these two fields is not enough. At least one has to have coin effective computable results beyond the general framework.

This is really a demanding and time-consuming job. The only wise way to cope with this difficulty seems for one to have one’s own style or philosophy; not to follow the fashion of the time to waste time in vain.

References


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