4 特性サイクルと超局所化

- $SS\mathcal{F} \subset T^*X$
  $= \bigcup C_a$
- $CC\mathcal{F} = \sum m_a C_a$ curve の構成.

6 operations. 6 functor formalism

$D^b_c(X,\Lambda)$ $\quad f_*\ f^*\ \ f_!\ f^!\ \ \otimes\ \mathcal{H}om$ adjoint

$\supset D_{\mathrm{ctf}}(X,\Lambda)$

4.1. 特性類 $\quad cc\mathcal{F} \in H^0(X, K_X)$ $\quad a: X \to \mathrm{Spec}\, k$

$= H^{2d}(X,\Lambda(d))$ $\quad K_X = a^!\Lambda.$

$$\mathcal{H}om(pr_1^*\mathcal{F}, pr_2^!\mathcal{G}) \xleftarrow{\sim} D_X\mathcal{F}\boxtimes\mathcal{G}$$

$$\begin{tikzcd}
pr_1^*\mathcal{F}\otimes(D_X\mathcal{F}\boxtimes\mathcal{G}) \arrow[r] \arrow[d, equal] & pr_2^!\mathcal{G} \\
pr_1^*(\mathcal{F}\otimes D_X\mathcal{F})\otimes pr_2^*\mathcal{G} \arrow[r] & pr_1^*K_X\otimes pr_2^*\mathcal{G} \arrow[u] \\
 & pr_2^!\Lambda \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

$\delta: X \to X\times X$

$$\mathcal{H}om(\mathcal{F},\mathcal{G}) \xrightarrow{\sim} \delta^!\mathcal{H}om(pr_1^*\mathcal{F}, pr_2^!\mathcal{G})$$

$$\delta^*(D_X\mathcal{F}\boxtimes\mathcal{G}) = D_X\mathcal{F}\otimes\mathcal{G}$$

$\mathcal{F}=\mathcal{G}$

$1_{\mathcal{F}}: \Lambda \to \mathcal{H}om(\mathcal{F},\mathcal{F})$, $\quad ev: D_X\mathcal{F}\otimes\mathcal{F} \to K_X$

$\mathcal{O}$

constructible $\cdot\ X = \coprod X_i$

$\dim X = 1$ 　　$\mathcal{F}|_{X_i}$ 局所定数

$\mathcal{F}$ 　 U

$$SS\mathcal{F} = \left\{\begin{matrix} T^*_X X \\ \phi \end{matrix}\right\} \cup T^*X \underset{X}{\times} (X-U)$$

$$CC\mathcal{F} = -\left(\operatorname{rk}\mathcal{F}\cdot T^*_X X + \sum_{x \in X} a_x\mathcal{F}\cdot T^*_x X\right)$$

$$a_x\mathcal{F} = \operatorname{rk}\mathcal{F} - \operatorname{rk}\mathcal{F}_x + \operatorname{Sw}_x\mathcal{F}.$$

$$\chi(X,\mathcal{F}) = (CC\mathcal{F}, T^*_X X).$$

$$= \operatorname{rk}\mathcal{F}\cdot\chi(X,\mathcal{O}_X) - \sum a_x\mathcal{F}.$$

$$\mathcal{F} = j_! g$$

$$\chi(U,g) = \operatorname{rk} g\cdot\chi(X,\mathcal{O}_X) - \sum_{x \in X-U}(\operatorname{rk} g + \operatorname{Sw}_x g)$$

$$= \operatorname{rk} g\, \chi(U,\mathcal{O}_X) - \sum_{x \in X} \operatorname{Sw}_x g.$$

$$\Lambda \to \mathcal{H}om(\mathcal{F},\mathcal{F}) \to \delta^! \mathcal{H}om(pr_1^*\mathcal{F}, pr_2^!\mathcal{F})$$

$$\downarrow$$

$$\delta^*(D_X\mathcal{F} \boxtimes \mathcal{F}) \to D_X\mathcal{F} \otimes \mathcal{F} \to K_X$$

$cc\mathcal{F}$ = 合成 $\in H^0(X, K_X)$

X proper $\Rightarrow$ Tr $H^0(X, K_X) \to \Lambda$ $\qquad a_! a^! \Lambda \to \Lambda$

$k = \bar{k}$ $\qquad cc\mathcal{F} \mapsto \chi(X, \mathcal{F})$

4.2 特殊化

$\mathbb{A}$ k. 代数閉

$$\begin{array}{ccccc} TX & \to & A_X(X \times X) & \leftarrow & X \times X \times \mathbb{G}_m \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & \mathbb{A}^1 & \leftarrow & \mathbb{G}_m \end{array}$$

nearby cycles functor.

$\hat{O}_{\mathbb{A}^1,0} = O_K = k[[t]]$ $\quad$ strict

$O_{\mathbb{A}^1_k,0}$ 局所化 $\qquad O_{\mathbb{A}^1,0}$ ~~Hensel~~ 化

$$\begin{array}{ccccccc} A_s & \xrightarrow{i} & A_S & \leftarrow & A_\eta & \leftarrow & A_{\bar\eta} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ s = \mathrm{Spec}\, k & \xrightarrow{i} & S = \mathrm{Spec}\, O_K & \leftarrow & \eta = \mathrm{Spec}\, K & \leftarrow & \bar\eta = \mathrm{Spec}\, K_{sep} \end{array}$$

$$\Psi\mathcal{F} = i^* \bar{j}_* (\mathcal{F}|_{A_{\bar\eta}})$$

法束への変形

$i: Z \to X$ smooth スキームの閉はめこみ 余次元 c

$T_Z X$ $\qquad X = \mathrm{Spec}\, A,\quad Z = \mathrm{Spec}\, A/I$

$$\bigoplus I^n/I^{n+1} = S_{A/I}(I/I^2)$$

Z上の rk c のベクトル束

$$\begin{tikzcd}
Z\times 0 \arrow[r] \arrow[d, "0\text{-section}"'] & Z\times \mathbb{A}^1 \arrow[d, phantom, "\cap"] & Z\times \mathbb{G}_m \arrow[l] \arrow[d] \\
T_Z X \arrow[r] \arrow[d] & A_Z X \arrow[d] & X\times \mathbb{G}_m \arrow[l] \arrow[d] \\
0 \arrow[r] & \mathbb{A}^1 & \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

の構成

$X\times \mathbb{A}^1 \supset Z\times\{0\}$ を blow up して $X\times\{0\}$ の proper transform をとり除く

$X = \mathrm{Spec}\, A.\quad Z = \mathrm{Spec}\, A/I$

$X\times\mathbb{A}^1 = \mathrm{Spec}\, A[u].\quad A[u] = \bigoplus_{n\geq 0} A u^n$

$A_Z X = \mathrm{Spec}\, A[u][\frac{I}{u}]\quad A[u][\frac{I}{u}] = \bigoplus_{n\geq 0} A u^n \oplus \bigoplus_{n>0} \frac{I^n}{u^n}$

$$= \Phi \circ pr_{12}^*$$

$$\nu : D_c^b(X \times X, \Lambda) \longrightarrow D_c^b(TX, \Lambda)$$

adjoint $\delta_! \Lambda \longrightarrow \mathcal{H} \longrightarrow \delta_* k_X$ on $X \times X$

$$\mathcal{H} = \mathcal{H}om(pr_1^* \mathcal{F}, pr_2^! \mathcal{F}) \simeq D_X \mathcal{F} \boxtimes \mathcal{F}$$

$\nu \delta_! \Lambda \longrightarrow \nu \mathcal{H}om(\mathcal{F}, \mathcal{F}) \longrightarrow \nu \delta_* k_X$ on TX

$$\nu\delta_!\Lambda = 0_! \nu \Lambda = 0_! \Lambda \qquad \nu\delta_* k_X = 0_* \nu k_X = 0_* k_X$$

4.3 超局所化

Fourier 変換 $\quad X / k \quad E$. vector 束 $/X$

$E^\vee$ 双対 vect 束 $/X$

$$f^\vee(t) = \int_{\mathbb{R}} f(x) \exp(2\pi i x t)\, dx$$

$$F \mathcal{F} = pr_{2!}(pr_1^* \mathcal{F} \otimes \mu^* \mathcal{L}_\psi)$$

$$\begin{array}{ccccc} E & \xleftarrow{pr_1} & E \times_X E^\vee & \xrightarrow{pr_2} & E^\vee \\ & & \downarrow \mu & & \\ & & \mathbb{A}^1 & & \end{array}$$

例1. $F\Lambda = \Lambda_0(-1)[-2]$, $F 0_* \Lambda = \Lambda$.

$T^*X \quad TX$ の双対 vect 束

$\mu = F \circ \nu$. $\quad \mu\mathcal{H}om = F \circ \nu\mathcal{H}om$

$e: TX \to X$. $e^{\vee}: T^{\vee}X \to X$

$$F\mathcal{O}_!\Delta \longrightarrow F\nu\mathcal{H}om(\mathcal{F},\mathcal{F}) \longrightarrow F\mathcal{O}_* k_X$$

$$\Delta \longrightarrow \mu\mathcal{H}om(\mathcal{F},\mathcal{F}) \longrightarrow e^{\vee *} k_X$$

合成 $\in H^0(T^{\vee}X, e^{\vee *}k_X) = H^0(X, k_X)$

$\longmapsto \ni cc$

$\mathrm{supp}\,\mu\mathcal{H}om(\mathcal{F},\mathcal{F}) = SS_{\mu}\mathcal{F} \subset T^{\vee}X$ closed subset

Verdier $\nu\mathcal{H}om(\mathcal{F},\mathcal{F})$ は monodromic.

定 $\forall t \in E$ $w_t: \mathbb{G}_m \longrightarrow E$ による ひきもどし

$\lambda \longmapsto \lambda t$

$w_t^*\mathcal{F}$ は locally constant constructible で $0, \infty$ で tamely ramified.

monodromic の Fourier 変換は monodromic

monodromic の supp は conical.

$\Rightarrow SS_{\mu}\mathcal{F}$ は conical.

Q1. $SS_{\mu}\mathcal{F} \subset SS\mathcal{F}$?

$s: SS_\mu \mathcal{F} \to T^*X.$

$$\Lambda \to \mu\mathcal{H}om(\mathcal{F}, \mathcal{F}) \to s^! e^{\nu_6} k_X$$

$$CC_\mu \mathcal{F} \in H^0(SS_\mu \mathcal{F}, s^! e^{\nu_6} k_X) = H^0_{SS_\mu \mathcal{F}}(T^*X, e^{\nu_6} k_X)$$

Q1 が示せたならば　　　$C = \bigoplus_{C_a} \Lambda$

$$CC_\mu \mathcal{F} = \sum \mu_a C_a \qquad \mu_a \in \Lambda.$$
$$CC\mathcal{F} = \sum m_a C_a$$

Q2　$\mu_a = m_a$　$\forall C_a \subset SS\mathcal{F}.$

$[CC_\mu \mathcal{F}] = CC\mathcal{F}$ なので $CC_\mu \mathcal{F}$ は指数定式をみたす。

4.4. evidence

補題1. $\mathcal{F}$ が locally constant ならば Q1, Q2 ともに OK.

$\mathcal{F}$ が locally cont $\Rightarrow$ $\mathcal{H}$ も locally cont

$$\nu\mathcal{H}om(\mathcal{F}, \mathcal{F}) = e^! \mathcal{H}om(\mathcal{F}, \mathcal{F})$$
$$\mu\mathcal{H}om(\mathcal{F}, \mathcal{F}) = 0_* \mathcal{H}om(\mathcal{F}, \mathcal{F})$$
$$\mathrm{Supp} \subset 0\text{-section}$$
$$\mathcal{F} = 0 \Rightarrow \phi$$

$$CC_\mu \mathcal{F} = \underline{(-1)}^{\dim X} \cdot \mathrm{rk}\mathcal{F} \cdot T^*_X X$$

$H^2(\mathbb{A}^1 \times \mathbb{A}^1, \mu^* \mathcal{L}_\psi)$ に 対する バ・ョのいたずる は -1 倍で作用.

命題 2 $\dim X = 1$ ならば Q1, Q2 ともに OK.

$U \subset X$ $\mathcal{F}$ が locally const な最大の open

補題1より $SS_x \mathcal{F} \subset T^*_x X \cup (T^*_X X \times_X (X - U)) = SS\mathcal{F}$.

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$$

$$CC_x \mathcal{F} = CC_x(i_* \mathcal{F}) + CC_x(j_! \mathcal{F})$$

$\mathcal{F} = j_! j^* \mathcal{F}$ としてよい.

$\mathcal{F}$ が tamely ramified なとき. 直接計算.

$\mathcal{F}$ が一般のとき étale local.

$$\left(\begin{array}{l} \mathbb{G}_m \text{ 上の locally constant} \\ \text{shf. } 0 \text{ で tamely ramified} \\ \text{monodromy 有限 } p\text{-Sylow normal} \end{array}\right) \xrightarrow{\sim} (G_{K_\infty} \text{ の表現})$$

$\mathcal{G}$ GOS formula

$$\chi(\mathbb{G}_m, \mathcal{G}) = -\underline{m_\infty}$$

index formula

$$-\underline{\mu_\infty}$$

命題3. Q1が正しければ Q2も正しい。

X proj として良い。
Lefschetz pencil をとれば
curve に帰着される.

$$\mu_a = \langle\langle c_\mu \mathcal{F}, df\rangle\rangle_x$$

$$= \langle\langle c_\mu f_* \mathcal{F}$$ での $T_0^*\mathbb{P}^1$ の係数

$$= -a_0(f_*\mathcal{F})$$

$$m_a = \langle\langle c_* \mathcal{F}, df\rangle\rangle_x$$

$$= -\mathrm{dimtot}\,\Phi\mathcal{F}_x$$

$$= -a_0(f_*\mathcal{F})_x$$