The second S-W class of ℓ-adic coh.

ℓ-adic coh of middle degree of a variety of even dim

$\to$ orthogonal rep'n of Gal gp

$\to$ St. Wh class

Compare them with the inv. of de Rham coh.

1st SW class determinant v.s disc.

2nd : e.g. $k/\mathbb{Q}_p$ finite $p \neq \ell$

sign of the local ε-const Deligne.

dim$=0$ Serre's formula ~~in~~ involving the trace form.

Plan.

1. Def 2. Conj 3. Evidences

1. Second SW class.

k field. X proper smooth $/k$ $n = \dim X$.

$\ell \neq \operatorname{char} k$ q integer

$H^q(X_{\bar k}, \mathbb{Q}_\ell)$ ℓ-adic rep'n of $G_k = \operatorname{Gal}(\bar k/k)$

$\det H^q = e_q \cdot \chi_\ell^{\frac{q \cdot b_q}{2}}$ $\chi_\ell : G_k \to \mathbb{Z}_\ell^\times$ ℓ-adic cyclo. ch

$b_q = \dim H^q$ even if q odd (Suh) e_q

$e_q^2 = 1$ conseq. of Weil conj indep. of ℓ

$q = n$ $H^q \times H^q \to \mathbb{Q}_\ell(-n)$ non deg bilinear

n. odd $\Rightarrow$ symplectic $\Rightarrow$ $e_n = 1$.

n. even $V = H^n(X_{\bar k}, \mathbb{Q}_\ell(\frac{n}{2}))$. orthogonal rep'n.

e.g. $n=0$ $X = \operatorname{Spec} L$. L/k fin sep. ext'n. $V = \operatorname{Ind}_{G_L}^{G_k} 1$.

$\det V : G_k \to \{\pm 1\}$ elt. of $H^1(G_k, \mathbb{Z}/2\mathbb{Z}) \underset{\uparrow}{\simeq} k^\times/k^{\times 2}$ char $k \neq 2$

$sw_2(V) \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$ ($\simeq Br_2(k)$ char $k \neq 2$) 2nd sw. class

the class of the pull-back of the central ext'n

$$\begin{array}{ccccccccc} & & & & & & G_k & & \\ & & & & & & \downarrow & & \\ 1 & \to & \{\pm 1\} & \to & \tilde O(V) & \to & O(V) & \to & 1 \end{array}$$

central ext'n of alg gp $/\mathbb{Q}_\ell$

defined by using the Clifford alg

$Cl(V) = T^{\cdot}(V)/(x \cdot x - q(x); x \in V)$ $\dim Cl(V) = \dim \Lambda^{\cdot}(V)$

$C(V) \subset Cl(V)^{\times}$ alg gp subgp gen by $V \cap Cl(V)^{\times} = \{x \in V \mid q(x) \neq 0\}$

$C(V) \to G_m$ can. gp hom sending $x \in V \cap Cl(V)^{\times}$ to $q(x)$

$\tilde{O}(V) = \mathrm{Ker}(C(V) \to G_m)$

$\tilde{O}(V) \to O(V)$ " " $x \in V \cap \tilde{O}(V)$ to the reflexion $v \mapsto v - 2b(v,x)x$

$b(x,y)$ sym bil. $q(x) = b(x,x)$

2. Conjecture.

$D = H^n_{dR}(X/k)$ fin. dim k-v.sp.

$\cup$ defines nondeg symm bil. form.

E.g. $n=0$ $X = \mathrm{Spec}\, L$ L/k finite sep. ext'n.

$D = L$. $\mathrm{Tr}_{L/k}(xy)$.

char $k \neq 2$

$d = \mathrm{disc}\, D \in k^{\times}/k^{\times 2} = H^1(G_k, \mathbb{Z}/2\mathbb{Z})$

$= \sum_{i=1}^{b_n} \{a_i\}$ $x_1, \dots, x_n$ orthogonal basis $a_i = q(x_i)$ $a \mapsto \{a\}$ $\hat{k}^{\times} \mapsto \hat{k}^{\times}/(k^{\times})^2$.

$hw_2 D \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$

$= \sum_{i<j} \{a_i, a_j\}$ $\{a,b\} = \{a\} \cup \{b\} \in H^2(G_k, \mathbb{Z}/2\mathbb{Z})$.

2 Conjecture. X proper smooth$/k$. $n = \dim X$ even

char $k \neq 2, \ell$

$$sw_2(H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\tfrac{n}{2})) + \{e, -1\} + \beta \cdot c_\ell$$
$$= hw_2(D) + \begin{cases} r \cdot \{d, -1\} + \binom{r}{2}\{-1,-1\} & n \equiv 0 \ (4) \\ (r + b_{dR,n} - 1)\{d,-1\} + \binom{r+b_{dR,n}}{2}\{-1,-1\} & \equiv 2 \ (4) \end{cases}$$
$$+ \{2, d\} + \eta(c_\ell - c_2)$$

1 ~~Prop~~ Prop. X proper smooth$/k$ $n = \dim X$ even char $k \neq 2, \ell$

$\det V$

$$= \mathrm{disc}\, D + \begin{cases} r \cdot \{-1\} & n \equiv 0 \ (4) \\ (r + b_{dR,n})\{-1\} & \equiv 2 \ (4) \end{cases}$$

$r = \sum_{q<n} (-1)^q b_{dR,q}$.

$e = \sum_{q<n} e_q \qquad \det H^q = e_q \cdot \chi_e^{\frac{q_?+b_q}{2}}$.

$\beta = \frac{1}{2} \sum_{q<n} (-1)^q (n-q) b_q$.

$c_e \qquad H^2(\mathbb{Z}_e^\times, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\chi_e^*} H^2(G_k, \mathbb{Z}/2\mathbb{Z})$

uniq non-trivial $\longmapsto c_e$

$\mathbb{Z}_e^\times \simeq \mathbb{Z}_p \times$ cyclic of even order. $\qquad$ 0 if char $k \neq 0$

$d = \operatorname{disc} D$.

$\eta = \sum_{q<\frac{n}{2}} (-1)^q \left(\frac{n}{2} - q\right) \chi(X, \Omega^q_{X/k})$

A complex version $\quad sw_2(H^\cdot) = hw_2(D^\cdot) + \{2, d\} + \eta(c_e - c_2)$

3. Evidence.

Theorem Conjecture is true if one of the following is satisfied

arith local H.P.
1. $k/\mathbb{Q}_p$ finite $p \neq 2, l$. $\exists \mathcal{X}/\mathcal{O}_k$ proj. reg flat, s.t X_F has at most odp as sing.

2. $k/\mathbb{Q}_p$ finite unr. $p = l > n+1$. good reduction. — direct link l-adic v.s deRh

3. $k = \mathbb{R}$. X proj.

geom. 4. $k \supset \bar{\mathbb{Q}}$

global 5. smooth hypersurface in $\mathbb{P}^{n+1}_k$. $l > n+1$

Rmq 1. Theorem $\underset{5, n=0}{\Rightarrow}$ Th of Serre $\quad Sw_2(\operatorname{Ind}_{G_L}^{G_k} 1) = hw_2(\operatorname{Tr}_{L/k} x^2) + \{d, 2\}$

2. $k = \mathbb{Q}_p$ $p > 3$. A abelian surface (good red) $\quad Sw_2(H^2(A_{\bar{k}}, \mathbb{Q}_2(1))) = 0 \quad \in Br_2(\mathbb{Q}_p)$

$\neq Sw_2(H^2(A_{\bar{k}}, \mathbb{Q}_p(1)))$

Sketch of Pf.

1. Picard-Lefschetz formula ~~[illegible]~~ $\{d, 2\}$ ~~+ formula for det~~ Prop

direct red'n
2. p-adic Hodge theory with integral coeff. (= Fontaine-Laffaille) + a similar argument to prove ~~[illegible]~~ Th of Serre

3. Hodge theory + polarization

4. Lefschetz principle $k \supset \mathbb{C}$. transcendental argument.

5. moduli space + 1~4.

$U/\mathbb{Z}$ moduli space.

$$\begin{array}{ll}
\text{PD} \quad \operatorname{Ker}(1 \oplus 2 \oplus 4) \subset H^2(U[\tfrac{1}{2e}], \mathbb{Z}/2\mathbb{Z}) & \xrightarrow{1} H^1(D[\tfrac{1}{2e}], \mathbb{Z}/2\mathbb{Z}) \\
\qquad \| & \xrightarrow{2} H^1(U \otimes \mathbb{F}_e, \mathbb{Z}/2\mathbb{Z}) \quad (l \neq 2) \\
H^2(\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\sim} & \xrightarrow{3} H^2(\mathbb{R}, \mathbb{Z}/2\mathbb{Z}) \\
 & \xrightarrow{4} H^2(U_{\bar{\mathbb{Q}}}, \mathbb{Z}/2\mathbb{Z})
\end{array}$$