characteristic cycle & singular support of l-adic sheaf

Analogy

l-adic sheaf / char $p>0$ — D-modules / complex mfd

wild ramification — irregular singularity

microlocal analysis

? $\longleftarrow$ Char(M)

cycle on T^*X cotangt bdle

How to define it?

Deligne's approach — vanishing cycles.

1 Vanishing cycles

X smooth sch of dim d. / k alg closed char $=p>0$.

K constructible cx of Λ-modules. $\Lambda/\mathbb{F}_\ell$ finite $\ell\neq p$.

$f: X\to C$. flat morphism to smooth curve $u\in X$ closed pt $v=f(u)$

$\phi_u(K,f)$ cx of Λ-mod $\phi^q_u(K,f)$. fin. dim. $=0$. except finitely many q.

- qualitative — local acyclicity $\phi_u(K,f)=0$.

local acyclicity of smooth morphism. K l.c. + f smooth $\Rightarrow$ l.a.

$\Rightarrow \phi$ measures how much K ramifies + f degenerates

- quantitative — total dim. ϕ^q_u rep of G_{K_v} local field at v

$=$ dim + Swan conductor

measure of wild ramification.

Use $\phi_u(\mathcal{F},f)$ to define. S.S / Char.

SS qualitative. closed <u>conic</u> subset of T^*X of dim d

stable under multiplication.

Char quantitative ~~supp~~ cycle supported on SS

linear combination of irred cpt.

~~Condit~~

Conditional result
Assume the existence of SS
$\Rightarrow \begin{cases} \cdot \text{ Definition of Char} \\ \cdot \text{ Milnor formula } (= \text{formula for } \dim \operatorname{tot} \phi_u) \\ \cdot \text{ index formula } (= \text{formula for } \chi.) \end{cases}$

Unconditional
$\dim \leqq 2 \Rightarrow$ existence of SS (last year)

2 Singular support.
$S = \bigcup S_i$ closed conic subset of T^*X of dim d
$T_i = S_i \cap T^*_X X$
0-section

$W \to X \times B$ unramified. reg of codim r
(= étale locally reg imm of codim r)
B smooth/k, $W \to B$ flat
(e.g. $f: X \to C$ $W = X \to X \times C$ graph)

Def 1. We say $W \to X \times B$ is <u>non char</u> w.r.t S if
(1) By $T^*_W(X \times B) \to T^*X$, the inv. image of S is a subset of the 0-section.
(conormal bdle)

(e.g if $S = T^*_X X$, non char $\Leftrightarrow$ $W \to B$ smooth)

(2) $W \to X \times B$ meets $T_i \times B$ properly.
i.e $\forall b \in B$ closed pt, $\forall Q \subset W_b \times_X T_i$ irr. cpt
$\dim Q = \dim T_i - r$.
(e.g for $W = X \to X \times C$ graph. $T_i \subset X \to C$. open map)

Def 2 We say S satisfies (SS_r) for K if.
for every $W \to X \times B$ codim $q \leqq r$,
non char w.r.t S $\Rightarrow$ local acyclicity of $W \to B$ rel. to the pull-back of K.

(e.g $X \to X \times B$ graph $\dim B = q \leq r$. + its family)

Example $j : U = X - D \to X$ $\quad D = \bigcup_{i=1}^{m} D_i$ div. w. SNC
$K = j_! \mathcal{F}$

(1) $\mathcal{F}$ tamely ramified along D.

$\Rightarrow SS K = \bigcup_I T^*_{X_I} X$ $\quad X_I = \bigcap_{i \in I} D_i$
conormal b'dle.

(2) $\dim X = 1$
$\Rightarrow \quad SS K = T^*_X X \cup T^*_D X$
0-section $\quad$ fiber.

$\dim \geq 2$ ramification thy $\Rightarrow$ outside codim ≥ 2
not necessarily Lagrangean.

3. Characteristic cycle
Theorem 1 (Milnor formula) Assume S satisfies (SS1) for K
Then $\exists !$ $Ch\, K$ supp'd on S with $\mathbb{Z}[\frac{1}{p}]$-coeff. s.t.

$$- \dim \operatorname{tot} \phi_u(K, f) = (Ch\, K, df)_u$$
for every $f : X \to C$ df'd on a nbd of u
s.t. u is an isolated non char. pt.

Thm 2 (index formula) Assume S satisfies (SS2) for K
and X projective. Then

$$\chi(X, K) = (Ch\, K, T^*_X X)_{T^*X}$$

Key ingredients in pf.
(1) cohomological formalism of vanishing cycles over an arbitrary base scheme
Vanishing topos $X \overleftarrow{\times}_S S$ and a generalization of the continuity of Swan conductor by Deligne-Laumon

(2) local geometric version of Radon transform and its variant

X quasi-proj $\mathcal{L}$ very ample

$E \subset \Gamma(X, \mathcal{L})$ finite dim s.t

$X \to \mathbb{P}(E^\vee) = \mathbb{P}(\text{"="} E^\vee - \{0\} / \mathbb{G}_m^\times)$ is an immersion

$q : X \times_{\mathbb{P}} \mathbb{H} \to \mathbb{P}^\vee = \mathbb{P}(E)$. univ. hyperplane section

$p \downarrow$

X $\quad \mathbb{H} = \{(x, H) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in H\}$

$S = \bigcup S_i \qquad T_i = S_i \cap T^*_X X \qquad \mathbb{P}(\tilde{S}) = \bigcup \mathbb{P}(\tilde{S}_i)$

$\subset X \times_{\mathbb{P}} \mathbb{H} = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P})$

$q : X \times_{\mathbb{P}} \mathbb{H} \to \mathbb{P}^\vee$ is univ locally acyclic rel to p^*K

outside $\mathbb{P}(\tilde{S}) \cup \mathbb{T}(S) \qquad \mathbb{T}(S) = \bigcup_{\dim T_i \geq 1} T_i \times \mathbb{P}_i^\vee$

$\mathbb{P}(E_i) = \mathbb{P}_i^\vee \subset \mathbb{P}^\vee = \mathbb{P}(E)$

$E_i = \ker(E \to \Gamma(T_i, \mathcal{L} \otimes \mathcal{O}_{T_i}))$

local Radon transform

$R_E K = R\Phi_q p^* K \quad$ on $(X \times_{\mathbb{P}} \mathbb{H}) \overleftarrow{\times}_{\mathbb{P}^\vee} \mathbb{P}^\vee$