# INTRODUCTION TO WILD RAMIFICATION OF SCHEMES AND SHEAVES

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In classical theory of algebraic numbers, the conductor-discriminant formula asserts that the discriminant of an extension of number fields is the product of local invariants of ramification called the conductor. The conductor of a Galois representation plays also a crucial role in the quantitative formulation of the Langlands correspondences.

In this course, we discuss more geometric aspects of ramification, due to the following reasons. Firstly, we have a clearer picture and more complete theory in the geometric case. Secondly, in the arithmetic case, even in the cases where results similar to the geometric case are obtained, it usually requires more sophisticated technics.

A prototype of the geometric ramification theory is the Grothendieck-Ogg-Shafarevich formula [10] that computes the Euler number of an $\ell$-adic sheaf on a curve over an algebraically closed field of positive characteristic different from $\ell$. The formula is a sheaf theoretic refinement of the Riemann-Hurwitz formula for a ramified covering of algebraic curves, which is a geometric counterpart of the conductor-discriminant formula, with the analogy between the discriminant of a number field and the genus of a curve. In the Grothendieck-Ogg-Shafarevich formula, the conductor appears as the local contribution of ramification.

We will discuss generalizations of the GOS formula through the following three approaches:

1. Ramified coverings and log products.
2. Characteristic classes and characteristic cycles.
3. Blow-up at the ramification locus in the diagonal.

They are related to each other but can be discussed independently at least in the beginning of the theory. All of the three approaches rely essentially on constructions using the product that make the theory in arithmetic case technically more complicated where we need to work with some substitutes.

Here follows a more concrete description of the course. Using ramified coverings and log products in the first approach, we introduce the Swan class of an $\ell$-adic sheaf ramified along the boundary. The Swan class is a generalization of the conductor and defined as an 0-cycle class supported on the boundary. It enables us to generalize the GOS formula computing the Euler number to higher dimension.

For an $\ell$-adic sheaf on a variety, its characteristic class is defined as a cohomology class using an abstract formalism and the Lefschetz trace formula applied to a compactification asserts that its trace computes the Euler number. Hence, a second approach to a generalization of the GOS formula is the computation of the characteristic class. The theory of $\mathcal{D}$-modules suggests that the characteristic cycle [11] defined in the cotangent bundle gives the class. We observe that this is the case for rank 1 sheaf at least in a certain favorable situation.

The conductor of a Galois representation of a local field is defined by the filtration of ramification groups on the Galois group. The definition of the filtration is generalized to a local field with imperfect residue field, first using rigid geometry. The interpretation of rigid geometry in terms of blow-up inspired the third approach. We define the filtration of ramification groups without using rigid geometry and observe how the groupoid structure on a blow-up of the product allows us to study the graded subquotients of the filtration.

The course is intended to make an introduction to the subjects discussed in a survey article [15]. More details are found in the following articles; [13] for the first approach, [9], [12], [7], [14] for the second approach and [5], [6], [14], [8] for the third approach.

The audiences are assumed to have some acquaintances to the following subjects:

Number theory ([1, Parties 1, 2]): Galois theory including infinite Galois extensions, discrete valuation rings, ...

Algebraic geometry ([4, Exposés I, V], [3, Sections 1, 2]): schemes, etale morphisms, etale sites, fundamental groups, cohomology, ...

Representations of finite groups ([2]).

We will briefly recall some basic terminologies on etale topology at the beginning of the course.

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