Perfectoid spaces and the weight-monodromy conjecture, after Peter Scholze

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1 Weight-monodromy conjecture

The Galois representation associated to the étale cohomology of a variety defined over a number field is a central subject of study in number theory. The Galois action on the Tate module of an elliptic curve and the Galois representation associated to a modular form that played the central role in the proof of Fermat's last theorem by Wiles and Taylor are typical examples.

To simplify the notation, we assume that a proper smooth variety X is defined over the rational number field $\mathbf{Q}$. We fix a prime number ℓ and consider the representation of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ acting on the ℓ-adic étale cohomology $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$. As is seen in the definition of the Hasse-Weil L-function, a standard method in the study of a Galois representation is to investigate it locally at each prime.

Let p be a prime number different from ℓ. If X has good reduction at p, the Galois representation $H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ at p is almost completely understood thanks to the Weil conjecture proved by Deligne [1]. Namely its restriction to the decomposition group $\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ at p is unramified and the characteristic polynomial $\det(1-F_pt : H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell))$ of the geometric Frobenius is determined by counting the number of points of the reduction of X modulo p defined over $\mathbf{F}_{p^n}$ for every $n \geqq 1$.

However, for a prime of bad reduction, an important piece, called the weight-monodromy conjecture, is still missing. To state it, let us recall briefly the structure of the absolute Galois group $\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$. Corresponding to the maximal unramified extension and the maximal tamely ramified extension $\mathbf{Q}_p \subset \mathbf{Q}_p^{\mathrm{ur}} = \mathbf{Q}_p(\zeta_m; p \nmid m) \subset \mathbf{Q}_p^{\mathrm{tr}} = \mathbf{Q}_p^{\mathrm{ur}}(p^{1/m}; p \nmid m) \subset \bar{\mathbf{Q}}_p$, the inertia subgroup and its wild part $\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p) \supset I \supset P \supset 1$ are defined. The quotient $\mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)/I$ is canonically identified with $\mathrm{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ and is topologically generated by the geometric Frobenius F_p. The quotient I/P by the pro-p Sylow subgroup P is non-canonically identified with the product of $\mathbf{Z}_{p'}$ for $p' \neq p$.

By the monodromy theorem of Grothendieck, there exists a nilpotent operator N on $V = H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ such that the restriction to an open subgroup of I is given by $\exp(t_\ell(\sigma)N)$ where $t_\ell \colon I \to \mathbf{Z}_\ell$ is a surjection. By elementary linear algebra, the nilpotent operator defines a unique finite increasing filtration W on V characterized by the property that $N(W_iV) \subset W_{i-2}V$ for every integer i and N^i induces an isomorphism $\mathrm{Gr}_i^W V = W_iV/W_{i-1}V \to \mathrm{Gr}_{-i}^W V$ for every $i \geqq 0$. Then the weight-monodromy conjecture is stated as follows.

Conjecture 1.1. *Let $F \in \mathrm{Gal}(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ be a lifting of the geometric Frobenius F_p. Then the eigenvalues of F acting on $\mathrm{Gr}_i^W H^q(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ are algebraic integers and the complex absolute values of the conjugates are $p^{(q+i)/2}$.*

Conjecture was proved for $q = 1$ by Grothendieck by studying the Néron model of an abelian variety. It is proved for $q = 2$ by Rapoport-Zink using the weight spectral sequence in a semi-stable case and in general by using alteration by de Jong. For an arbitrary q, the Weil conjecture and alteration by de Jong imply that the eigenvalues of F acting on $\mathrm{Gr}_i^W V$ are algebraic integers and the complex absolute values of the conjugates are $p^{n/2}$ for an integer n independent of the conjugates. The function field analogue is proved by Deligne in the course of the proof of the Weil conjecture in [1].

P. Scholze introduced a new method in [7] to study the weight-monodromy conjecture and proved it for smooth complete intersections in smooth toric varieties, stated later as Theorem 6.2. The method is to construct and study the following diagram and reduce it to the function field case already proved by Deligne. In the diagram, k, k', K and $K^\flat$ denote finite extensions of $\mathbf{Q}_p, \mathbf{F}_p((t)), \mathbf{Q}_p(\zeta_{p^\infty})$ and of $\mathbf{F}_p((t))(t^{1/p^\infty})$ respectively.

$$\begin{array}{ccc} (\text{algebraic varieties}/k) & & (\text{algebraic varieties}/k') \\ {\scriptstyle ?}\big\downarrow & & \big\downarrow{\scriptstyle \text{perfection}} \\ (\text{perfectoid spaces}/K) & \xrightarrow{\simeq} & \left(\text{perfectoid spaces}/K^\flat\right). \end{array} \tag{1.1}$$

The upper categories stay in the realm of algebraic geometry while the lower ones are in that of rigid geometry in the sense of R. Huber [6]. A point is that we have a canonical equivalence of categories on the lower line. Another point is that we have a canonical isomorphism

$$\mathrm{Gal}(\bar{K}/K) \leftarrow \mathrm{Gal}(\bar{K}^\flat/K^\flat). \tag{1.2}$$

Since some important special cases were first introduced by Fontaine-Wintenberger [4], the isomorphism (1.2) has been used effectively in the study of p-adic Galois representations of p-adic fields. It allows us to expect to deduce results in characteristic 0 from the corresponding results in characteristic $p > 0$.

A problem is that we do not have a natural functor for the left vertical arrow in (1.1) with ?, in general. Another problem is that we may lose some information by going down. The morphisms $\mathrm{Gal}(\bar{k}/k) \leftarrow \mathrm{Gal}(\bar{K}/K)$ and $\mathrm{Gal}(\bar{k}'/k') \leftarrow \mathrm{Gal}(\bar{K}^\flat/K^\flat)$ defined by inclusions $k \subset K$ and $k' \subset K^\flat$ induce isomorphisms on the quotients by the wild inertia subgroups. Since the weight-monodromy conjecture can be regarded as a statement on the actions of these quotients, we do not lose too much information by going down.

The purpose of the lecture is to explain the diagram and sketch the proof of the weight-monodromy conjecture for smooth complete intersections in toric varieties. For these varieties, one can construct the left vertical arrow with ? and recover enough information to prove the weight-monodromy conjecture.

The contents of the notes are summarized as follows. In Section 2, we introduce a perfectoid field and the tilting functor associating to a perfectoid field of characteristic 0 a perfectoid field of characteristic $p > 0$. The canonical isomorphism (1.2) is obtained as a special case of the almost purity theorem discussed in Section 5.

We also introduce perfectoid algebras in Section 2. The perfectoid spaces are defined by patching their associated adic spaces in Section 4, after recalling briefly some foundations on adic spaces. The key fact that the categories of perfectoid algebras over a perfectoid field K and its tilt $K^\flat$ are equivalent to each other is proved using the language of almost commutative algebra recalled in Section 3. The idea of the proof of the equivalence will be sketched at the end of the section.

In Section 5, we recall a crucial generalization of the almost purity theorem of Faltings and compare the étale topology in characteristic $p > 0$ and characteristic 0. Finally, we state the main result on the weight-monodromy conjecture and sketch the proof in Section 6.

For the full detail of proof, we refer to the original article [7]. A survey [3] is written by Fontaine.

2 Perfectoid fields and perfectoid algebras

We begin with recalling the definition of perfectoid fields and the tilting construction. We also state an isomorphism of Galois groups of a perfectoid field of characteristic 0 and its associated perfectoid field of characteristic $p > 0$, whose sketch of proof is postponed to the Section 5.

Let K be a field. A mapping $v\colon K \to \mathbf{R} \cup \{\infty\} = (-\infty, \infty]$ is called a(n additive) valuation of K (of height 1) if $v(a+b) \geqq \min(v(a), v(b))$ and $v(ab) = v(a) + v(b)$ for $a, b \in K$, if $v(a) = \infty$ is equivalent to $a = 0$ and if $v(K) \supsetneqq \{0, 1\}$. A field K equipped with a valuation of height 1 will be called a valuation field. The subring $\mathcal{O}_K = \{a \in K \mid v(a) \geqq 0\}$ is called the valuation ring of $\mathcal{O}_K$ and $\mathfrak{m} = \{a \in K \mid v(a) > 0\}$ is the maximal ideal of $\mathcal{O}_K$. Choosing a real number $0 < a < 1$, we define a metric $d(x, y) = a^{v(x-y)}$ on K. The topology is independent of the choice of a.

A valuation field K of characteristic 0 is said to be of mixed characteristic $(0, p)$ if the residue field $\mathcal{O}_K/\mathfrak{m}$ is of characteristic $p > 0$.

Definition 2.1. *Let K be a valuation field such that the restriction $v|_{K^\times}\colon K^\times \to \mathbf{R}$ has dense image and assume that K is either of characteristic $p > 0$ or of mixed characteristic $(0, p)$. Then K is called a* perfectoid field *if K is complete and if the Frobenius endomorphism $\mathcal{O}_K/p\mathcal{O}_K \to \mathcal{O}_K/p\mathcal{O}_K\colon x \mapsto x^p$ is a surjection.*

A perfectoid field K of characteristic $p > 0$ is a perfect field and its valuation ring $\mathcal{O}_K$ is a perfect ring. Let K be a perfectoid field of mixed characteristic $(0, p)$. Then the inverse limit $\varprojlim \mathcal{O}_K/p\mathcal{O}_K$ with respect to the Frobenius endomorphism is a complete valuation ring and its fraction field $K^\flat$ is a perfectoid field of characteristic p, called the *tilt* of K. For $a = (a_n) \in \mathcal{O}_{K^\flat}$, the limit $a^\sharp = \lim_{n\to\infty} (\tilde{a}_n)^{p^n} \in \mathcal{O}_K$, defined by taking liftings, is independent of the choices and induces a mapping $K^\flat \to K$ compatible with the multiplications and the valuations.

Example 2.2. *The p-adic completion K of $\mathbf{Q}_p(\zeta_{p^\infty})$ is a perfectoid field of mixed characteristic $(0, p)$. Its tilt $K^\flat$ is isomorphic to the t-adic completion of $\mathbf{F}_p((t))(t^{1/p^\infty})$.*

The following fact is fundamental.

Proposition 2.3 ([7, Theorem 3.7]). *Let K be a perfectoid field of mixed characteristic $(0, p)$. Then, a finite extension L of K is also a perfectoid field and its tilt $L^\flat$ is a finite (separable) extension of $K^\flat$. Further, the functor*

(2.1) $$\text{(finite separable extensions of } K) \to \text{(finite separable extensions of } K^\flat)$$

sending L to $L^\flat$ is an equivalence of categories.

Proposition 2.3 is proved as a special case of Theorem 5.1 and makes an important step of the proof. The equivalence of categories (2.1) induces a canonical isomorphism (1.2) of absolute Galois groups. It gives a generalization of the theory of fields of norms by Fontaine-Wintenberger [4].

We define perfectoid algebras over a perfectoid field. We fix a perfectoid field K of characteristic $p > 0$ or of mixed characteristic $(0, p)$. Let $\mathcal{O}_K$ denote its valuation ring and we also fix a non-zero element ϖ satisfying $0 < v(\varpi) \leqq v(p)$. We take a real number $0 < a < 1$ and define a norm $|x| = a^{v(x)}$ on K. A K-algebra R is said to be a K-Banach algebra if it is complete and separated with respect to a norm $|\ |$ on the K-vector space R compatible with multiplication. An element x of R is said to be power-bounded if the subset $\{x^n \mid n \geqq 0\}$ is bounded with respect to a norm. Let A denote the subring of R consisting of power-bounded elements.

Definition 2.4. *Let K be a perfectoid field of characteristic $p > 0$ or of mixed characteristic $(0, p)$. We say that a Banach K-algebra R is a* perfectoid K-algebra *if the subring $A \subset R$ consisting of power-bounded elements is bounded and if the Frobenius endomorphism $A/\varpi \to A/\varpi\colon x \mapsto x^p$ is surjective. A morphism of perfectoid K-algebras is a continuous morphism of K-algebras.*

Similarly as for perfectoid fields, we define tilting construction. Let K be a perfectoid field, R be a perfectoid K-algebra and $A \subset R$ be the subring as above. Then, we set

$$A^\flat = \varprojlim A/\varpi A \tag{2.2}$$

with respect to the Frobenius endomorphism. It is naturally an $\mathcal{O}_{K^\flat}$-algebra. Thus $R^\flat = A^\flat \otimes_{\mathcal{O}_{K^\flat}} K^\flat$ is defined as a Banach $K^\flat$-algebra, called the *tilt* of R.

Lemma 2.5 ([7, Proposition 5.9]). *Let K be a perfectoid field of* characteristic $p > 0$ *and R be a Banach K-algebra such that the subring A consisting of power-bounded elements is open and bounded. Then, the following conditions are equivalent:*

(1) *R is a perfectoid K-algebra.*
(2) *R is perfect.*
(3) *A is perfect.*

Example 2.6. 1. *If k is a complete discrete valuation field of characteristic $p > 0$ with a uniformizer t, the fraction field K of the t-adic completion of the perfection $\mathcal{O}_k^{1/p^\infty}$ is a perfectoid field. If A is a flat $\mathcal{O}_k$-algebra of finite type, the t-adic completion of the perfection A^{1/p^∞} tensored K is a perfectoid algebra over K.*

2. *The ϖ-adic completion R of $K[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}]$ is a perfectoid K-algebra, denoted $K\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$. Its tilt $R^\flat$ is $K^\flat\langle T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}\rangle$.*

Now we state a key result.

Theorem 2.7 ([7, Theorem 5.2]). *Let K be a perfectoid field of mixed characteristic $(0, p)$. Then, for a perfectoid K-algebra R, its tilt $R^\flat$ is a perfectoid $K^\flat$-algebra. Further the functor*

$$(\text{perfectoid } K\text{-algebras}) \to (\text{perfectoid } K^\flat\text{-algebras}) \tag{2.3}$$

sending R to $R^\flat$ is an equivalence of categories.

3 Almost commutative algebra

A basic idea on almost commutative algebra in the context of perfectoid extension of a complete discrete valuation ring is the following. Let k be a complete discrete valuation field and $\mathcal{O}_k$ be the valuation ring. We consider the factorization

$$\begin{array}{ccc} (\mathcal{O}_k\text{-modules}) & \xrightarrow{\otimes_{\mathcal{O}_k} k} & (k\text{-vector spaces}) \\ \downarrow & \nearrow & \\ (\mathcal{O}_k\text{-modules})/(\text{torsion modules}) & & \end{array}$$

of the scalar extension functor by the quotient category. Then the slant arrow is an equivalence of category.

Now, let K denote a perfectoid field and $\mathcal{O}_K$ be the valuation ring. Similarly, we consider the factorization

$$\begin{array}{ccc} (\mathcal{O}_K\text{-modules}) & \xrightarrow{\otimes_{\mathcal{O}_K} K} & (K\text{-vector spaces}). \\ \downarrow & \nearrow & \\ (\mathcal{O}_K\text{-modules})/(\text{almost zero modules}) & & \end{array}$$

Here an $\mathcal{O}_K$-module is said to be almost zero if every element is annihilated by the maximal ideal $\mathfrak{m}_K$, satisfying $\mathfrak{m}_K = \mathfrak{m}_K^2$. This time, the vertical arrow is very close to an equivalence of categories. This is analogous to ignoring infinitesimal in classical calculus. Moreover, one can develop a theory of commutative algebra in the lower tensor abelian category, which is called *almost* commutative algebra.

We work in a more general setting. Let A be a commutative ring and let $\mathfrak{m}$ be a flat ideal of A satisfying $\mathfrak{m}^2 = \mathfrak{m}$. We say that an A-module M is *almost zero* if $\mathfrak{m}M = 0$. The essential image of the natural fully faithful functor

$$(A/\mathfrak{m}\text{-modules}) \to (A\text{-modules})$$

is the subcategory consisting of almost zero modules.

Lemma 3.1. *For an A-module M, the following conditions are equivalent.*

(1) *M is almost zero.*

(2) *$Tor_q^A(\mathfrak{m}, M) = 0$ for every $q \geqq 0$.*

(3) *$Ext_A^q(\mathfrak{m}, M) = 0$ for every $q \geqq 0$.*

Proof. Since $\mathfrak{m}M$ is the image of the composition $\mathfrak{m} \otimes_A M \to \mathfrak{m} \otimes_A Hom_A(\mathfrak{m}, M) \to M$, either of the vanishings $\mathfrak{m} \otimes_A M = 0$ and $Hom_A(\mathfrak{m}, M) = 0$ implies $\mathfrak{m}M = 0$.

A free resolution of the A-module $\mathfrak{m}$ defines spectral sequences

$$E^2_{p,q} = Tor_p^{A/\mathfrak{m}}(Tor_q^A(\mathfrak{m}, A/\mathfrak{m}), M) \Rightarrow Tor_{p+q}^A(\mathfrak{m}, A/\mathfrak{m} \otimes_{A/\mathfrak{m}} M),$$

$$E_2^{p,q} = Ext^p_{A/\mathfrak{m}}(Tor^A_{-q}(\mathfrak{m}, A/\mathfrak{m}), M) \Rightarrow Ext_A^{p+q}(\mathfrak{m}, Hom_{A/\mathfrak{m}}(A/\mathfrak{m}, M))$$

for an $A/\mathfrak{m}$-module M. Since the A-module $\mathfrak{m}$ is assumed flat, they induce isomorphisms $Tor_p^{A/\mathfrak{m}}(\mathfrak{m} \otimes_A (A/\mathfrak{m}), M) \to Tor_p^A(\mathfrak{m}, M)$ and $Ext^p_{A/\mathfrak{m}}(\mathfrak{m} \otimes_A (A/\mathfrak{m}), M) \to Ext^p_A(\mathfrak{m}, M)$. Since $\mathfrak{m} \otimes_A (A/\mathfrak{m}) = \mathfrak{m}/\mathfrak{m}^2 = 0$, the assertion follows. □

Either of the conditions (2) and (3) implies that the subcategory consisting of almost zero modules is closed under extensions.

We say that a morphism of A-modules $f\colon M \to N$ is an *almost isomorphism* if the kernel and the cokernel of f are almost zero. For morphisms $f\colon L \to M$, $g\colon M \to N$ of A-modules, if two of f, g and $g \circ f$ are almost isomorphisms, so is the third.

Lemma 3.2. 1. *For an A-module M, the canonical morphisms $\mathfrak{m} \otimes_A M \to M \to Hom_A(\mathfrak{m}, M)$ are almost isomorphisms.*

2. *For a morphism of A-modules $f\colon M \to N$, the following conditions are equivalent.*

(1) *f is an almost isomorphism.*

(2) *$f_*\colon \mathfrak{m} \otimes_A M \to \mathfrak{m} \otimes_A N$ is an isomorphism.*

(3) *$f_*\colon Hom_A(\mathfrak{m}, M) \to Hom_A(\mathfrak{m}, N)$ is an isomorphism.*

Proof. 1. The exact sequence $0 \to \mathfrak{m} \to A \to A/\mathfrak{m} \to 0$ induces exact sequences $Tor_1^A(A/\mathfrak{m}, M) \to \mathfrak{m} \otimes_A M \to M \to (A/\mathfrak{m}) \otimes_A M \to 0$ and $Hom_A(A/\mathfrak{m}, M) \to M \to Hom_A(\mathfrak{m}, M) \to Ext^1_A(A/\mathfrak{m}, M)$ and the assertion follows.

2. Since (1) in Lemma 3.1 implies (2) and (3) in Lemma 3.1 respectively, (1) implies (2) and (3) respectively. Conversely, the left (resp. right) square of the commutative diagram

$$\begin{array}{ccccc} \mathfrak{m} \otimes_A M & \longrightarrow & M & \longrightarrow & Hom_A(\mathfrak{m}, M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{m} \otimes_A N & \longrightarrow & N & \longrightarrow & Hom_A(\mathfrak{m}, N) \end{array}$$

shows that (2) (resp. (3)) implies (1) by 1. and the remark preceding Lemma. □

We define the category of *almost A-modules* to be the quotient category of that of A-modules by the subcategory consisting of almost zero A-modules. We have a canonical functor

$$(A\text{-modules}) \to (\text{almost } A\text{-modules}) \tag{3.1}$$

sending an A-module M to the associated almost A-module M^a. The category (almost A-modules) is an abelian category and inherits tensor products and internal Hom from the category (A-modules). Consequently, we can do linear algebra as well as multi-linear algebra and commutative algebra in the category.

For A-modules M and N, we call a morphism in the category of almost A-modules *an almost morphism* and let $Hom_{A^a}(M^a, N^a)$ denote the A-module of almost morphisms. We set ${}_*M = \mathfrak{m} \otimes_A M$ and $N_* = Hom_A(\mathfrak{m}, N)$.

Lemma 3.3. 1. *The functors $M \mapsto {}_*M$ and $N \mapsto N_*$ are adjoint to each other. The canonical morphisms ${}_*M \to M$ and $N \to N_*$ induce isomorphisms of functors ${}_*({}_*M) \to {}_*M$ and $N_* \to (N_*)_*$.*

2. *The canonical functor* (3.1) *admits left and right adjoint functors*

$$(\text{almost } A\text{-modules}) \to (A\text{-modules}), \tag{3.2}$$

induced by the functors $M \mapsto {}_*M$ and $N \mapsto N_*$ respectively. For A-modules M and N, we have a canonical isomorphism $Hom_{A^a}(M^a, N^a) \to Hom_A(M, N)_*$.

3. *The functors* (3.2) *are fully faithful and the essential images consist of A-modules M such that the canonical morphism $M \to {}_*M$ is an isomorphism and N such that $N \to N_*$ is an isomorphism respectively.*

Proof. 1. The canonical isomorphisms $Hom_A(\mathfrak{m} \otimes_A M, N) \to Hom_A(M, Hom_A(\mathfrak{m}, N))$ define an adjunction. Since the multiplication induces an isomorphism $\mathfrak{m} \otimes_A \mathfrak{m} \to \mathfrak{m}$, we obtain an isomorphism ${}_*({}_*M) = \mathfrak{m} \otimes_A \mathfrak{m} \otimes_A M \to \mathfrak{m} \otimes_A M = {}_*M$. The isomorphism $N_* \to (N_*)_*$ follows from this and the adjunction.

2. By Lemma 3.2, the functors $M \mapsto {}_*M$ and $N \mapsto N_*$ induce functors (3.2). By the definition of the quotient category and Lemma 3.2, the almost isomorphisms ${}_*M \to M$ and $N \to N_*$ induce an isomorphism $Hom_A({}_*M, N_*) \to Hom_{A^a}(M^a, N^a)$. By 1., we obtain canonical isomorphisms $Hom_A({}_*M, N_*) \to Hom_A({}_*({}_*M), N) \to Hom_A({}_*M, N)$ and $Hom_A({}_*M, N_*) \to Hom_A(M, (N_*)_*) \to Hom_A(M, N_*)$.

Further, we have a canonical isomorphism $Hom_A({}_*M, N) = Hom_A(\mathfrak{m} \otimes_A M, N) \to Hom_A(\mathfrak{m}, Hom_A(M, N)) = Hom_A(M, N)_*$.

3. Since $M \to M_*$ is an almost isomorphism and $N \mapsto N_*$ induces the right adjoint, we obtain canonical isomorphisms $Hom_{A^a}(M^a, N^a) \leftarrow Hom_{A^a}(M_*^a, N^a) \to Hom_A(M_*, N_*)$. The description of the essential image follows from the isomorphism $N \to (N_*)_*$.

The assertion for the functor ${}_*M$ is proved similarly. □

Let R be an A-algebra. We say that an R-module M is *almost locally free of finite rank* if the canonical map $M \otimes_R Hom_R(M, R) \to End_R(M)$ is an almost isomorphism. For an almost locally free R-module M of finite rank, the *trace map* is defined to be the composition

$$End_R(M) \to End_R(M)_* \xleftarrow{\sim} (M \otimes_R Hom_R(M, R))_* \to R_*,$$

where the last map is induced by the evaluation map $x \otimes f \mapsto f(x)$.

We say that a surjection $R \to S$ of commutative A-algebras *defines an almost open immersion* if S is an almost locally free R-module of finite rank. We say that a morphism $R \to S$ of commutative A-algebras is *almost finite étale* if S is an almost locally free R-module of finite rank and if the surjection $S \otimes_R S \to S$ defines an almost open immersion.

A commutative ring R over A is an object of the category of (A-modules) equipped with the multiplication $R \otimes_A R \to R$ satisfying a certain set of axioms. Similarly, one defines an almost commutative ring R over A as an object of the category of (almost A-modules) equipped with an almost multiplication $R \otimes_A R \to R$ satisfying the corresponding set of axioms. It is established in [5] that one obtains a completely parallel theory.

We will sketch the proof of the equivalence of categories (2.3). The functor $\otimes_{\mathcal{O}_K} K$: ($\mathcal{O}_K$-modules) $\to$ (K-vector spaces) induces a functor $\otimes_{\mathcal{O}_K^a} K$: (almost $\mathcal{O}_K$-modules) $\to$ (K-vector spaces). Let (Perf/K) and (Perf/$K^\flat$) denote the categories of perfectoid algebras. Theorem 2.7 is proved by constructing a diagram

$$(3.3) \qquad (\mathrm{Perf}/K) \leftarrow (\mathrm{Perf}/\mathcal{O}_K^a) \to (\mathrm{Perf}/(\mathcal{O}_K/\varpi\mathcal{O}_K)^a) \leftarrow (\mathrm{Perf}/\mathcal{O}_K^{\flat a}) \to (\mathrm{Perf}/K^\flat)$$

of equivalences of categories. First, we define the categories in the diagram (3.3). By abuse of notation, for an $\mathcal{O}_K^a$-algebra A and for a non-zero element $\varpi \in \mathfrak{m}_K$, let $A/\varpi^{1/p}A$ denote the quotient by the principal ideal generated by an element of valuation $v(\varpi)/p$. An almost $\mathcal{O}_K$-module M is said to be complete if the canonical morphism $M \to (\varprojlim_n M_*/\varpi^n M_*)^a$ is an isomorphism.

Definition 3.4. *Let K be a perfectoid field of characteristic $p > 0$ or of mixed characteristic $(0, p)$. Let $\mathcal{O}_K$ be the valuation ring and ϖ be a non-zero element satisfying $0 < v(\varpi) \leqq v(p)$.*

1. *We say that a* ϖ*-adically complete flat* $\mathcal{O}_K^a$*-algebra* A *is a perfectoid* $\mathcal{O}_K^a$*-algebra if the morphism* $A/\varpi^{1/p}A \to A/\varpi A\colon x \mapsto x^p$ *is an isomorphism. A morphism of perfectoid* $\mathcal{O}_K^a$*-algebras is a morphism of* $\mathcal{O}_K^a$*-algebras.*

2. *We say that a flat* $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$*-algebra* $\bar{A}$ *is a perfectoid* $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$*-algebra if the morphism* $\bar{A}/\varpi^{1/p}\bar{A} \to \bar{A}\colon x \mapsto x^p$ *is an isomorphism. A morphism of perfectoid* $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$*-algebras is a morphism of* $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$*-algebras.*

The following Lemma defines the arrows in (3.3) and shows that the first and the last ones are equivalences of categories.

Lemma 3.5. *Let* K *be a perfectoid field and* $\mathcal{O}_K$ *be the valuation ring.*

1.([7, Proposition 5.5]) *Let* R *be a perfectoid algebra over* K *and* A *be the subring consisting of power-bounded elements. Then, the Frobenius morphism induces an isomorphism* $A/\varpi^{1/p}A \to A/\varpi A$ *and* A^a *is a perfectoid algebra over* $\mathcal{O}_K^a$*.*

2.([7, Lemma 5.6]) *Let* A *be a perfectoid algebra over* $\mathcal{O}_K^a$ *and equip the* K*-algebra* $R = A \otimes_{\mathcal{O}_K^a} K$ *a Banach* K*-algebra structure such that* $A_* \subset R$ *is open and bounded. Then,* R *is a perfectoid algebra over* K *and* $A_* \subset R$ *is the subring consisting of power-bounded elements.*

To prove that the middle arrows are equivalences of categories, we need to find liftings of perfectoid $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$-algebras and their morphisms. This is done by using the theory of cotangent complexes adjusted to the context of almost commutative algebras developed in [5]. It is eventually reduced to that the cotangent complex of a perfect $\mathbf{F}_p$-algebra vanishes. One also needs to check that the composition of (3.3) is actually given by the construction (2.2).

4 Perfectoid spaces

So far, we have studied only local pieces. We make a global construction using the language of adic spaces in the sense of R. Huber [6]. A building block of an adic space is a locally ringed space $\mathrm{Spa}(R, R^+)$ defined for an affinoid k-algebra (R, R^+).

Let k denote a complete valuation field. We call a topological k-algebra R a *Tate* k*-algebra* if there exists a subring R_0 (over $\mathcal{O}_k$) such that aR_0 for $a \in k^\times$ form a basis of open neighborhoods of 0. For a Tate k-algebra R, let A denote the subring consisting of power-bounded elements. A pair (R, R^+) of a Tate k-algebra R and an open and integrally closed subring $R^+ \subset A$ (over $\mathcal{O}_k$) is called an *affinoid* k*-algebra*.

For an affinoid k-algebra (R, R^+), the underlying set of $\mathrm{Spa}(R, R^+)$ is defined as the set of (equivalence classes of) continuous valuations v satisfying $v(f) \geqq 0$ for $f \in R^+$. For a ring R and a totally ordered additive group Γ, a mapping $v\colon R \to \Gamma \cup \{\infty\}$ is called a(n additive) *valuation* if the following conditions are satisfied; $v(xy) = v(x) + v(y)$ and $v(x+y) \geqq \min(v(x), v(y))$ for $x, y \in R$, $v(0) = \infty$ and $v(1) = 0$. If R is a topological ring, a valuation v is said to be *continuous* if $v^{-1}((g, \infty]) \subset R$ is open for every $g \in \Gamma$ such that $g = v(x)$ for some $x \in R$.

For a valuation v, define the value group Γ_v to be the subgroup of Γ generated by $\{g \in \Gamma \mid g = v(x) \text{ for some } x \in R\}$ and the support as a prime ideal of R by $\mathfrak{p}_v = \{x \in R \mid v(x) = \infty\}$. Then, the valuation v induces a valuation of the fraction field of $R/\mathfrak{p}_v$. Valuations v and v' are said to be equivalent, if there exists an isomorphism of totally ordered groups $\Gamma_v \to \Gamma_{v'}$ compatible with v and v'.

For an affinoid k-algebra (R, R^+), we define the set $X = \mathrm{Spa}(R, R^+)$ to be the equivalence classes of continuous valuations v of R such that $v(R^+) \subset [0, \infty]$. We equip X a topology with a basis consisting of *rational subsets* $U\Big(\dfrac{f_1, \ldots, f_n}{g}\Big) = \{v \in X \mid v(f_i) \geqq v(g) \text{ for } i = 1, \ldots, n\}$ defined for $f_1, \ldots, f_n, g \in R$ satisfying $(f_1, \ldots, f_n) = R$.

We define the structure presheaf $\mathcal{O}_X$ on X. Let $f_1, \ldots, f_n, g \in R$ be elements satisfying $(f_1, \ldots, f_n) = R$. We define a topology on the ring $R\Big[\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big] = R\Big[\dfrac{1}{g}\Big]$ by a basis of open neighborhood of 0 consisting of $aR_0\Big[\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big]$ for $a \in k^\times$. Let $R\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ be the completion of $R\Big[\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big]$ and let $R^+\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ denote abusively the completion of the integral closure of $R^+\Big[\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big]$ in $R\Big[\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big]$. Then, the morphism $R \to R\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ induces a homeomorphism

$$\mathrm{Spa}\left(R\Big\langle\frac{f_1}{g}, \ldots, \frac{f_n}{g}\Big\rangle, R^+\Big\langle\frac{f_1}{g}, \ldots, \frac{f_n}{g}\Big\rangle\right) \to U\Big(\frac{f_1, \ldots, f_n}{g}\Big) \subset X = \mathrm{Spa}(R, R^+).$$

Further, the topological rings $R\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ and $R^+\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ depend only on the rational subset $U\Big(\dfrac{f_1, \ldots, f_n}{g}\Big) \subset X = \mathrm{Spa}(R, R^+)$.

We define a presheaf $\mathcal{O}_X$ on X by requiring $\mathcal{O}_X(U) = R\Big\langle\dfrac{f_1}{g}, \ldots, \dfrac{f_n}{g}\Big\rangle$ for rational subsets $U = U\Big(\dfrac{f_1, \ldots, f_n}{g}\Big)$. For each point x of $X = \mathrm{Spa}(R, R^+)$, the (equivalence class of) continuous valuation of R induces a(n equivalence class of) continuous valuation of the local ring $\mathcal{O}_{X,x}$. We regard $X = \mathrm{Spa}(R, R^+)$ as a topological space equipped with the presheaf $\mathcal{O}_X$ of topological rings together with the (equivalence class of) continuous valuation of the local ring at each point.

If the presheaf $\mathcal{O}_X$ is a sheaf, we call $X = \mathrm{Spa}(R, R^+)$ equipped with these structures an *affinoid adic space*. An *adic space* X is defined to be a topological space equipped with an sheaf $\mathcal{O}_X$ of topological rings together with a(n equivalence class of) continuous valuation of the local ring at each point, that is locally isomorphic to an affinoid adic space.

For a perfectoid affinoid algebra (R, R^+), the presheaf $\mathcal{O}_X$ on $X = \mathrm{Spa}(R, R^+)$ is a sheaf. Let K be a perfectoid field. We say that an affinoid K-algebra (R, R^+) is a *perfectoid* affinoid K-algebra if R is a perfectoid K-algebra. The tilting functor (2.3) induces an equivalence of categories

$$(4.1) \qquad \text{(perfectoid affinoid } K\text{-algebras)} \to \text{(perfectoid affinoid } K^\flat\text{-algebras)}$$

sending (R, R^+) to $(R^\flat, R^{\flat+})$. Here $R^{\flat+}$ denotes the open and integrally closed subalgebra satisfying $\mathfrak{m}R^{\flat\circ} \subset R^{\flat+} \subset R^{\flat\circ}$ and corresponding to $\mathfrak{m}R^\circ \subset R^+ \subset R^\circ$.

Theorem 4.1 ([7, Theorem 6.3]). *Let K be a perfectoid field, (R, R^+) be a perfectoid affinoid K-algebra and let $X = \mathrm{Spa}(R, R^+)$ be the associated topological space equipped with the structures as above.*

1. *The presheaf $\mathcal{O}_X$ is a sheaf.*

2. *Assume K is of mixed characteristic $(0,p)$ and let $K^\flat$ and $(R^\flat, R^{\flat+})$ be the tilts of K and (R, R^+). Then, there exists a unique homeomorphism $\flat\colon X \to X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$ compatible with the construction of rational subsets U and isomorphisms $\mathcal{O}_X(U)^\flat \to \mathcal{O}_{X^\flat}(U^\flat)$.*

Theorem 4.1.1 is a part of an analogue for perfectoid algebras of Tate's acyclicity theorem in rigid geometry. Theorem 4.1.2 is proved by using the equivalence of categories established in Theorem 2.7 and some approximation property. Theorem 4.1.1 is reduced to the case of characteristic $p > 0$ by Theorem 4.1.2. In the latter case, it is proved by reducing eventually to Tate's acyclicity theorem in the classical case.

Theorem 4.1 enables us to define perfectoid spaces.

Definition 4.2. *An adic space over a perfectoid field K is called a* perfectoid space *if it is locally isomorphic to* $\mathrm{Spa}(R, R^+)$ *for an perfectoid affinoid K-algebra (R, R^+).*

Theorem 2.7 and Theorem 4.1 immediately imply an equivalence of categories

$$\text{(perfectoid spaces over } K) \to \text{(perfectoid spaces over } K^\flat) \tag{4.2}$$

attaching to a perfectoid space X over K its tilt $X^\flat$ over $K^\flat$. Further Theorem 4.1 implies a homeomorphism $X \to X^\flat$ compatible with tilting construction on the structure sheaves.

To prove (cases of) the weight-monodromy conjecture, we still need to understand the compatibility of étale topology with the equivalence of categories (4.2).

5 Almost purity theorem and étale topology

The most important point in the theory is the following generalization of the almost purity theorem of Faltings [2].

Theorem 5.1 ([7, Theorem 7.9]). *Let K be a perfectoid field, R be a perfectoid K-algebra and A be the perfectoid $\mathcal{O}_K^a$-algebra associated to the subring of R consisting of power-bounded elements. Then, a finite étale R-algebra S is a perfectoid K-algebra and the $\mathcal{O}_K^a$-algebra B associated to the subring of S consisting of power-bounded elements is a perfectoid $\mathcal{O}_K^a$-algebra and is almost finite étale over A.*

We sketch the proof. Let R be a perfectoid K-algebra and let A be the perfectoid $\mathcal{O}_K^a$-algebra associated to the subring consisting of power-bounded elements. Setting $\bar{A} = A/\varpi A$ and we consider the following diagram of categories

$$(\mathrm{F\acute{E}t}/R) \leftarrow (\mathrm{F\acute{E}t}/A) \to (\mathrm{F\acute{E}t}/\bar{A}) \leftarrow (\mathrm{F\acute{E}t}/A^\flat) \to (\mathrm{F\acute{E}t}/R^\flat) \tag{5.1}$$

consisting of (almost) finite étale algebras, similar to the diagram (3.3). Using the theory of almost commutative algebra [5], one checks ([7, Theorem 4.17, Proposition 5.22]) that the middle two arrows are equivalences of categories, that a finite étale A-algebra is a perfectoid $\mathcal{O}_K^a$-algebra and that a finite étale $\bar{A}$-algebra is a perfectoid $(\mathcal{O}_K/\varpi\mathcal{O}_K)^a$-algebra, respectively. This implies that the middle three categories in (5.1) are subcategories of the corresponding categories in (3.3) and that the middle two arrows are compatible to each other.

Using Lemma 2.5 in characteristic $p > 0$, it is proved rather directly ([7, Proposition 5.23]) that the last arrow in (5.1) is an equivalence of categories and is compatible with that in (3.3). By what is already proven and by Theorem 2.7, the proof of Theorem 5.1 is reduced to showing that the composite functor from the right end to the left end of (5.1) is essentially surjective. This is proved first in the case where R is a field. Then, applying this to the residue field of each point and using that the local rings are henselian, we show that we obtain an etale covering locally. Then, by using the sheaf property, Theorem 4.1.1, we conclude the proof.

An isomorphism of étale sites follows directly from Theorem 5.1. First, we define the étale site. A morphism $X \to Y$ of adic spaces is said to be étale if locally on Y, there exists an open covering by affinoids $V = \mathrm{Spa}(R)$ and an almost finite étale R-algebra S such that $X \times_Y V \to V$ factors through an open immersion $X \times_Y V \to U = \mathrm{Spa}(S)$ over V.

Definition 5.2. *Let X be a perfectoid space. Then, the underlying category of the étale site $X_{\text{ét}}$ consists of perfectoid spaces étale over X. A family of morphisms in $X_{\text{ét}}$ is a covering if the family of underlying continuous mappings is a covering.*

Theorem 5.1 and Theorem 4.1.2 imply the following.

Corollary 5.3 ([7, Theorem 7.12]). *The tilting induces an isomorphism $X_{\text{ét}} \to X^\flat_{\text{ét}}$ of the étale sites.*

This completes the construction of the bottom arrow in the diagram (1.1).

6 Complete intersections in toric varieties

We recall the definition of toric varieties. Let P be a free abelian group of finite rank and $N = Hom(P, \mathbf{Z})$ be the dual. A cone $\sigma \subset N_{\mathbf{R}}$ is said to be rational if it is spanned by a finitely many elements of N. It is said to be strongly convex if it does not contain a line. A sub cone τ of a rational cone σ is called a face if there exists $a \in P$ such that $f(a) \geqq 0$ for every $f \in N$ and $\tau = \{f \in \sigma \mid f(a) = 0\}$. For a rational cone $\sigma \subset N_{\mathbf{R}}$, let $\sigma^\vee \subset P_{\mathbf{R}}$ denote the dual $\{a \in P_{\mathbf{R}} \mid f(a) \geqq 0 \text{ for } f \in \sigma\}$ and set $P_\sigma = P \cap \sigma^\vee$.

A fan Σ is a finite set of strongly convex rational cones of $N_{\mathbf{R}}$ such that a face τ of an element σ of Σ is an element of Σ and the intersection $\sigma \cap \tau$ of elements σ, τ of Σ is a face of σ and of τ. We say a fan Σ is proper if the union $\bigcup_{\sigma \in \Sigma} \sigma$ is equal to $N_{\mathbf{R}}$. We say a fan Σ is smooth, if the monoid $N \cap \sigma$ is isomorphic to the product of copies of $\mathbf{N}$ for every $\sigma \in \Sigma$.

For a fan Σ and a field k, we define the toric variety $X_{\Sigma,k}$ by patching $X_{\sigma,k} = \mathrm{Spec}\, k[P_\sigma]$ along $X_{\sigma\cap\tau,k}$. The toric variety $X_{\Sigma,k}$ has a natural action of the torus $T = \mathrm{Spec}\, k[P] \subset X_{\Sigma,k}$ defined by $k[P_\sigma] \to k[P_\sigma] \otimes k[P] = k[P_\sigma \times P]$ induced by $P_\sigma \to P_\sigma \times P\colon a \mapsto (a, a)$. If Σ is a proper fan, the toric variety $X_{\Sigma,k}$ is proper. If Σ is a smooth fan, the toric variety $X_{\Sigma,k}$ is smooth. If τ is a one dimensional face of a smooth fan Σ, the ideals of $k[P_\sigma]$ generated by $\{a \in P_\sigma \mid f(a) > 0 \text{ for } f \in \tau\}$ define a smooth irreducible divisor D_τ of $X_{\Sigma,k}$.

Example 6.1. *Set $P = \mathrm{Ker}(\mathrm{sum}\colon \mathbf{Z}^{n+1} \to \mathbf{Z})$ and $N = \mathbf{Z}^{n+1}/\Delta\mathbf{Z}$. For a subset $\sigma \subsetneqq \{0, \ldots, n\}$, let σ also denote abusively the cone of $N_{\mathbf{R}}$ spanned by the images of the standard basis $e_i \in \mathbf{Z}^{n+1}$ for $i \in \sigma$. Let Σ be the set of cones σ associated to subsets $\sigma \subsetneqq \{0, \ldots, n\}$.*

For $i = 0, \ldots, n$, set $\sigma_i = \{j \mid j \neq i\}$. Then, P_{σ_i} is generated by $x_j - x_i$ where x_j denote the standard basis of $\mathbf{Z}^{n+1} \supset P$. Thus the toric variety $X_{\Sigma,k}$ is defined by patching $\mathrm{Spec}\, k\left[\frac{X_j}{X_i}\right]$ *and is nothing but the projective space $\mathbf{P}^n_k$.*

Let K be a perfectoid field and Σ be a proper smooth fan. For a face $\sigma \in \Sigma$, set $P_\sigma^{1/p^\infty} = P_{\mathbf{Z}[\frac{1}{p}]} \cap \sigma^\vee$ and $K\langle P_\sigma\rangle, \mathcal{O}_K\langle P_\sigma\rangle, K\langle P_\sigma^{1/p^\infty}\rangle$ and $\mathcal{O}_K\langle P_\sigma^{1/p^\infty}\rangle$ be the ϖ-adic completions. Then, we define an adic space $X^{\mathrm{ad}}_{\Sigma,K}$ and a perfectoid space $X^{\mathrm{perf}}_{\Sigma,K}$ by patching $\mathrm{Spa}(K\langle P_\sigma\rangle, \mathcal{O}_K\langle P_\sigma\rangle)$ and $\mathrm{Spa}(K\langle P_\sigma^{1/p^\infty}\rangle, \mathcal{O}_K\langle P_\sigma^{1/p^\infty}\rangle)$ respectively. The construction of $X^{\mathrm{perf}}_{\Sigma,K}$ corresponds to the vertical arrow with ? in the diagram (1.1).

Assume that K is of mixed characteristic $(0,p)$. Then the tilt of $X^{\mathrm{perf}}_{\Sigma,K}$ is $X^{\mathrm{perf}}_{\Sigma,K^\flat}$ and we obtain morphisms of étale sites;

$$X^{\mathrm{ad}}_{\Sigma,K,\text{ét}} \longleftarrow X^{\mathrm{perf}}_{\Sigma,K,\text{ét}} \xrightarrow{\flat} X^{\mathrm{perf}}_{\Sigma,K^\flat,\text{ét}} \longrightarrow X^{\mathrm{ad}}_{\Sigma,K^\flat,\text{ét}} \tag{6.1}$$

By Corollary 5.3, the middle arrow is an isomorphism. Since a surjective radicial morphism induces an isomorphism on the étale site, the right arrow is also an isomorphism. For a prime number $\ell \neq p$, the right arrow induces an isomorphism $H^q(X^{\mathrm{ad}}_{\Sigma,\bar{K},\text{ét}}, \mathbf{Q}_\ell) \to H^q(X^{\mathrm{perf}}_{\Sigma,\bar{K},\text{ét}}, \mathbf{Q}_\ell)$ by the proper base change theorem. This means that we do not lose too much information by going down in the diagram (1.1).

Let k be a field and Σ be a smooth proper fan. A closed subscheme Y of $X_{\Sigma,k}$ defined by a non-zero section of an invertible sheaf defined by a linear combination of D_τ for one-dimensional faces is called a hypersurface of $X_{\Sigma,k}$.

Theorem 6.2 ([7, Theorem 9.6]). *Let Y be a smooth closed subscheme of codimension c of a smooth projective toric variety $X_{\Sigma,k}$ over a finite extension k of $\mathbf{Q}_p$. If there exist hypersurfaces $H_1, \ldots, H_c$ of $X_{\Sigma,k}$ such that the underlying set of Y is equal to that of the intersection $H_1 \cap \ldots \cap H_c$, then the weight-monodromy conjecture is true for Y.*

We take a perfectoid extension K of k, for example the completion of $k(\zeta_{p^\infty})$. We prove Theorem 6.2 by constructing a proper smooth variety Z of dimension $\dim Y$ over $K^\flat$ defined over a dense subfield $k_0 \subset K^\flat$ that is a function field of one variable over $\mathbf{F}_p$ and a generically finite morphism $Z \to X_{\Sigma,K^\flat}$ and an injection $H^q(Y_{\bar{K}}, \mathbf{Q}_\ell) \to H^q(Z_{\bar{K}^\flat}, \mathbf{Q}_\ell)$ compatible with the canonical isomorphism $\mathrm{Gal}(\bar{K}/K) \leftarrow \mathrm{Gal}(\bar{K}^\flat/K^\flat)$ such that the image is a direct summand.

By the theory of étale cohomology of adic space, there exists an open neighborhood $\widetilde{Y}$ of Y^{ad}_K such that the pull-back $H^q(\widetilde{Y}_{\bar{K},\text{ét}}, \mathbf{Q}_\ell) \to H^q(Y_{\bar{K},\text{ét}}, \mathbf{Q}_\ell)$ is an isomorphism for every $q \geqq 0$. Let $\pi\colon X^{\mathrm{ad}}_{\Sigma,K^\flat} \to X^{\mathrm{ad}}_{\Sigma,K}$ denote the composition of continuous mappings similarly defined as (6.1). Then, by approximation, we find a closed subscheme $Z_0 \subset X_{\Sigma,K^\flat}$ of codimension c defined over a subfield $k_0 \subset K^\flat$ that is a function field of one variable over $\mathbf{F}_p$ such that $Z^{\mathrm{ad}}_{0,K^\flat}$ is contained in $\flat(\widetilde{Y})$. By a theorem of de Jong, we find an alteration $Z \to Z_0$ proper smooth over k_0. The weight-monodromy conjecture for Z is known by Deligne in [1].

The diagram

$$\begin{array}{ccc} H^q(X_{\Sigma,\bar{K},\text{ét}},\mathbf{Q}_\ell) & \xrightarrow{\simeq} & H^q(X_{\Sigma,\bar{K}^\flat,\text{ét}},\mathbf{Q}_\ell) \\ \downarrow & & \downarrow \\ H^q(\widetilde{Y}_{\bar{K},\text{ét}},\mathbf{Q}_\ell) & \longrightarrow & H^q(\flat(\widetilde{Y}_{\bar{K}})_{\text{ét}},\mathbf{Q}_\ell) \\ \simeq\downarrow & & \downarrow \\ H^q(Y_{\bar{K},\text{ét}},\mathbf{Q}_\ell) & & H^q(Z_{\bar{K}^\flat,\text{ét}},\mathbf{Q}_\ell) \end{array} \tag{6.2}$$

defines a morphism $H^q(Y_{\bar{K}},\mathbf{Q}_\ell) \to H^q(Z_{\bar{K}^\flat},\mathbf{Q}_\ell)$ compatible with the isomorphism (1.2). The Poincaré duality implies that, in order to show that it satisfies the required property, it suffices to show that it is non-zero for $q = 2\dim Y$. If it was zero, the composition of the right vertical arrows would be zero. This contradicts the fact that the $\dim Y$-th power of the class of an ample divisor of $X_{\Sigma,\bar{K}^\flat}$ is non-zero in $H^{2\dim Y}(Z_{\bar{K}^\flat,\text{ét}},\mathbf{Q}_\ell)$. Since the weight-monodromy conjecture is known for $H^q(Z_{\bar{K}^\flat,\text{ét}},\mathbf{Q}_\ell)$ by Deligne [1], it also holds for a direct summand $H^q(Y_{\bar{K},\text{ét}},\mathbf{Q}_\ell)$.

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