Ramification groups

1. Def'n.
2. Result 1.
3. More def'n
4. Result 2.
5. Idea of Pf.

1. K complete discrete val. fd.
L/K finite Galois ext'n.
lower numbering fil. $G_i := \mathrm{Ker}(G \to \mathrm{Aut}\, \mathcal{O}_L/m_L^{i})$.
log $G_{i\,\log} = \mathrm{Ker}(G \to \mathrm{Aut}(L^\times/1+m_L^i))$

$$G_{i+1} \subset G_{i\,\log} \subset G_i$$
$$\overset{\text{T}}{=} \text{ if } F \text{ is perfect}$$

with A. Abbes
upper numbering $\mathcal{O}_L = \mathcal{O}_K[X]/f$.
$G = \{x \in \mathcal{O}_{\bar K}^n \mid f(x) = 0\}$ $G_i = G \cap \{\mathrm{ord}(x - x_0) \geq i\}$

$G^r = G \cap \left(\text{conn cpt of } \{x \in \mathcal{O}_{\bar K}^n \mid \mathrm{ord}\, f(x) \geq r\} \text{ containing } 1 \in G\right)$
Swan-conductor

algebraic vs rigid blow up $\longleftrightarrow$ shrink radius

$Q = \mathrm{Spf}\, A \leftarrow \mathrm{Spf}\, \mathcal{O}_L$ A smooth$/\mathcal{O}_K$
A/I

K'/K finite $\mathrm{Spf}\; A \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}\left[\frac{I}{\pi^r}\right] = Q_{\mathcal{O}_{K'}}^{[r]}$ $\pi_r \in \mathcal{O}_{K'}$ $\mathrm{ord}\, \pi_r = r$

normalizat'n $= Q_{\mathcal{O}_{K'}}^{(r)}$

Reduced fiber thm B.L.R $Q_{\mathcal{O}_{K'}}^{(r)} \otimes_{\mathcal{O}_{K'}} \bar F$ reduced
$0 = r_0 \leq r_1 \leq \cdots \leq r_n$ G-R Stable under b.c
t.s.t. $= Q_{\bar F}^{(r)} \longrightarrow Q_{\bar F}^{\mathrm{orig}}$ f.n.g. vector space
$G \twoheadrightarrow \pi_0(Q_{\bar F}^{(r)}) = G/G^r$ indep of choices.
$G^1 = I$. $G^{1+} = P$. G^r is const in $(r_i, r_{i+1}]$, (r_n, ∞)

Theorem 1. L/K finite Galois extension. $r>1$. $\in\mathbb{Q}$

Assume that either p is not a uniformizer or

F is perfect $\quad \Omega=\Omega(K)$

1. (L. Xiao) $\quad Gr^r G$ abelian & killed by p

2. There exists a can. inj

$$\mathrm{char}\ \mathrm{Hom}(Gr^r G, \mathbb{F}_p) \to \mathrm{Hom}_{\bar F}(m_{\bar K}^r/m_{\bar K}^{r+}, \varinjlim_{e} \Omega_{\bar F}(\Omega \otimes \bar F))$$

$p \mid e \quad \Omega\otimes_F \bar F$

$\Omega = \Omega^1_{O_K/m_K^2} \otimes_{O_K} F$ if

$0 \to m_K/m_K^2 \to \Omega^1 \otimes \bar F \to \Omega^1_F \to 0$ $\quad (m_{\bar K}^r/m_{\bar K}^{r+}) \otimes \bar F$ if F perfect

known. F perfect, $\mathrm{ch}\,K=p>0$, log ...

logarithmic variant. $e=e(L/K)$ $\quad K_e$ log smooth ext of ram index e.

$p\nmid e \quad K_e = K[t]/(t^e-\pi_K)$ $\quad \Omega(K)$

$p\mid e \quad \mathrm{Frac}\big(O_K[u^{\pm 1}, t]/(ut^e-\pi_K)\big)^{\wedge}_{(t)}$

$$G^r_{\log} = \mathrm{Gal}(LK_e/K_e)^{er}$$

Abelian (Kato) $\quad$ fil on $G \leftarrow$ fil on $X_K = \mathrm{Hom}(G_K^{ab}, \mathbb{Q}/\mathbb{Z})$
$= H^1(K, \mathbb{Q}/\mathbb{Z}) = H^2(K, \mathbb{Z})$

$(\ ,\ ): X_K \times K^\times \to \mathrm{Br}(K) \qquad H^2(K,\mathbb{Z}) \times H^0(K, \mathbb{G}_m) \to H^2(K, \mathbb{G}_m)$

$n\in\mathbb{N}$, $n\geq 1$

$$\mathrm{Fil}^n X_K = \{x\in X_K \mid (x, 1+m_K^n m_{K'})_{K'} = 0 \quad \forall K'/K\}$$

rsw $\quad Gr^n X_K \to \mathrm{Hom}_F(m_K^n/m_K^{n+1}, \Omega^1_{O_K}(\log)\otimes_{O_K} F)$

$0 \to \Omega^1_F \to \Omega^1(\log)\otimes F \xrightarrow{\mathrm{res}} F \to 0$ $\quad 0\to \Omega^1_F \to \ast \to F \to 0$

$1+m_K^n/m_K^{n+1}$ $\quad \mathrm{Br}_n(K)\{t,\ \}$ $\quad \Omega^1_{O_K}(\log)\otimes_{O_K} F$ $\quad \mathrm{Br}(K) \to \mathrm{Br}^{ur}(K)$

$t\ m_K^n/m_K^{n+1} \quad \Omega^1_{K}(\log) \to \Omega^1_F \to \Omega^1_F \to 0$ $\quad \mathrm{Br}(K) \to H^1_K \to \mathbb{Z} \to 0$

$0 \to \mathbb{G}_m \to \ldots$

Th2 (K-S) L/K finite abelian

(no cond. on K)

1. log fil = Kato's fil

2 log analogue of ch = rsw.

Known ch $K = p > 0$ using ASW.

Sketch of Pf. Similar

Reduction to monogenic case.

1. Construct K_1/K with following properties.

(i) F_1 perfect $e(K_1/K) = 1$

(ii) can morph $\Omega^1_{O_K/\mathbb{Z}_p} \to \Omega^1_{O_{K_1}/\mathbb{Z}_p}$ induces an inj

$$S^\circ(\Omega^1_{O_K/\mathbb{Z}_p} \otimes \bar{F}) \to S^\circ(\Omega^1_{O_{K_1}/\mathbb{Z}_p} \otimes_{O_{K_1}} \bar{F}_1) \quad \text{"} m_{K_1}/m_{K_1}^2 \otimes_{F_1} \bar{F}_1 .$$

Prop 1. $L_1 = L \otimes K_1$ ~~is a Galois ext of K_1 and~~ $e = 1$.

$G^r = G_1^r$

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$$\begin{array}{ccc} \mathrm{Hom}(G^r_r G, \mathbb{F}_p) & \overset{\exists 1}{\dashrightarrow} & \mathrm{Hom}(m_K^r/m_K^{r+}, S^{p^e}(\Omega^1 \otimes \bar{F})) \\ \cup & & \downarrow \text{(ii)} \\ \mathrm{Hom}(G^r_r G_1, \mathbb{F}_p) & \longrightarrow & \mathrm{Hom}(m_{K_1}^r/m_{K_1}^{r+}, m_{K_1}/m_{K_1}^2 \otimes_{F_1} \bar{F}) \\ & & \exists \end{array}$$

2 Construct K_2/K

(i) $[F_2 : F_2^p] = p$. $e(L_2/K_2) = 1$. Ξ_{pp}

(ii) $\Omega^1_{O_K}(\log) \otimes \bar{F} \xrightarrow{\text{inj}} \Omega^1_{F_2} \otimes_{F_2} \bar{F}_2 \subset \Omega^1_{O_{K_2}}(\log) \otimes_{O_{K_2}} \bar{F}_2$