~~s.t. ...~~

$n \geq r \geq 0$ ~~integer~~ $d_0, \ldots, d_r > 1$ integers

$f_0, \ldots, f_r$ homogeneous ply of degree $d_0, \ldots, d_r$ of $T_0, \ldots, T_n$

$X \subset \mathbb{P}^{n+1}$ $f_0 = \ldots = f_r = 0$ smooth

Assume $n \equiv r$ we consider

$\det H^{n-r}(X, \mathbb{Q}_\ell(\frac{n-r}{2}))$

Defin. $\mathrm{disc}(f_0, \ldots f_r)$ homogeneous pol of coeff of $f_0, \ldots f_r$. well-defined upto ± 1

If $n \equiv r$. one can det. sign by requiring

$\mathrm{disc} \equiv \square$ modulo 4.

and $\det = \sqrt{\mathrm{disc}}$. if $\mathrm{char} \neq 2$

$t^2 + t = b/a^2$ $\quad$ $\mathrm{disc} = a^2 + 4b$ if $d_n = 2$

1. Def of disc. and outline of Pf.
2. Reduction to hyp-surface sections
3. Dual varieties
4. Boundary

2 translate pbm on c.i to that on h.s s'n. — Smoothness — Cohomology disc det

3 advantage of h.s. s'n = apply gen. thy of dual varieties

4 ~~Study the~~ properties of disc. det. are deduced from their behavior at boundary ← gen on bdy.

c.i & h.ss have odp.

1 $E = \mathbb{Z}^{n+2}$ $\quad S^\cdot E = \mathbb{Z}[T_0, \dots, T_n]$ $\quad \mathbb{P}^{n+1} = \mathbb{P}(E) = \mathrm{Proj}\, S^\cdot E$

$S^d E = \bigoplus \mathbb{Z} T^I$ $\quad (S^d E)^\vee = \bigoplus \mathbb{Z} C_I$

$d_0, \dots, d_r$

$V = \bigoplus S^{d_i} E$ $\quad V^\vee = \bigoplus_{i=0}^{r} \bigoplus_{|I|=d_i} C_I^{(k)}$ $\quad P^\vee = \mathbb{P}(V^\vee)$

$X \subset \mathbb{P}^{n+1} \times P^\vee$ $\quad \{F_0 = \dots = F_r = 0\}$

$f \searrow \downarrow \mathrm{pr.}$ $\quad F_i = \sum_{|I|=d_i} T^I C_I^{(i)}$

$P^\vee$

$U \subset P^\vee$ max. open s.t. f is smooth (of rel dim $n-r$)

(2.2) 32

Prop 1 $\exists\, m > 0$ $\left(= (n-r+2)\binom{n+2}{r} d^r (d-1)^{n-r+1} \text{ if } d_0 = \dots = d_r = d \right)$

$\exists$ disc $\in \Gamma(P^\vee, \mathcal{O}(m))$ geom. irred hom. pol of $C_I^{(i)}$ of degree m with coeff in $\mathbb{Z}$ uniquely det'd up to ± 1

s.t. $U = P^\vee \setminus D$ $\quad D = (\mathrm{disc} = 0)$

Prop 2 Let p be a prime

1. Except for $p=2$ & $n-r$ even, disc mod p is geom. irred
2. Assume $p=2$ & $n-r$ even. Then m is even, $\exists\, A, B$ s.t. disc $= A^2 + 4B$ & A mod 2 geom. irred

Thm Assume $n-r$ even

1. char $k \neq 2$ $\quad \det V = \sqrt{\mathrm{disc}}$
2. $k = 2$ $\quad t^2 + t = B/A^2$

Idea of Pf. Def'n a p. sm sch $T \subset P = P(V)$

1. ~~We put $T = \mathbb{P}(\mathcal{E})$ $\mathcal{E} = \mathcal{O}(d_0) \oplus \cdots \oplus \mathcal{O}(d_r) / \mathbb{P}^{n+1}$~~
~~$T \subset \mathbb{P}(V)$ $V = \Gamma(P, \mathcal{O}(1))$~~
~~$= \Gamma(\mathbb{P}^{n+1}_{\mathbb{Z}}, \mathcal{E})$~~

and define a h.s. sm $Y \subset T$ by
$(F_0, \ldots, F_r) = 0$.
Study the rel'n $X \subset P$ and $Y \subset T$
smoothness. cohomology. etc

2. $H^1(U, \mathbb{Z}/2\mathbb{Z}) \to H^1(I_\eta, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$
disc, det $\longmapsto$ unique non zero elt.
ker $= 0$.

2 $X \subset \mathbb{P}^{n+1} \times P^\nu$
$\mathcal{E} = \oplus \mathcal{O}(d_i)$ $\hat{\mathcal{E}} = \oplus \mathcal{O}(d_i, 1)$ $\nu \le r+1$

$$\begin{array}{c} T = \mathbb{P}(\mathcal{E}) \times P^\nu = \mathbb{P}(\hat{\mathcal{E}}). \\ \downarrow \\ \mathbb{P}^{n+1} \times P^\nu \end{array}$$

$$\begin{array}{ccc} s = (F_0, \ldots, F_r) \in \Gamma(\mathbb{P}^{n+1} \times P^\nu, \hat{\mathcal{E}}) & & X = D(s) \\ \downarrow & & \\ \tilde{s} \in \Gamma(\mathbb{P}(\hat{\mathcal{E}}), \mathcal{O}(1)) & & Y = D(\tilde{s}) \end{array}$$

__Prop__

$$\begin{tikzcd} Y \arrow[r, hook] \arrow[d, "q"'] & T \arrow[d, "p"] \\ X \arrow[r, hook] \arrow[dr] & P \arrow[d] \\ & S \end{tikzcd}$$

__Prop 1__ TFAE
(1) $X \to S$ smooth $X \subset P$ reg im of codim $r+1$
(2) $Y \to S$ smooth $Y \subset P$ reg im of cod 1.

__Prop 2__ $R^i p_* \mathbb{Q}_\ell \to R^i q_* \mathbb{Q}_\ell$ isom unless $i \neq 2r$

$i = 2r$

$$\begin{array}{ccc} R^i p_* \mathbb{Q}_\ell & \to & R^i q_* \mathbb{Q}_\ell \\ \downarrow \wr & & \downarrow \wr \\ \mathbb{Q}_{\ell, P}(-r) & \to & \mathbb{Q}_{\ell, X}(-r) \end{array}$$

~~Cor~~ Define

$$\begin{array}{ccc} X \hookrightarrow P & & Y \to T \\ f \searrow \ \downarrow \bar{f} & & g \searrow \ \downarrow \bar{g} \\ S & & S \end{array}$$

$R^i f_* \mathrm{prim} = \mathrm{Coker}\,(R^i \bar{f}_* \to R^i f_*)$

$R g_*$.

$R^i f_* \mathrm{prim} = 0$ unless $i = n - r$

$R^i g_* \mathrm{p} = 0$ " $i = n + r$.

and $R^{n+r} g_* \mathrm{p} \xrightarrow{\sim} R^{n-r} f_* \mathrm{p}\,(-r)$

3 $T = \mathbb{P}(\mathcal{E}) \times P^\vee \to P \times P^\vee$
$\downarrow$
$P^\vee$

~~Prop~~. S ~~regular flat / $\mathbb{Z}$~~ $T \to S$ proj smooth

$$\begin{array}{c} T \to P = \mathbb{P}(V) \\ \downarrow \\ S \end{array}$$

closed imm, not line subsp

$P^\vee = \mathbb{P}(V^\vee)$

$\mathbb{P}^\vee(N_{T/P}) \subset \mathbb{P}^\vee(\Omega_{P/S}|_T) \to P \times P^\vee$
$\searrow \quad \downarrow$
~~S~~ $P^\vee$

univ. h.s. sn or $\Omega^{(1)}_{P/S} \to V \otimes \mathcal{O}_P \to \mathcal{O}(1)$

<u>Prop</u>. Assume S regular flat / ~~Am~~ $\mathbb{Z}$

Then the image $D = T^\vee \subset P^\vee$ is a Cartier divisor . flat over S.

If ~~Ts~~ $T \to S$ has geom reduced fiber $\Rightarrow^{\text{so is}}$ $D \to S$ ~~then~~

Def ~~Dis~~ Discriminant Prop 1

$$H^1(U[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(I_{\bar\eta}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$$

det, disc $\longmapsto$ $\cdot$ $\neq 0$

disc. irreducibility.

det P.L formula

$$\det - \mathrm{disc} \in \ker = H^1(P[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z}) = H^1(\mathbb{Z}[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z}) = \langle -1, 2 \rangle$$

$$H^1(U, \mathbb{Z}/2\mathbb{Z}) \hookrightarrow H^1(U, \mu_2) \longrightarrow H^1(U[\tfrac{1}{2}], \mathbb{Z}/2\mathbb{Z})$$

det disc

$$\ker(\operatorname{Im}(H^1(U, \mu_2) \to H^1(\ \ )) \to H^1(I_{\bar\eta})) = \langle -1 \rangle.$$

$$\ker(H^1(U) \to H^1(I_{\bar\eta})) = H^1(P, \mathbb{Z}/2\mathbb{Z}) = H^1(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = 0$$

Suffices to show that disc is in the image of

inj. $H^1(U, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^1(U, \mu_2)$

i.e disc $= A^2 + 4B$

4. $$\begin{array}{ccccc} \Delta = \mathbb{D}^\vee(N_{T/P}) & \hookrightarrow & \mathbb{P}^\vee(\Omega_{P/S}|_T) & \hookrightarrow & T \times P^\vee \\ \downarrow & & \downarrow & \swarrow & \\ D & \hookrightarrow & P^\vee & & \end{array}$$

$W \subset \Delta$ max open subsch w is an odp of the fiber $\mathbb{P}^\vee(\Omega_{P/S}|_T) \to P^\vee$

o.d.p $f = 0$ $\bar f \in m_w^2/m_w^3$ defines a smooth quad in $\mathbb{P}(m_w/m_w^2)$

$W' \subset W$ w is the unique s.p.

- $W'[\tfrac{1}{2}] \hookrightarrow P^\vee[\tfrac{1}{2}]$ imm. $\Rightarrow$ invsed.
- $W'_{\mathbb{Z}/4\mathbb{Z}} \longrightarrow P^\vee_{\mathbb{Z}/4\mathbb{Z}}$ radicial. degree 2.