Euler-Poincaré characteristic and ramification of an ℓ-adic sheaf

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Abstract

We introduce the characteristic class and the Swan class for an ℓ-adic sheaf. We show that they compute the Euler characteristic and give their relations.

Plan

1. Characteristic class.
2. Swan class and the Grothendieck-Ogg-Shafarevich formula.
3. Relation between the characteristic class and the Swan class.

1 Characteristic class.

Let F be a perfect field of characteristic $p > 0$. Let X be a separated scheme of finite type over F and $\mathcal{F}$ be an ℓ-adic sheaf on X. Then the characteristic class

$$C(\mathcal{F}) \in H^0(X, K_X)$$

is defined as follows. Here and in the following $K_X = a^!\Lambda$ where $a : X \to F$. Hence if X is smooth of dimension d, the characteristic class $C(\mathcal{F})$ is defined in $H^{2d}(X, \Lambda(d))$.

We consider

$$\begin{aligned} 1 \in Hom(\mathcal{F}, \mathcal{F}) &= H^0_X(X \times X, R\mathcal{H}om(p_2^*\mathcal{F}, p_1^!\mathcal{F})) \\ &= H^0_X(X \times X, R\mathcal{H}om(p_1^*\mathcal{F}, p_2^!\mathcal{F})). \end{aligned}$$

By the natural pairing, $R\mathcal{H}om(p_2^*\mathcal{F}, p_1^!\mathcal{F}) \otimes R\mathcal{H}om(p_1^*\mathcal{F}, p_2^!\mathcal{F}) \to K_{X\times X}$, their pairing is defined and gives the characteristic class as

$$C(\mathcal{F}) = \langle 1, 1\rangle = H^0_X(X \times X, K_{X\times X}) = H^0(X, K_X).$$

If X is smooth of dimension d and $\mathcal{F}$ is smooth of rank r, we have $C(\mathcal{F}) = r \cdot (-1)^d c_d(\Omega^1_{X/F})$.

If X is proper, the Lefschetz trace formula in SGA 5 gives

$$\mathrm{Tr} C(\mathcal{F}) = \chi(X_{\bar{F}}, \mathcal{F}).$$

2 Swan class and the G-O-S formula

Let $j : U \to X$ be an open immersion and consider $j_!\mathcal{F}$ where $\mathcal{F}$ is a smooth sheaf on U. We define the Swan class Sw $\mathcal{F}$ in $CH_0(X \setminus U)_{\mathbb{Q}}$.

Assume for simplicity that there is a finite étale Galois covering $V \to U$ trivializing $\mathcal{F}$. In general, we consider the Brauer trace.

First recall the case of curves. Let G be the Galois group and take $\sigma \in G$. Let $X \supset U$ and $Y \supset V$ be the compactification. Then the graph $\Gamma_\sigma \subset Y \times Y$ looks like

$$\cdots .$$

In the log brow-up $(Y \times Y)'$, it looks like

$$\cdots .$$

We define

$$s_{V/U}(\sigma) = \begin{cases} -(\Gamma_\sigma, \Delta_Y)_{(Y\times Y)'} & \text{if } \sigma \neq 1 \\ -\sum_{\tau \neq \sigma} s_{V/U}(\tau) & \text{if } \sigma = 1. \end{cases}$$

The Swan class is defined by

$$\text{Sw } \mathcal{F} = \frac{1}{|G|} \sum_{\sigma \in G} f_* s_{V/U}(\sigma) \text{Tr}(\sigma : M)$$

where M is the representation of G corresponding to $\mathcal{F}$. This is an exact geometric reformulation of the classical definition. The Hasse-Arf theorem tells us that Sw $\mathcal{F}$ is in $CH_0(X \setminus U)$ even we have a denominator in the defining equation.

In higher dimension, we need to take an alteration. But, it works and we define Sw $\mathcal{F}$ in $CH_0(X \setminus U)_{\mathbb{Q}}$.

Conjecture 1 Sw $\mathcal{F}$ *is in the image of* $CH_0(X \setminus U)$.

OK if $\dim U \leq 2$. Reduction to rank 1. Explicit computation below.

Theorem 2 *We have*

$$\chi_c(U_{\bar{F}}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U_{\bar{F}}, \Lambda) - \deg \text{Sw } \mathcal{F}.$$

The key ingredient is a Lefschetz trace formula for an open variety. Changing notation, $U \subset X$, $\tilde{\Gamma} \subset (X \times X)'$ and let $(D \times X)'$ and $(X \times D)'$ be the proper transforms.

Theorem 3 *Assume* $\tilde{\Gamma} \cap (D \times X)' \subset \tilde{\Gamma} \cap (X \times D)'$. *Then, we have*

$$\text{Tr}(\Gamma^* : H^*(U_{\bar{F}}, \mathbb{Q}_\ell)) = (\tilde{\Gamma}, \Delta)_{(X\times X)'}.$$

3 Relation between the characteristic class and the Swan class.

Conjecture 4

$$C(j_!\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot C(j_!\Lambda) - \mathrm{clSw}\ \mathcal{F}.$$

OK under a technical assumption. It is satisfied if dim $U \leq 2$.

Conjecture 4 is a refinement of Theorem 2.

Rank 1 case.

Kato defined a divisor $D_{\mathcal{F}}$. He also defined a 0-cycle class $c_{\mathcal{F}} \in CH_0(X \setminus U)$ by

$$c_{\mathcal{F}} = (-1)^{d-1}\{c_*(\Omega^1_{X/F}(\log D))(1 - D_{\mathcal{F}})^{-1}D_{\mathcal{F}})\}_{\dim 0}$$

assuming the cleaness condition. It is satisfied if dim $U \leq 2$ after a blow-up.

Theorem 5 *Assume $\mathcal{F}$ satisfies the cleaness condition. Then, we have*

$$C(j_!\mathcal{F}) = C(j_!\Lambda) - c_{\mathcal{F}}.$$

Conjecture 6 *Under the same assumption, we have*

$$\mathrm{Sw}\ \mathcal{F} = c_{\mathcal{F}}.$$

One can prove Conjecture 6 under some additional technical conditions. The conditions are satified if dim $U \leq 2$.

Sketch of Proof of Theorem 5.

Assume for simplicity dim $U = 1$. Let $D = \mathrm{Sw}\mathcal{F} = c_{\mathcal{F}}$ be the Swan divisor. Let $(X \times X)^{(D)} \to (X \times X)'$ be the blow up of D in the log diagonal $X \subset (X \times X)'$. Then, it induces an immersion $X \to (X \times X)^{(D)}$ and $(X \times X)^{(D)}$ is smooth on a neighborhood of X. Then, using the fact that $\mathcal{H}om(p_2^*\mathcal{F}, p_1^*\mathcal{F})$ is extended to a smooth sheaf on the neighborhood, one can conclude that $C(j_!\mathcal{F}) = (X, X)_{(X\times X)^{(D)}}$. Similarly, we get $C(j_!\Lambda) = (X, X)_{(X\times X)'}$. Since $(X, X)_{(X\times X)^{(D)}} = (X, X)_{(X\times X)'} - D$, the assertion follows.