Characteristic cycle

k perfect field char $p \geqq 0$

X smooth $/k$ $\dim X = n$

Λ finite field char $\ell \neq p$

$\mathcal{F}$ constructible complex of Λ-modules.

$\operatorname{supp} \mathcal{F} \subset X$ closed

dim $2n$.

Singular support $SS\mathcal{F} \subset T^*X$ cotangent b'dle
Beilinson
closed conical. = stable under $\mathbb{G}_m$-action

$SS\mathcal{F} = \bigcup_a C_a$ irred. component. $\dim C_a = n$

Characteristic cycle

$CC\mathcal{F} = \sum_a m_a C_a$ $m_a \in \mathbb{Z}$.

$m_a > 0$ if $\mathcal{F}$ a perverse sheaf

1. Classical example
2. Compatibility with direct image.

Example $\dim X = 1$ T^*X line b'dle over curves

C_a either 0-section or fiber

$D \subset X$ smallest s.t $\mathcal{F}|_{X-D}$ locally constant coh. sheaves $\neq 0$

$SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in D} T^*_x X$. Lagrangian

0-section fiber not in higher dim

$CC\mathcal{F} = -(\operatorname{rk}\mathcal{F} \cdot T^*_X X + \sum_{x \in D} a_x\mathcal{F} \cdot T^*_x X)$

$a_x\mathcal{F}$ Artin conductor

$= \operatorname{rk}\mathcal{F} - \operatorname{rk}\mathcal{F}_{\bar{x}} + \operatorname{Sw}_x \mathcal{F}$ Swan conductor

wild ramification

2. Direct image

X, Y smooth $/k$. $f: X \to Y$ proper morphism $/k$

dim n, m.

$\mathcal{F}$ on X. $Rf_*\mathcal{F}$ on Y

$SS\mathcal{F}$, $CC\mathcal{F}$ on T^*X $\quad$ $SSRf_*\mathcal{F}$, $CCRf_*\mathcal{F}$ on T^*Y

$$\begin{array}{ccccc} T^*X & \xleftarrow{g} & X\times_Y T^*Y & \xrightarrow{p} & T^*Y \\ \cup & & \cup & & \cup \\ C & & g^{-1}(C) & & p(g^{-1}(C)) \\ & & & & \| \\ & & & & f_\circ(C) \end{array}$$

p. proper

algebraic correspondence

closed conical $\quad$ closed conical

Beilinson $SSRf_*\mathcal{F} \subset f_\circ SS\mathcal{F}$.

$A = \sum m_a C_a$ $\quad g^! A \in CH_m(g^{-1}(C))$ $p_! g^! A \in CH_m(f_\circ(C))$ modulo rat. eq.

$\quad\quad f_! A$.

If $\dim f_\circ(C) \leqq m \Rightarrow CH_m(f_\circ(C)) = Z_m(f_\circ(C))$ free abelian gp

satisfied if char $k = 0$

Conjecture $\quad CCRf_*\mathcal{F} = f_! CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$

$$\Rightarrow \quad \begin{array}{ccc} K(X,\Lambda) & \longrightarrow & CH_0(X) = CH_n(T^*X) \\ f_! \downarrow & \circlearrowleft & \downarrow f_* \\ K(Y,\Lambda) & \longrightarrow & CH_0(Y) \end{array} \quad \mathcal{F} \mapsto [CC\mathcal{F}]$$

Grothendieck gp

(cf. finite X, Y proj Umezaki–Yang–Zhao)

Theorem. Assume X, Y proj smooth, $f: X \to Y$ proj,

$\dim f_\circ SS\mathcal{F} \leqq m = \dim Y$. Then

$$CCRf_*\mathcal{F} = f_! CC\mathcal{F} \quad \text{as cycles w.o rat. eq.}$$

2-1. Examples.

2-2 Sketch of Pf.

Example 1. $Y = \mathrm{Spec}\, k$ $\dim \mathrm{fo}\, SS\mathcal{F} = 0$

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$$ index formula

Euler # intersection #.

Further if $\dim X = 1$. Grothendieck - Ogg - Shafarevich.

2 Y curve. (need not be proj).

$$CCRf_*\mathcal{F} = -(\underbrace{\cancel{Rf_*\mathcal{F}}}_{\chi(X_{\bar\eta}, \mathcal{F})} \cdot T^*_Y Y + \sum_y a_y Rf_*\mathcal{F} \cdot T^*_y Y)$$

$$a_y Rf_*\mathcal{F} = \chi(X_{\bar\eta}, \mathcal{F}) - \chi(X_{\bar y}, \mathcal{F}) + \mathrm{Sw}_y H^*(X_{\bar\eta}, \mathcal{F})$$

$$-a_y Rf_*\mathcal{F} = \underset{\text{int. \# supp'd on } X_y}{(CC\mathcal{F}, df)_{T^*X, X_y}}$$ conductor formula.

$f: X \to Y \ni y$ t local parameter

$df = f^* dt$ section of T^*X on a nbd of fiber X_y

Further if $\mathcal{F} = \Lambda$.

$$-a_y Rf_*\Lambda \; (-1)^n (T^*_X X, df)_{T^*X, X_y}$$

localized Chern class of $\Omega^1_{X/Y}$

geometric case of Bloch's conductor formula.

Outline of proof

G.O.S. $\Rightarrow$ index formula

$\Rightarrow$ conductor formula $\Rightarrow$ general case

Goal $-a_y = (CC\mathcal{F}, df)_{T^*X, X_y}$

index formula $\Rightarrow$ $\sum_y -a_y = \sum_y (\quad)_{T^*X, X_y}$

~~How~~ to Deduce the equalities for individual terms for that for the sum.

Fix $y=y_0$ & find $Y'\to Y$ s.t

- étale at y_0
- at $y'\mapsto y\neq y_0$ for a modification of the pull-back of $\mathcal{F}$

$$\text{L.H.S} = \text{R.H.S} = 0$$

an analogue of stable reduction thm.

By graph, we may assume $f: X\to Y$ proj. smooth.
By devissage, may assume $\mathcal{F}$ is a perverse sheaf,

Fix $y=y_1\neq y_0$

$$\begin{array}{ccc} U & \subset & X \\ \downarrow & \square & \downarrow \\ V & \subset & Y \ni y_1 \end{array} \qquad \begin{array}{ccc} j': U' & \to & X' \\ \downarrow & & \downarrow \\ V' & \to & Y' \end{array}$$

modification $j'_{!*}\mathcal{F}_{U'}$

L.H.S easier

Action of inertia gp I_y on nearby cycles $R\Psi\mathcal{F}$.

Trivial action $\Leftrightarrow$ $\underbrace{R\Phi}_{\text{vanishing cycles}} j_{!*}\mathcal{F}=0 \Rightarrow \text{LHS}=0$

$\uparrow$ after finite ramified extension (Λ. finite)

R.H.S more difficult

transversality $f: X\to Y$. $C\subset T^*X$ closed conical

$$X\times_Y T^*Y \xrightarrow{df} T^*X \supset C$$

f C-transversal $\quad df^{-1}(C)\subset$ 0-section $\cup X$

$\Leftrightarrow$ f smooth on a nbd of the base ($C\cap T^*_X X$) of C

$C=SS\mathcal{F}$ $\qquad$ C-transversal $\Rightarrow R\Phi\mathcal{F}=0$

$\qquad\qquad\qquad\qquad\Vert \qquad\qquad \uparrow$

$\qquad\qquad\qquad \text{R.H.S}=0 \qquad$ Def of $SS\mathcal{F}$

Find Y' s.t $f': X'\to Y'$ is C'-trans for $C'=SS\mathcal{F}'$ $\quad \mathcal{F}'=j'_{!*}\mathcal{F}_U$

analogue of stable red thm.

induction on ~~rel~~ rel. dim.

Deligne's method + Th. finitude + $R\Phi=0$

$\Rightarrow$ isolated char pt. (+ Milnor formula) $\Rightarrow$ transversality

$\Vert$

Def of $CC\mathcal{F}$