# Wild ramification and the characteristic cycle of an $\ell$-adic sheaf

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**Abstract**

We measure the wild ramification of an l-adic etale sheaf by introducing blow-ups of the self-product at the ramification locus in the diagonal.

Using the geometric construction, we define the characteristic cycle of an $\ell$-adic sheaf as a cycle on the logarithmic cotangent bundle and prove that the intersection with the 0-section gives the characteristic class, under a certain condition.

## 1 Ramification along a divisor

Let $k$ be a perfect field of characteristic $p > 0$, $X$ be a smooth scheme of dimension $d$ over $k$ and $U = X \setminus D$ be the complement of a divisor $D$ with simple normal crossings. We consider a smooth $\ell$-adic sheaf $\mathcal{F}$ on $U$.

We construct a commutative diagram

$$\begin{array}{ccccc} X \times X & \longleftarrow & (X \times X)' & \longleftarrow & (X \times X)^{[R]} \\ & & \cup & & \cup \\ & & (X \times X)^{\sim} & \longleftarrow & (X \times X)^{(R)}, \end{array}$$

where $R = r_1 D_1 + \cdots + r_m D_m$ is a linear combination of the irreducible components $D_1, \ldots, D_m$ of $D$ with rational coefficients $r_i \geq 0, r_i \in \mathbb{Q}$. For simplicity in this note, we will assume $r_i > 0, r_i \in \mathbb{Z}$.

We define the log blow up

$$(X \times X)' \to X \times X$$

to be the blow-up at $D_1 \times D_1, D_2 \times D_2, \ldots, D_m \times D_m$. We define the log product $(X \times X)^{\sim} \subset (X \times X)'$ to be the complement of the proper transforms of $D \times X$ and $X \times D$. The diagonal map $\delta : X \to X \times X$ is uniquely lifted to the log diagonal map $\tilde{\delta} : X \to (X \times X)^{\sim}$. The conormal sheaf $\mathcal{N}_{X/(X \times X)^{\sim}}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)$ of rank $d$.

We define

$$(X \times X)^{[R]} \to (X \times X)'$$

to be the blow-up at the divisor $R \subset X$ in the log diagonal $X \subset (X \times X)'$. We define an open subscheme $(X \times X)^{(R)} \subset (X \times X)^{\sim} \times_{(X \times X)'} (X \times X)^{[R]}$ to be the complement of the proper transforms of the exceptional divisors of $(X \times X)^{\sim}$. The log diagonal map $\delta' : X \to (X \times X)'$ is uniquely lifted to a closed immersion $\delta^{(R)} : X \to (X \times X)^{(R)}$. The projections $(X \times X)^{(R)} \to X$ are smooth. The conormal sheaf $\mathcal{N}_{X/(X \times X)^{(R)}}$ is canonically identified with the locally free $\mathcal{O}_X$-module $\Omega^1_X(\log D)(R)$.

We consider the commutative diagram

$$\begin{array}{ccc} U \times U & \xrightarrow{j^{(R)}} & (X \times X)^{(R)} \\ {\scriptstyle \delta_U}\uparrow & & \uparrow{\scriptstyle \delta^{(R)}} \\ U & \xrightarrow{j} & X \end{array}$$

of open immersions and the diagonal immersions.

**Definition 1** *Let $\mathcal{F}$ be a smooth sheaf on $U = X \setminus D$. We define a smooth sheaf $\mathcal{H}$ on $U \times U$ by $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$. Let $R = \sum_i r_i D_i \geq 0$ be an effective divisor with rational coefficients.*

*We say that the log ramification of $\mathcal{F}$ along $D$ is bounded by $R+$ if the identity $1 \in \mathrm{End}_U(\mathcal{F}) = \Gamma(U, \mathcal{E}nd_U(\mathcal{F})) = \Gamma(X, j_* \mathcal{E}nd_U(\mathcal{F}))$ is in the image of the base change map*

$$\Gamma(X, \delta^{(R)*} j^{(R)}_* \mathcal{H}) \longrightarrow \Gamma(X, j_* \mathcal{E}nd_U(\mathcal{F})) = \mathrm{End}_U(\mathcal{F}). \tag{1.1}$$

Definition 1 is related to the filtration by ramification groups in the following way. Let $D_i$ be an irreducible component and $K_i$ be the fraction field of the completion $\widehat{\mathcal{O}}_{X,\xi_i}$ of the local ring at the generic point $\xi_i$ of $D_i$. We will often drop the index $i$ in the sequel. The sheaf $\mathcal{F}$ defines an $\ell$-adic representation $\mathcal{F}_{\bar{\eta}_i}$ of the absolute Galois group $G_{K_i} = \mathrm{Gal}(\overline{K}_i/K_i)$. The filtration $G^r_{K,\log} \subset G_K, r \in \mathbb{Q}, r > 0$ by the logarithmic ramification groups is defined. We put $G^{r+}_{K,\log} = \overline{\bigcup_{q>r} G^q_{K,\log}}$.

**Lemma 2** *The following conditions are equivalent.*

(1) *There exists an open neighborhood of $\xi_i$ such that the log ramification of $\mathcal{F}$ along $D$ is bounded by $R+$.*

(2) *The action of $G^{r_i+}_{K_i,\log}$ on $\mathcal{F}_{\bar{\eta}_i}$ is trivial.*

The open subscheme $U \times U \subset (X \times X)^{(R)}$ is the complement of the inverse image $E = (X \times X)^{(R)} \times_X D$. The inverse image $E$ is canonically identified with the vector bundle $\mathbf{V}(\Omega^1_X(\log D)(R)) \times_X D$.

**Proposition 3** *Assume that the log ramification is bounded by $R+$. Then, for every geometric point $\bar{x}$ of $D$, the restriction $(j_*\mathcal{H})|_{E_{\bar{x}}}$ on the geometric fiber is isomorphic to the direct sum $\bigoplus_f \mathcal{L}_f^{\oplus n_f}$ where $\mathcal{L}_f$ is a smooth rank one sheaf defined by the Artin-Schreier equation $T^p - T = f$ and $f$ denotes a linear form on the vector space $E_{\bar{x}}$.*

Proposition 3 has the following consequence. Let $D_i$ be an irreducible component of $D$. The graded piece $\mathrm{Gr}^{r_i}_{\log} G_{K_i} = G^{r_i}_{K,\log}/G^{r_i+}_{K,\log}$ is abelian. The restriction of $\mathcal{F}_{\bar\eta_i}$ to $G^{r_i}_{K,\log}$ is decomposed into direct sum of characters $\bigoplus_\chi \chi^{n_\chi}$. The fiber $\Theta^{(r_i)}_{\log} = E^+ \times_{D^+} \xi_i$ at the generic point $\xi_i$ is a vector space over the function field $F_i$ of $D_i$. The restriction of $j_*\mathcal{H}$ on the geometric fiber $\Theta^{(r_i)}_{\log,\overline{F}_i}$ is decomposed as $\bigoplus_\chi \mathrm{End}_{I_i}(\mathcal{F}_{\bar\eta_i}) \otimes \mathcal{L}_\chi$ where $\mathcal{L}_\chi$ is a smooth rank one sheaf defined by the Artin-Schreier equation $T^p - T = f_\chi$ where $f_\chi = \mathrm{rsw}\ \chi$ is a linear form on $\Theta^{(r_i)}_{\log,\overline{F}_i}$ called the refined Swan character of $\chi$.

**Theorem 4** *The graded quotient $\mathrm{Gr}^r_{\log} G_K$ is annihilated by $p$ and the map*

$$\mathrm{Hom}(\mathrm{Gr}^r_{\log} G_K, \mathbb{F}_p) \longrightarrow \mathrm{Hom}_{\overline{F}_i}(\Theta^{(r)}_{\log}, \overline{F}_i) \tag{1.2}$$

*sending a character $\chi$ to the refined Swan character $f_\chi = \mathrm{rsw}\ \chi$ is an injection.*

# 2 Characteristic cycle

For a non-trivial character $\chi : \mathrm{Gr}^r_{\log} G_K \to \mathbb{F}_p$, the refined Swan character $\mathrm{rsw}\ \chi : \Theta^{(r)}_{\log} \to \overline{F}_i$ defines an $\overline{F}_i$-rational point $[\mathrm{rsw}\ \chi] : \mathrm{Spec}\ \overline{F}_i \to \mathbf{P}(\Omega^1_X(\log D)^*)$. We define a reduced closed subscheme $T_\chi \subset \mathbf{P}(\Omega^1_X(\log D)^*)$ to be the Zariski closure $\overline{\{[\mathrm{rsw}\ \chi](\mathrm{Spec}\ \overline{F}_i)\}}$ and let $L_\chi = \mathbf{V}(\mathcal{O}_{T_\chi}(1))$ be the pull-back to $T_\chi$ of the tautological sub line bundle $L \subset T^*X(\log D) \times_X \mathbf{P}(\Omega^1_X(\log D)^*)$. The inclusion $T_\chi \to \mathbf{P}(\Omega^1_X(\log D)^*)$ corresponds to a surjection $\Omega^1_X(\log D)^* \otimes \mathcal{O}_{T_\chi} \to \mathcal{O}_{T_\chi}(1)$ and hence defines a commutative diagram

$$\begin{array}{ccccc} L_\chi & \longrightarrow & T^*X(\log D) \times_X D_i & \longrightarrow & T^*X(\log D) = \mathbf{V}(\Omega^1_X(\log D)^*) \\ \downarrow & & \downarrow & & \downarrow \\ T_\chi & \xrightarrow{\pi_\chi} & D_i & \longrightarrow & X. \end{array} \tag{2.1}$$

We put

$$SS_\chi = \frac{1}{[T_\chi : D_i]} \pi_{\chi *}[L_\chi] \tag{2.2}$$

in $Z_d(T^*X(\log D) \times_X D_i)_{\mathbb{Q}}$.

Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf on $U = X \setminus D$ and $R = \sum_i r_i D_i$ be an effective divisor with rational coefficients $r_i \geq 0$. In the rest of talk, we assume that $\mathcal{F}$ satisfies the following conditions:

(R) The log ramification of $\mathcal{F}$ along $D$ is bounded by $R+$.

(C) For each irreducible component $D_i$ of $D$, the closure $\overline{S_{\mathcal{F}} \times F_i}$ is finite over $D_i$ and the intersection $\overline{S_{\mathcal{F}} \times F_i} \cap D_i$ with the 0-section is empty.

The conditions imply $\mathcal{F}_{\bar\eta_i} = \mathcal{F}^{(r_i)}_{\bar\eta_i}$ for every irreducible component $D_i$ of $D$.

**Definition 5** *Let $\mathcal{F}$ be a smooth $\Lambda$-sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

*For an irreducible component $D_i$ of $D$ with $r_i > 0$, let $\mathcal{F}_{\bar{\eta}_i} = \sum_\chi n_\chi \chi$ be the direct sum decomposition of the representation induced on $\mathrm{Gr}^{r_i}_{\log} G_{K_i}$. We define the characteristic cycle by*

$$CC(\mathcal{F}) = (-1)^d \left( \text{rank } \mathcal{F} \cdot [X] + \sum_{i, r_i > 0} r_i \cdot \sum_\chi n_\chi \cdot [SS_\chi] \right) \tag{2.3}$$

*in $Z^d(T^*X(\log D))_{\mathbb{Q}}$.*

**Theorem 6** *Let $X$ be a smooth scheme over $k$ and $D$ be a divisor with simple normal crossings. Let $\mathcal{F}$ be a smooth $\ell$-adic sheaf on $U = X \setminus D$ satisfying the conditions* (R) *and* (C).

*Then we have*

$$(CC(\mathcal{F}), X)_{T^*X(\log D)} = C(j_!\mathcal{F})$$

*where the right hand side denotes the characteristic cycle of $j_!\mathcal{F}$. In particular, if $X$ is proper, we have $(CC(\mathcal{F}), X)_{T^*X(\log D)} = \chi_c(U_{\bar{k}}, \mathcal{F})$.*

Questions: 1. How we deal with more than one $R$?
2. What we can do in mixed characteristic case?
3. Does the same construction work to study irregular singularities of $\mathcal{D}$-modules?