Second St-Wh. class

k field X proper smooth. $\dim = n \,/ k$ $\ell \neq \operatorname{char} k \neq 2$

$H^n(X_{\bar{k}}, \mathbb{Q}_\ell)$ orthogonal or symplectic acc. to parity of n

n even ~~$\det H^n = (\text{char. of order} \le 2)\cdot\left(-\frac{n \cdot b_n}{2}\right)$~~

$\uparrow$
$e_n = \det H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\frac{n}{2}))$

$e_n \in H^1(k, \mathbb{Z}/2\mathbb{Z}) = k^\times/k^{\times 2}$

Theorem (JAG)

$$e_n = \{d\} + \begin{cases} r \cdot \{-1\} & n \equiv 0 \ (4) \\ (r + b_{dR,n})\{-1\} & \equiv 2 \ (4) \end{cases}$$

$r = \sum_{q<n} (-1)^q b_{dR,q}$. d disc. of $H^n_{dR}(X/k)$.

What one can say on classes in H^2.

- def'n of sw_2
- State conj.
- Evidences : - X smooth hyp of dim n & $\ell > n+1$

  ($n=0$ Theorem of Serre)

Def'n of sw_2

π profinite gp. A finite abelian gp

$1 \to A \to E \to \pi \to 1$ central extension

$$\underset{\substack{\text{trivial act} \\ \text{identity}}}{H^1(\pi, \pi)} \longrightarrow H^2(\pi, A) \ni [E]$$

$\rho: \pi \to O(V)$ orthogonal ℓ-adic rep'n

$1 \to \mu_2 \to \widetilde{O}(V) \to O(V) \to 1$ central ext'n of alg. gp.

$\uparrow$
$\mathrm{Pin}(V)$

$Cl(V)$ Clifford algebra $T^\cdot(V)/(x \otimes x - q(x))$

$\dim Cl(V) = \dim \Lambda^\cdot(V) = 2^{\dim V}$

$C(V)^\times \subset Cl(V)^\times$ subgp gen by $x \in V$ st $q(x) \neq 0$

$\ni 1 \quad C(V)^\times \to \mathbb{G}_m \quad x \mapsto q(x)$

$\widehat{O}(V) = \ker\,(C(V)^\times \to \mathbb{G}_m)$.

$\widehat{O}(V) \to O(V) \quad x \mapsto (y \mapsto y - 2\cancel{q}\,b(x,y)\cdot x)$

$\cancel{q(x)\cdot 2}$

$\cancel{}\ \ b(x,x) = q(x) = 1$

$\ker(\widehat{O}(V) \to O(V)) = \langle x_1 \cdots x_n \rangle$ order 2

$x_1, \ldots, x_n$ orthonormal basis

$sw_2(\rho) = \rho^*([\widehat{O}(V)]) \in H^2(\pi, \mathbb{Z}/2) = H^2(k, \mathbb{Z}/2\mathbb{Z})$ if $\pi = G_k$

D k-v.sp of finite dim with sym. non-deg bilin form

$\operatorname{disc} D = \sum \{a_i\}$ $\qquad a_i = \langle x_i, x_i \rangle$ orthogonal basis

$hw(D) = \sum_{i<j} \{a_i, a_j\} \in H^2(k, \mathbb{Z}/2\mathbb{Z})$

Formulate the Conj. sw_2 hw_2 $\quad e^b =$

X/k $\quad q$ integer $\quad \det H^q = e_q \cdot (-\frac{b_q \cdot q}{2})$

$\boxed{q \text{ odd} \Rightarrow b_q \text{ even}}$ (even if X is not assumed projective)

Thm 2.2.2 (Suh Junecue)

$e = \sum_{\substack{q<n \\ \text{even}}} e_q$

n even $\quad \beta = \frac{1}{2} \sum_{q<n} (n-q)\cdot b_q$

$sw_2(H^n_\ell) = sw_2(H^n(X_{\bar{k}}, \mathbb{Q}_\ell(\tfrac{n}{2})))$

c_ℓ $\cancel{}$ image in $H^2(k, \mathbb{Z}/2\mathbb{Z})$ of

~~[illegible]~~ the gen. of

$H^2(\pi_1(\operatorname{Spec} \mathbb{Z}[\tfrac{1}{\ell}])^{ab}, \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$

<u>Conj</u> $sw_2(H^n_\ell) + \{e, -1\} + \beta \cdot c_\ell$

$$= hw_2(H^n_{dR}) + \begin{cases} r\cdot\{d, -1\} + \binom{n}{2}\{-1,-1\} & n \equiv 0\ (4) \\ (r + b_{dR,n} - 1)\cdot\{d, -1\} + \binom{r + b_{dR,n}}{2}\{-1,-1\} & n \equiv 2\ (4) \end{cases}$$

$+ \{2, d\} + \eta\,(c_\ell - c_2)$

$$\eta = \sum_{q < \frac{n}{2}} (-1)^q \left(\frac{n}{2} - q\right) \chi(X, \Omega^q), \quad r = \sum_{q < n} (-1)^q b_{dR, q} \frac{L3}{q}$$

$$d = \operatorname{disc} H^n_{dR}(X/k)$$

$Sw H^2_\ell$ dep'd on ℓ

e.g. $K = \mathbb{Q}_p$ X/k good red $p \neq 2$.

$\ell \neq p \Rightarrow$ ~~*~~ $Sw_{2q} = 0$

$Sw_{2,p} \neq 0$

$X = A$ abelian ~~***~~ surface ~~***~~ $\beta = -1$

$/\mathbb{R}$

complex version that looks simpler

<u>Theorem</u> Conj is true in the following cases.

1. $K/\mathbb{Q}_p$ finite $p \neq 2, \ell$ $\exists X_{\mathcal{O}_K}$ proj reg. flat model s.t. X_F has at most isolated ordinary double pts
2. $K/\mathbb{Q}_p$ unramified finite ext. $p = \ell > n+1$. good red
3. $K = \mathbb{R}$ X proj
4. $k \supset \overline{\mathbb{Q}}$
5. X smooth hypersurface of dim n. $\ell > n+1$

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2 p-adic ~~*~~ Hodge theory Fontaine-Lafaille

3 polarization on the Hodge struct

4 transcendental

5 put everything together on the univ. family