Automorphic forms and ℓ-adic representations 4

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In Carayol's note [4], a geometric construction of the Galois representations associated to Hilbert modular forms and the compatibility with the local Langlands correspondence are discussed. In loc. cit., the compatibility is established in the case $\ell \neq p$ where the Galois representation is an ℓ-adic representation and p is the prime divided by the prime $\mathfrak{p}$ of the totally real field where the restriction to the decomposition group is considered. The purpose of this note is to sketch the proof of the compatibility in the remaining case $p = \ell$.

In this note, we only discuss the compatibility in the case where the Galois representation is constructed geometrically. Namely, we assume the condition $(*)$ in Theorem 1 in the text. We only give the main ideas of the proof and refer for the detail to [10]. In other cases, there are alternative arguments using congruences. They cover the cases where the level is prime to $\mathfrak{p}$ [12] or the residual representation is absolutely irreducible [6] Theorem (4.3). However, the general case still remains open.

1 Compatibility for $p = \ell$

The compatibility is stated as follows.

Theorem 1 [Deligne, Langlands, Carayol, S.] *Let F be a totally real number field and $\pi = \bigotimes_v \pi_v$ be a cuspidal automorphic representation of $GL_2(\mathbb{A}_F)$ associated to a Hilbert modular form of parallel weight $k \geq 2$ satisfying the following condition:*

$(*)$ *If the degree $[F : \mathbb{Q}] = g$ is even, there exists a finite place v of F where π_v is essentially square integrable.*

Let L be a number field on which the finite part $\bigotimes_{v\nmid\infty} \pi_v$ is defined and, for a finite place λ of L, let V_λ be the λ-adic represenation of the absolute Galois group $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$ associated to f.

For a finite place $\mathfrak{p}$ of F, let $D_{\mathfrak{p}}(V_\lambda)$ denote the representation of the Weil-Deligne group $WD_{F_{\mathfrak{p}}} = W_{F_{\mathfrak{p}}} \ltimes \mathbf{G}_a$ associated to the restriction of V_λ to the decomposition group $G_{F_{\mathfrak{p}}} \subset G_F$ and let $\sigma(\pi_{\mathfrak{p}})$ be the F-semi-simple representation of the Weil-Deligne group associated to the $\mathfrak{p}$-component $\pi_{\mathfrak{p}}$ by the local Langlands correspondence. Then, there exists an isomorphism

$$D_{\mathfrak{p}}(V_\lambda)^{F\text{-}ss} \to \sigma(\pi_{\mathfrak{p}})$$

of F-semi-simple representations of the Weil-Deligne group $WD_{F_{\mathfrak{p}}}$ defined over L.

Under the condition ($*$), the construction of the Galois representation V_λ is discussed in [4]. The representation $D_{\mathfrak{p}}(V_\lambda)$ of the Weil-Deligne group associated to a local Galois representation is defined in Breuil's note [1]. The superscript F-ss denotes the Frobenius semi-simplification.

One can weaken the assumption that f is of parallel weight k by the parity condition that f is of weight $(k_1, \dots, k_g)$ where $k_1 \equiv \cdots \equiv k_g \bmod 2$. However, for simplicity, we assume the parallel weight condition in this note. For $k = 2$, the proof is much simpler.

In the rest of this section, we explain some preliminary reduction steps. Let l and p denote the prime numbers divided by λ and $\mathfrak{p}$ respectively. The case $p \neq \ell$ has been discussed in [4]. We will derive the case $p = \ell$ from the case $p \neq \ell$. Take a finite place μ of L above a prime number $\ell \neq p$. Then, since we already know an isomorphism $D_{\mathfrak{p}}(V_\mu)^{F\text{-ss}} \to \sigma(\pi_{\mathfrak{p}})$, it suffices to show an isomorphism

$$D_{\mathfrak{p}}(V_\lambda)^{F\text{-ss}} \to D_{\mathfrak{p}}(V_\mu)^{F\text{-ss}}. \tag{1.1}$$

In other words, we compare the p-adic representation with the ℓ-adic representation.

By a standard argument on two-dimensional representations of the Weil-Deligne group, the isomorphism (1.1) is a consequence of the assertions (1) and (2) below. Let $n : W_{F_{\mathfrak{p}}} \to \mathbb{Z}$ denote the canonical surjection.

(1) For an arbitrary element $\sigma \in W_{F_{\mathfrak{p}}}$ such that $n(\sigma) \geq 0$, we have an equality

$$\operatorname{Tr}(\sigma : D_{\mathfrak{p}}(V_\lambda)) = \operatorname{Tr}(\sigma : D_{\mathfrak{p}}(V_\mu)). \tag{1.2}$$

(2) For the monodromy operator N, we have an equivalence

$$N \text{ on } D_{\mathfrak{p}}(V_\lambda) \text{ is } 0 \quad \Leftrightarrow \quad N \text{ on } D_{\mathfrak{p}}(V_\mu) \text{ is } 0. \tag{1.3}$$

2 The key ingredients in the proof

We sketch the main steps of the proof. More details will be discussed in the later sections respectively.

Step 1. The construction given in [4] is not geometric enough for our purpose. We need a more geometric construction. Namely, we construct a projective smooth variety X over $F_{\mathfrak{p}}$ and an algebraic correspondence Γ on X with coefficients in L satisfying the following properties: Γ^* acts on the étale cohomology of $X_{\overline{F_{\mathfrak{p}}}}$ as an idempotent and we have

$$\Gamma^* \cdot H^q(X_{\overline{F_{\mathfrak{p}}}}, L_\lambda) = \begin{cases} V_\lambda(-(g-1)(k-2)) & \text{if } q = (2g-1)(k-2)+1 \\ 0 & \text{otherwise} \end{cases}$$

for every finite place λ of L. This construction shows in particular that, if $\mathfrak{p}$ does not divide the level, the representation $D_{\mathfrak{p}}(V_\lambda)$ is pure of weight $k-1$.

Step 2. The equality (1.2) will follow from the equality

$$\text{(2.1)} \qquad \mathrm{Tr}(\sigma \circ \Gamma^* : D_{\mathfrak{p}} H^*(X_{\overline{F_{\mathfrak{p}}}}, L_\lambda)) = \mathrm{Tr}(\sigma \circ \Gamma^* : D_{\mathfrak{p}} H^*(X_{\overline{F_{\mathfrak{p}}}}, L_\mu))$$

for finite places λ, μ of L. This is proved for an arbitrary X and Γ in [9] for $\lambda \nmid p$ and $\mu \nmid p$. It is a consequence of a functoriality of the weight spectral sequence of Rapoport-Zink [8] and of the Lefschetz trace formula, that is independent of ℓ.

However, if $\lambda \mid p$ or $\mu \mid p$, we need to assume a condition on Γ, at least for the moment. This conditional result suffices for our purpose but we need to verify that the condition is actually satisfied. More precisely, we show that Γ is a linear combination of algebraic correspondences extended to finite étale correspondences on a semi-stable model. A recent work of Tsuji in progress on a new construction of the weight spectral sequence of Mokrane [7] seems to imply the required functoriality in the proof of the general case.

Step 3. Given Step 2, it suffices to show that (the traces of) the representations of the Weil group determine the monodromy operator. Thus, it *would* follow from the monodromy-weight conjecture.

Conjecture 2 *Let M be the unique increasing filtration on $D_{\mathfrak{p}} H^q(X_{\overline{F_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ characterized by the following conditions:*

(1) $M_q = D_{\mathfrak{p}} H^q(X_{\overline{F_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ and $M_{-q-1} = 0$.

(2) $NM_r \subset M_{r-2}$ for $r \in \mathbb{Z}$.

(3) $N^r : \mathrm{Gr}_r^M \to \mathrm{Gr}_{-r}^M$ is an isomorphism for $r \in \mathbb{N}$.

Then, for a lifting $\sigma \in W_{F_{\mathfrak{p}}}$ of the geometric Frobenius, every eigenvalue α of σ acting on $\mathrm{Gr}_r^M D_{\mathfrak{p}} H^q(X_{\overline{F_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ is pure of weight $q + r$ for $r \in \mathbb{Z}$.

We say that α is pure of weight m if the complex absolute values of its conjugates are $N\mathfrak{p}^{m/2}$. It is known that there exists a unique increasing filtration W on $D_{\mathfrak{p}} H^q(X_{\overline{F_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ stable under the action of the Galois group $G_{F_{\mathfrak{p}}}$ satisfying (1) and (2) and the following condition:

(4) For a lifting $\sigma \in W_{F_{\mathfrak{p}}}$ of the geometric Frobenius, every eigenvalue α of σ acting on $\mathrm{Gr}_r^W D_{\mathfrak{p}} H^q(X_{\overline{F_{\mathfrak{p}}}}, \mathbb{Q}_\ell)$ is pure of weight $q + r$.

This is a consequence of the Weil conjecture and its cristalline version, the weight spectral sequences of Rapoport-Zink and of Mokrane, Tsuji's comparison theorem and de Jong's alteration. Thus Conjecture 2 asserts that the filtration W satisfies the condition (3).

Let α and β be the eigenvalues of a lifting σ of the geometric Frobenius acting on $D_{\mathfrak{p}}(V_\lambda)$, that is known to be independent of λ in Step 2. The monodromy-weight conjecture for $D_{\mathfrak{p}}(V_\lambda)$ implies that we have either of the following cases.

(i) Both α and β are pure of weight $k - 1$. In this case, we have $N = 0$.

(ii) One of the two is pure of weight k and the other is of weight $k - 2$. In this case, we have $N \neq 0$.

Thus, together with Step 2, the monodromy-weight conjecture for $D_{\mathfrak{p}}(V_\lambda)$ implies the equivalence (1.3). The implication in (i) follows from a property of N in the definition of the Weil-Deligne group. Thus the non-trivial assertion is the implication in (ii).

The monodromy-weight conjecture remains open in general. However, it is proved for $D_{\mathfrak{p}}(V_\lambda)$, by studying the weight spectral sequence in detail.

3 Variant of the Kuga-Sato variety

In the case $g = [F : \mathbb{Q}]$ is even, we may assume that there exists a finite place $v_0 \neq \mathfrak{p}$ where π_{v_0} is essentially square integrable, by taking a quadratic base change splitting at v_0 in the condition $(*)$. Let B be the quaternion algebra over F satisfying the following conditions:

There exists only one infinite place where B splits. If g is odd, B is unramified at every finite place. If g is even, v_0 is the unique finite place where B is ramified.

We briefly recall the geometric construction in [4]. We consider the Shimura curve $M = M(G, X)$ defined by the reductive group $G = \mathrm{Res}_{F/\mathbb{Q}} B^\times$ and the $G(\mathbb{R})$-conjugacy class $X = \mathbb{C} \setminus \mathbb{R}$ of the map $S(\mathbb{R}) = \mathbb{C}^\times \to G(\mathbb{R}) = GL_2(\mathbb{R}) \times \mathbb{H}^{\times\, g-1} : x + yi \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}, 1, \ldots, 1 \right)$. Its canonical model is defined over the reflex field F. Enlarging L if necessary, we may assume that the representation $\bigotimes_{i:F\to\mathbb{R}} \mathrm{Sym}^{k-2} : G \to GL_{(k-1)g}$ is defined over L and we consider the corresponding smooth L_λ-sheaf W_λ on M. Then, the Galois representation V_λ is defined by decomposing the étale cohomology $H^1(M_{\overline{F}}, W_\lambda)$ by the action of the finite adeles $G(\mathbb{A}_f)$.

The construction of W_λ is geometric in the sense that it is defined using a Barsotti-Tate group $(E_n)_n$ on M. However, it is not geometric in the sense that it is not a part of a higher direct image of a proper smooth family of varieties parametrized by M. This is due to the fact that M is a so-called exotic model and is not a Shimura variety of PEL-type. However, the argument by Carayol to show that the Barsotti-Tate group $(E_n)_n$ is extended to the integral model of M shows that it admits a geometric construction, in a stricter sense as follows.

We introduce more Shimura varieties. Let E_0 be a quadratic imaginary field split at p and put $E = FE_0$. We consider the reductive group $G'' = \mathrm{Res}_{F/\mathbb{Q}}(B^\times \cdot E^\times)$ defined as the Weil restriction of $B^\times \cdot E^\times \subset (B \otimes_F E)^\times$. We define the fiber product $G' = G'' \times_{\mathbf{G}_{m,F}} \mathbf{G}_m$ with respect to the product $B^\times \cdot E^\times \to F^\times$ of the reduced norm $B^\times \to F^\times$ and the norm $E^\times \to F^\times$. Let T and T_0 denote the tori over $\mathbb{Q}$ defined by $E^\times$ and $E_0^\times$. We consider the Shimura varieties $M'' = M(G'', X), M' = M(G', X^+), N = M(T, *)$ and $N_0 = M(T_0, *_0)$. Here $X^+ \subset X$ denotes the conjugacy classes of $S(\mathbb{R}) = \mathbb{C}^\times \to G'(\mathbb{R}) \subset G''(\mathbb{R}) = GL_2(\mathbb{R}) \cdot \mathbb{C}^\times \times (\mathbb{H}^\times \cdot \mathbb{C}^\times)^{g-1} : z = x + yi \mapsto \left(\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^{-1}, z^{-1}, \ldots, z^{-1} \right)$, the symbol $*$ denotes the map $S(\mathbb{R}) = \mathbb{C}^\times \to T(\mathbb{R}) = \mathbb{C}^{\times g} : z \mapsto (z^{-1}, 1, \ldots, 1)$ and $*_0$ is the inverse $S(\mathbb{R}) = \mathbb{C}^\times \to T_0(\mathbb{R}) = \mathbb{C}^\times$. Their canonical models are defined over the reflex fields E, E, E and E_0 respectively.

We consider a diagram

$$\begin{array}{ccccccc} M & \xleftarrow{\mathrm{pr}_1} & M\times N & \xrightarrow{\alpha} & M'' & \longleftarrow & M' \\ & & \Big\downarrow{\scriptstyle \beta} & & & & \\ & & N_0 & & & & \end{array}$$

of Shimura varieties. The map $\alpha : M\times N \to M''$ is induced by the map $G\times T \to G'' : (b,e) \mapsto b\otimes N_{E/E_0}(e)\cdot e^{-1}$ and the map β is defined by $N_{E/E_0}\circ \mathrm{pr}_2$.

The Shimura varieties M' and N_0 are of PEL-type and carry a universal family $A' \to M'$ of abelian varieties and a universal family $A_0 \to N_0$ of CM elliptic curves. The universal family $A' \to M'$ is naturally extended to a family $A'' \to M''$. We consider the fiber product $f : X \to M\times N$ of the pull-backs $(\alpha\circ \mathrm{pr}_{12})^* A''^{g(k-2)}$ and $\beta^* A_0^{(g-1)(k-2)}$ over $M\times N$ as a variant of Kuga-Sato variety.

One defines an algebraic correspondence Γ on X as a linear combination of endomorphisms of the universal abelian varieties and of the universal elliptic curves and permutations of factors as in [11]. It acts on $R^q f_* L_\lambda$ as the projector to a direct summand $\mathrm{pr}_1^* W_\lambda \otimes \mathrm{pr}_2^* L_\lambda(-d') \subset R^d f_* L_\lambda$ for $q = d = (2g-1)(k-2)$ and for $d' = (g-1)(k-2)$ and acts as 0 for the other q. By decomposing $\Gamma^*\cdot H^{d+1}(X_{\overline{\mathbb{Q}}}, L_\lambda) = H^1(M_{\overline{\mathbb{Q}}}, W_\lambda)\otimes H^0(N_{\overline{\mathbb{Q}}}, L_\lambda(-d'))$ by the action of the adele group $(G\times T)(\mathbf{A}_f)$, we recover the Galois representation $V_\lambda \subset H^1(M_{\overline{E}}, W_\lambda)$. This completes Step 1.

4 Weight spectral sequences

Recall that we say that a scheme $X_{\mathcal{O}}$ over the integer ring $\mathcal{O}$ of a p-adic field K is strictly semi-stable if it is Zariski locally étale over Spec $\mathcal{O}[T_0,\dots,T_d]/(T_0\cdots T_m - \varpi)$ for $0 < m \le d$ and for a prime element ϖ of $\mathcal{O}$. Let $X_{\mathcal{O}}$ be a proper strictly semi-stable scheme over $\mathcal{O}$. Let $Y_1,\dots,Y_n$ be the irreducible components of the closed fiber and put $Y^{(i)} = \coprod_{1\le j_0<\cdots<j_i\le n} Y_{j_0}\cap\cdots\cap Y_{j_i}$. The scheme $Y^{(i)}$ is proper and smooth of dimension $d-i$ over the residue field $k = \mathcal{O}/(\varpi)$. By [8] and [7], we have spectral sequences

$$E_1^{s,t} = \bigoplus_{j=\max(0,-s)}^{\infty} H^{t-2j}(Y_{\bar{k}}^{(s+2j)}, \mathbb{Q}_\ell(-j)) \;\Rightarrow\; H^*(X_{\overline{K}}, \mathbb{Q}_\ell), \tag{4.1}$$

$$E_1^{s,t} = \bigoplus_{j=\max(0,-s)}^{\infty} H^{t-2j}_{\mathrm{cris}}(Y^{(s+2j)}/W)(-j)\otimes_W K_0 \;\Rightarrow\; D_K H^*(X_{\overline{K}}, \mathbb{Q}_p) \tag{4.2}$$

for $\ell \ne p = \mathrm{char}\ k$, called the weight spectral sequences. Here $W = W(k)$ is the ring of Witt vectors and $K_0 \subset K$ is its fraction field.

We consider a general algebraic correspondence Γ on the generic fiber X_K. There exist algebraic correspondences $\Gamma^{(i)}$ on $Y^{(i)}$ such that $\Gamma^{(i)*}$ on the E_1-terms are compatible with Γ^* on the limit, for every $\ell \ne \mathrm{char}\ k$. This means that the alternating sum $\mathrm{Tr}(\sigma\circ\Gamma^* : H^*(X_{\overline{K}}, \mathbb{Q}_\ell))$ for $\sigma \in W_K, n(\sigma)\ge 0$ is a linear combination of

$\mathrm{Tr}(F^{n(\sigma)} \circ \sigma^*_{\mathrm{geom}} \circ \Gamma^{(i)*} : H^*(Y^{(i)}_{\bar{k}}, \mathbb{Q}_\ell(j)))$, where F denotes the Card(k)-th power Frobenius and σ_{geom} is a geometric endomorphism of $Y^{(i)}$. The latter is computed by using the Lefschetz trace formula, that is indepenent of ℓ. In other words, one can compute the traces of actions of the Galois group in geometric terms. For $p = \mathrm{char}(k)$, the same argument works if we assume that Γ is extended to a finite étale algebraic correspondence on $X_{\mathcal{O}}$, since Mokrane's weight spectral sequence is compatible with the trace map for finite étale morphisms.

We go back to our case. Applying the stable reduction theorem to the Shimura curve M, we obtain a strictly semi-stable model of $M \times N$ for a fixed level. By the modular interpretation of A' and A_0, the Kuga-Sato family $X \to M \times N$ is extended to an abelian scheme over the strictly semi-stable model. Thus the total space also has a strictly semi-stable model. The relevant algebraic correspondence Γ is a linear combination of endomorphisms of abelian schemes whose degree are prime-to-p and of Hecke operators whose level are prime-to-p. Thus they are extended to finite étale algebraic correspondences on the strictly semi-stable model. This completes Step 2.

5 Weight-monodromy conjecture

We show that $D_{\mathfrak{p}}V_\lambda$ satisfies the monodromy-weight conjecture. To simplify the notation, we sketch the proof for $\lambda \nmid p$. The case $\lambda \mid p$ is proved similarly with suitable modifications. Since V_λ is a direct summand, it suffices to show the conjecture for $H^1(M_{\overline{K}}, W_\lambda)$. If $k = 2$, W_λ is the constant sheaf on a curve M. Hence the monodromy-weight conjecture is known in this case. Thus we may assume $k > 2$.

By cutting down the weight spectral sequences by Γ^*, we obtain a spectral sequence converging to $D_{\mathfrak{p}}H^*(M_{\overline{E}}, W_\lambda)$ whose E_1-terms are given by

$$\begin{array}{llll}
E_1^{-1,2} = H^0(Y^{(1)}_{\bar{k}}, W_\lambda(-1)) & \overset{d^{-1,2}}{\rightarrow} & H^2(Y^{(0)}_{\bar{k}}, W_\lambda) & \\
 & & H^1(Y^{(0)}_{\bar{k}}, W_\lambda) & \\
 & & H^0(Y^{(0)}_{\bar{k}}, W_\lambda) & \overset{d^{0,0}}{\rightarrow} \quad E_1^{1,0} = H^0(Y^{(1)}_{\bar{k}}, W_\lambda)
\end{array}$$

and their cristalline counterpart. The boundary map $d^{-1,2}$ is the Gysin map and $d^{0,0}$ is the restriction map.

We distinguish 2-types of the components of $Y^{(0)}$: ordinary or supersingular. For an ordinary component, we prove $H^0 = H^2 = 0$, as an analogue of Igusa's theorem. For a supersingular component, the sheaf W_λ is geometrically constant. Thus the complexes $d^{0,0} : E_1^{0,0} \to E_1^{1,0}$ and $d^{-1,2} : E_1^{-1,2} \to E_1^{0,2}$ admit purely combinatorial description. We know that the monodromy operator N is induced by the identity $E_1^{-1,2}(1) \to E_1^{1,0}$. The combinatorial description allows us to conclude that N being isomorphism on the E_1-terms implies the same on the E_2-terms.

The filtration W is defined by the weight spectral sequence that degenerates at E_2. Hence, the fact that N is an isomorphism on the E_2-terms implies that the filtration W satisfies the condition (3) in Conjecture 2 and is equal to the filtration M. This completes Step 3.

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