Ramification theory of schemes in mixed characteristic case

(joint work with K. Kato)

juin 28, 2005

Abstract

We define generalizations of classical invariants of ramification, for coverings on a variety of arbitrary dimension over a local field of mixed characteristic. For an ℓ-adic sheaf, we define its Swan class as a 0-cycle class supported on the closed fiber. We present a formula for the Swan conductor of cohomology and its relative version.

Let K be a complete discrete valuation field of characteristic 0 and F be the residue field of K. We assume F is a perfect field of characteristic $p > 0$.

Let U be a separated smooth scheme purely of dimension d of finite type over K. Let $f : V \to U$ be a finite étale morphism. The goal of this talk is to introduce a map

$$((\ ,\Delta_{\overline{V}}))^{\log} : Z_d(V \times_U V) \longrightarrow F_0 G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{0.1}$$

and to show that this map gives generalizations of classical invariants of ramification.

0.1 source

Since $V \to U$ is assumed finite étale, the fiber product $V \times_U V$ is also finite étale over U and hence is smooth of dimension d over K. Thus, $Z_d(V \times_U V)$ is the free abelian group generated by the classes of irreducible components of $V \times_U V$. In particular, if U is connected and V is a Galois covering of Galois group G, it is identified with the free abelian group $\mathbb{Z}[G]$.

0.2 target

For a noetherian scheme X, the Grothendieck group of the category of coherent O_X-modules is denoted by $G(X)$. Let $F_n G(X) \subset G(X)$ denote the topological filtration generated by the classes of modules of dimension of support at most n.

For V as above, we define $\mathcal{C}_V$ to be the category, whose objects are proper flat schemes Y over O_K containing V as a dense open subscheme. A morphism $Y' \to Y$ in $\mathcal{C}_V$ is a morphism $Y' \to Y$ over O_K inducing the identity on V. We put

$$F_0G(\overline{V}_F) = \varprojlim_{\mathcal{C}_V} F_0G(Y_F). \tag{0.2}$$

The transitions maps are proper push-forwards.

For a map $f : V \to U$ of separated smooth schemes of finite type over K, the push-forward map

$$f_* : F_0G(\overline{V}_F) \longrightarrow F_0G(\overline{U}_F) \tag{0.3}$$

is defined. In particular, taking $U = \text{Spec } K$, the degree map

$$\deg : F_0G(\overline{V}_F) \longrightarrow \mathbb{Z} \tag{0.4}$$

is defined. For a finite flat map $f : V \to U$ of separated smooth schemes of finite type over K, the flat pull-back map

$$f^* : F_0G(\overline{U}_F) \longrightarrow F_0G(\overline{V}_F) \tag{0.5}$$

is defined by the flattening theorem of Raynaud-Gruson.

Contents

1 Classical case

Let K and F be as above. Let L be a finite separable extension and consider $f : V = \text{Spec } L \to U = \text{Spec } K$. Then, the target group $F_0G(\overline{V}_F)$ of the map (0.1) is $F_0G(\text{Spec } O_L \otimes_{O_K} F) = \mathbb{Z}$. The map $f^* : F_0G(\overline{U}_F) = \mathbb{Z} \to F_0G(\overline{V}_F) = \mathbb{Z}$ is the multiplication by the ramification index $e_{L/K}$ and the map $f_* : F_0G(\overline{V}_F) = \mathbb{Z} \to F_0G(\overline{U}_F) = \mathbb{Z}$ is the multiplication by the residual degree $f_{L/K}$. If L is a Galois extension of Galois group G, the source group $Z_d(V \times_U V)$ is $\mathbb{Z}[G]$.

1.1 different

For a finite separable extension L of K, the different and its wild part are defined by

$$
\begin{aligned}
&(1.1) \qquad & D_{L/K} &= \text{length}_{O_L} \Omega^1_{O_L/O_K}, \\
&(1.2) \qquad & D^{\log}_{L/K} &= D_{L/K} - (e_{L/K} - 1).
\end{aligned}
$$

We have $D^{\log}_{L/K} \geq 0$. The equality is equivalent to $p \nmid e_{L/K}$. For an intermediate extension $K \subset M \subset L$, we have chain rules

$$
\begin{aligned}
&(1.3) \qquad & D_{L/K} &= D_{L/M} + e_{L/M} D_{M/K} \\
&(1.4) \qquad & D^{\log}_{L/K} &= D^{\log}_{L/M} + e_{L/M} D^{\log}_{M/K}.
\end{aligned}
$$

1.2 Artin and Swan characters

If L is a Galois extension of Galois group G, the Artin character and the Swan character are defined by

$$
\begin{aligned}
&(1.5) \qquad & a_{L/K}(\sigma) &= a_G(\sigma) \\
& & &= \begin{cases} D_{L/K} & \text{if } \sigma = 1 \\ -\text{length}_{O_L} O_L/(\sigma(x) - x; x \in O_L) & \text{if } \sigma \neq 1, \end{cases} \\
&(1.6) \qquad & s_{L/K}(\sigma) &= s_G(\sigma) \\
& & &= \begin{cases} D^{\log}_{L/K} & \text{if } \sigma = 1 \\ -\text{length}_{O_L} O_L/(\sigma(x)/x - 1; x \in O_L \setminus \{0\}) & \text{if } \sigma \neq 1. \end{cases}
\end{aligned}
$$

We have $s_G(\sigma) = 0$ unless the order of σ is a power of p. We also have $s_G(\sigma) = s_G(\sigma')$ if $\langle \sigma \rangle = \langle \sigma' \rangle$.

For a subgroup $H \subset G$ with corresponding intermediate extension M, we have

$$
(1.7) \qquad s_H(\sigma) = \begin{cases} s_G(1) - e_{L/M} D^{\log}_{M/K} & \text{if } \sigma = 1 \\ s_G(\sigma) & \text{if } \sigma \neq 1 \end{cases}
$$

and a similar equality for the Artin character, for $\sigma \in H$. For a quotient group $G \to \overline{G}$ with corresponding intermediate extension M, we have

$$e_{L/M} \cdot s_{\overline{G}}(\sigma) = \sum_{\tau \in G, \sigma = \bar{\tau}} s_G(\tau) \tag{1.8}$$

and a similar equality for the Artin character, for $\sigma \in \overline{G}$. In particular, putting $\overline{G} = 1$, we obtain $\sum_{\sigma \in G} a_G(\sigma) = \sum_{\sigma \in G} s_G(\sigma) = 0$.

1.3 Swan conductor

Let M be an ℓ-adic representation of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. Let L be a finite Galois extension of K of Galois group G such that the reduction modulo ℓ of the restriction to $G_L \subset G_K$ is trivial. Then, the Swan conductor of M is defined by

$$\mathrm{Sw}(M) = \frac{f_{L/K}}{|G|} \sum_{\sigma \in G} s_{L/K}(\sigma) \mathrm{Tr}(\sigma : M) \tag{1.9}$$

$$= \frac{f_{L/K}}{|G|} \sum_{\sigma \in G} s_{L/K}(\sigma) (\dim M^{\sigma} - \frac{\dim M^{\sigma}/M^{\sigma^p}}{p-1}). \tag{1.10}$$

It is independent of L by (1.8). We can use the second equality as the definition for a mod-ℓ representation.

The Hasse-Arf theorem asserts that $\mathrm{Sw}(M)$ is an integer. We have $\mathrm{Sw}(M) \geq 0$ and the equality holds if and only if the restriction to the p-Sylow subgroup P_K of the inertia subgroup $I_K \subset G_K$ is trivial. The equality (1.7) implies the induction formula

$$\mathrm{Sw}(\mathrm{Ind}_{G_L}^{G_K} M) = f_{L/K}(\dim M \cdot D_{L/K}^{\log} + \mathrm{Sw}(M)). \tag{1.11}$$

2 Definition of the map (0.1)

2.1 Logarithmic diagonal

Let U be a separated smooth scheme of dimension d of finite type over K and X be a separated regular flat scheme of finite type over O_K containing U as the complement of a divisor D with simple normal crossing. Let $(X \times_{O_K} X)^\sim$ be the log self-product and the log diagonal closed immersion $\Delta_X : X \to (X \times_{O_K} X)^\sim$. Its generic fiber $X_K \to (X_K \times_K X_K)^\sim$ is a regular immersion of codimension d.

We give a local description. Assume $X = \mathrm{Spec}\ A$ and D is defined by $\prod_{i \in I} t_i$. Then, we have

$$(X \times_{O_K} X)^\sim = A \otimes_{O_K} A[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i)\ (i \in I))$$

and the log diagonal map is defined by the map

$$A \otimes_{O_K} A[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i)\ (i \in I)) \to A$$

sending $a \otimes 1$ and $1 \otimes a$ to $a \in A$ and U_i to 1 for $i \in I$.

2.2 Logarithmic localized intersection product

Theorem 1 *Let the notation be as above. Then,*

1. There exists a unique map

$$((\ ,\Delta_X))^{\log} : G((X \times_{O_K} X)^{\sim}) \longrightarrow G(X_F) \tag{2.1}$$

such that for a coherent module $\mathcal{F}$ and an integer $q > d$, we have

$$(([\mathcal{F}], \Delta_X))^{\log} = (-1)^q ([\mathcal{T}or_q^{O_{(X\times_{O_K} X)^{\sim}}}(\mathcal{F}, O_X)] - [\mathcal{T}or_{q+1}^{O_{(X\times_{O_K} X)^{\sim}}}(\mathcal{F}, O_X)]).$$

2. The map (2.1) *induces a map*

$$((\ ,\Delta_X))^{\log} : G((X_K \times_K X_K)^{\sim}) \longrightarrow G(X_F). \tag{2.2}$$

Further, it maps $F_{d+i}G((X_K \times_K X_K)^{\sim})$ into $F_iG(X_F)$ for $i \in \mathbb{Z}$.

Now we define the map (0.1).

Corollary 2 *Let $f : U \to V$ be a finite étale morphism. Then, there exists a unique map*

$$((\ ,\Delta_{\overline{V}}))^{\log} : Z_d(V \times_U V) \longrightarrow F_0G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q} \tag{0.1}$$

that makes the diagram

$$\begin{array}{ccc}
Z_d(V \times_U V) & \xrightarrow{((\ ,\Delta_{\overline{V}}))^{\log}} & F_0G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\downarrow {\scriptstyle \otimes_{O_{V\times_K V}} O_{W\times_K W}} & & \downarrow {\scriptstyle \text{projection}} \\
F_dG(W \times_U W) & \searrow & F_0G(Y_F) \otimes_{\mathbb{Z}} \mathbb{Q} \\
\uparrow {\scriptstyle \text{restriction}} & & \uparrow {\scriptstyle \frac{1}{[W:V]} g_*} \\
F_dG((Z_K \times_{X_K} Z_K)^{\sim}) & \xrightarrow[((\ ,\Delta_Z))^{\log}]{} & F_0G(Z_F)
\end{array} \tag{2.3}$$

commutative for an arbitrary diagram

$$\begin{array}{ccccc}
W & \xrightarrow{\subset} & Z & & \\
{\scriptstyle g}\downarrow & & \downarrow & \searrow & \\
V & \xrightarrow{\subset} & Y & & \\
{\scriptstyle f}\downarrow & & & & \\
U & & \xrightarrow{\subset} & & X
\end{array} \tag{2.4}$$

of schemes over O_K satisfying the following conditions:

(2.4.1) *X is a proper flat scheme over O_K containing U as the complement of a Cartier divisor B. The generic fiber X_K is smooth and the divisor B_K has simple normal crossings.*

(2.4.2) *Y is a proper flat scheme over O_K containing V as a dense open subscheme. Namely, Y is an object of $\mathcal{C}_V$.*

(2.4.3) *Z is a proper regular scheme over O_K containing W as the complement of a divisor D with simple normal crossings.*

(2.4.4) *The quadrangles are Cartesian.*

(2.4.5) *The proper map $g : W \to V$ is generically finite of constant degree $[W : V]$.*

3 Ramification theory

Let K and F be as above. Let $f : V \to U$ be a finite étale morphism of separated smooth schemes purely of dimension d of finite type over K. We define generalizations of classical invariants recalled in Section 1 to higher dimension by using the map

$$((\ , \Delta_{\overline{V}}))^{\log} : Z_d(V \times_U V) \longrightarrow F_0G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q}. \tag{0.1}$$

3.1 different

We define the wild different by

$$D^{\log}_{V/U} = (([V \times_U V] - [\Delta_V], \Delta_{\overline{V}}))^{\log} = f^*(([\Delta_U], \Delta_{\overline{U}}))^{\log} - (([\Delta_V], \Delta_{\overline{V}}))^{\log} \tag{3.1}$$

in $F_0G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q}$. For a finite étale morphism $g : V' \to V$, we have obviously a chain rule

$$D_{V'/V} = D_{V'/V} + g^* D_{V/U}. \tag{3.2}$$

3.2 Swan character class

If V is a Galois covering of Galois group G, we define the Swan character class $s_{V/U}(\sigma) \in F_0G(\overline{V}_F) \otimes_{\mathbb{Z}} \mathbb{Q}$ by

$$s_{V/U}(\sigma) = s_G(\sigma) = \begin{cases} D^{\log}_{V/U} & \text{if } \sigma = 1, \\ -((\Gamma_\sigma, \Delta_{\overline{V}}))^{\log} & \text{if } \sigma \neq 1. \end{cases} \tag{3.3}$$

We have $s_G(\sigma) = 0$ unless the order of σ is a power of p. We expect but do not know $s_G(\sigma) = s_G(\sigma')$ if $\langle\sigma\rangle = \langle\sigma'\rangle$. We have equalities analogous to (1.7) and (1.8).

3.3 Swan class

Let $\mathcal{F}$ be a smooth ℓ-adic sheaf on U. We take a finite étale Galois covering $f : V \to U$ trivializing the reduction $\overline{\mathcal{F}}$. Let G be the Galois group and M be the representation of G corresponding to $\overline{\mathcal{F}}$. Then, we define the Swan class $\mathrm{Sw}\mathcal{F} \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}$ and the naive Swan class $\mathrm{Sw}^{\mathrm{naive}}\mathcal{F} \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}(\zeta_{p^\infty})$ by

$$\mathrm{Sw}\mathcal{F} = \frac{1}{|G|}\sum_{\sigma\in G} f_* s_G(\sigma)(\dim M^\sigma - \frac{\dim M^\sigma/M^{\sigma^p}}{p-1}), \tag{3.4}$$

$$\mathrm{Sw}^{\mathrm{naive}}\mathcal{F} = \frac{1}{|G|}\sum_{\sigma\in G_{(p)}} f_* s_G(\sigma)\mathrm{Tr}^{Br}(\sigma : \dim M) \tag{3.5}$$

where $G_{(p)} = \{\sigma \in G|$ the order of σ is a power of $p\}$ and Tr^{Br} denotes the Brauer trace. They are independent of V by an analogue of (1.8). The Swan class $\mathrm{Sw}\mathcal{F}$ is the image of the naive Swan class by the projection $\mathbb{Q}(\zeta_{p^\infty}) \to \mathbb{Q}$.

We expect the following generalization of the Hasse-Arf theorem.

Conjecture 3 *The Swan class* $\mathrm{Sw}\mathcal{F} \in F_0G(\overline{U}) \otimes_{\mathbb{Z}} \mathbb{Q}$ *is in the image of* $F_0G(\overline{U})$

Theorem 4 *Conjecture 3 is true if* $\dim U = 1$.

Idea of Proof. By the induction formula below, it is reduced to the rank 1 case. In the rank 1 case, one can compute the Swan class explicitly in terms of the ramification divisor in the sense of Kato. ■

Conjecture 3 implies a conjecture of Serre:

The Artin central function for an isolated fixed point is a character.

We have a conductor formula.

Theorem 5 *If* $V = \mathrm{Spec}\ K$, *we have*

$$\mathrm{Sw}R\Gamma_c(U_{\overline{K}}, \mathcal{F}) = \mathrm{deg}\mathrm{Sw}\mathcal{F} - \mathrm{rank}\mathcal{F} \cdot \deg((\Delta_U, \Delta_{\overline{U}}))^{\log}. \tag{3.6}$$

Further if $\mathcal{F} = \mathbb{Q}_\ell$, *we obtain*

$$\mathrm{Sw}R\Gamma_c(U_{\overline{K}}, \mathbb{Q}_\ell) = -\deg((\Delta_U, \Delta_{\overline{U}}))^{\log}. \tag{3.7}$$

Idea of Proof. A logarithmic Lefschetz trace formula for open variety and the associativity for the localized intersection product. ■

We expect have the following relative version.

Conjecture 6 *Let* $f : U \to V$ *be a smooth morphism of relative dimension* d *of separated smooth schemes of finite type over* K. *We assume that there exist a proper smooth scheme* X *over* V *containing* U *as the complement of a divisor* D *with relative simple normal crossings.*

Then, for a smooth $\bar{\mathbb{Q}}_\ell$*-sheaf or a smooth* $\bar{\mathbb{F}}_\ell$*-sheaf* $\mathcal{F}$, *we have*

$$\mathrm{rank}Rf_!\mathcal{F} \cdot ((\Delta_V, \Delta_{\overline{V}}))^{\log} - \mathrm{Sw}Rf_!\mathcal{F} = f_*(\mathrm{rank}\mathcal{F} \cdot ((\Delta_U, \Delta_{\overline{U}}))^{\log} - \mathrm{Sw}\mathcal{F}). \tag{3.8}$$

We can prove Conjecture 6 if $\dim V = 0$.

Note that we have

$$\mathrm{rank} Rf_!\mathcal{F} = \mathrm{rank}\mathcal{F} \cdot \mathrm{rank} Rf_!\mathbb{Q}_\ell = \mathrm{rank}\mathcal{F} \cdot (-1)^d \mathrm{deg}\ c_d(\Omega^1_{X/V}(\log D)).$$

The equality (3.8) is equivalent to the combination of

$$(3.9)\quad \mathrm{Sw} Rf_!\mathcal{F} = f_*\mathrm{Sw}\mathcal{F} + \mathrm{rank}\mathcal{F} \cdot \mathrm{Sw} Rf_!\mathbb{Q}_\ell,$$
$$(3.10)\quad \mathrm{Sw} Rf_!\mathbb{Q}_\ell = (-1)^d \mathrm{deg}\ c_d(\Omega^1_{X/V}(\log D)) \cdot ((\Delta_V, \Delta_{\overline{V}}))^{\log} - f_*((\Delta_U, \Delta_{\overline{U}}))^{\log}.$$

If $d = 0$, in other words, if $f : V \to U$ is finite étale, the right hand side of (3.10) is $f_* D^{\log}_{V/U}$.