An ℓ-adic Riemann-Roch formula. jt wk w/ K. KATO.
p-adic? log geometry

Conductor formula (5 years ago at Conf. for Illusie).

K local field. (c.d.v.f char $K=0$, res fd F perfect char $p>0$.)

U/K smooth sep'd of f.t. $\mathcal{F}$ smooth ℓ-adic on U, $\ell \neq p$

$$Sw_K H^*_c(U_{\bar K}, \mathcal{F}) = rk\,\mathcal{F} \cdot Sw_K H^*_c(U_{\bar K}, \mathbb{Q}_\ell) + \deg Sw_U \mathcal{F}.$$

Swan conductor — Swan class.

Arithmetic analogue of the GOS formula in higher dim

$$\chi_c(U_{\bar F}, \mathcal{F}) = rk\,\mathcal{F} \cdot \chi_c(U_{\bar F}, \mathbb{Q}_\ell) - \deg Sw_U \mathcal{F}$$

Generalization to constructible sheaves.

- U
- $\mathcal{F}$
- f

Relative version for an arbitrary morphism /K $f: V \to U$ of sep'd schemes of fin. type /K.

$\mathcal{F}$ constructible $\Rightarrow Sw_U \mathcal{F}$; $\overline{Sw}_U \mathcal{F}$; $\overline{Sw}_V Rf_! \mathcal{F} = f_! \overline{Sw}_U \mathcal{F}$

Main ingredients.

- Definition of the Swan class.
  (1) Smooth sheaf : k-th localized log. int'n product
  (2) const. sheaf : Excision formula.
- Proof of the formula.
  Dévissage $\to$ { • Induction formula, • Relative curve }
  log Lefschetz trace formula for open varieties.

Another topic

Integrality of the Swan class

- generalization of the Hasse-Arf thm dim 2
- Proved by reducing to the rk 1 case.
- Implies a conj of Serre on the Artin rep'n for an isolated fixed pt in dim 2.

Plan. · Def'n of the Swan class
· Riemann–Roch formula and its proof.

1. Def'n of the Swan class.

Classical case: Swan conductor

$G = Gal(L/K)$. L/K totally ramified. M rep of G

$$Sw_K M = \frac{1}{|G|} \sum_{\substack{\sigma \neq 1 \\ \sigma \in G}} s_G(\sigma) \left(Tr(\sigma : M) - \dim M \right).$$

$$s_G(\sigma) = - ord_L \left(\frac{\sigma(\pi_L)}{\pi_L} - 1 \right).$$

$f : V \to U$ finite étale Galois covering of smooth sep'd sch of f.t./K. $\sigma \in G = Gal(V/U)$, $\neq 1$

$((\Gamma_\sigma, \Delta))$

First approximation

Geom case /F $V \hookrightarrow Y$ smooth compactification, $V \downarrow U$

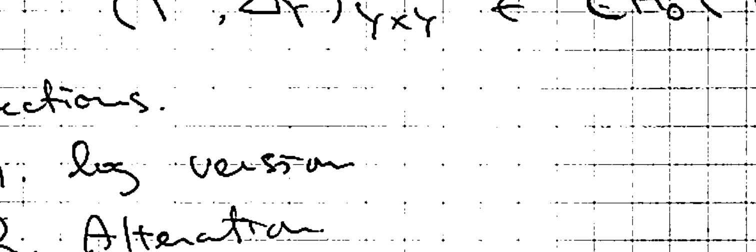
$$(\bar{\Gamma}, \Delta_Y)_{Y \times Y} \in CH_0(Y \setminus V).$$

Corrections.

1. log version
2. Alteration
3. limit of compactification
4. $/S = Spec\, O_K$. K-theory / Chow gp.

1. log version $V = Y \setminus D$ $D = D_1 \cup \cdots \cup D_n$ simple N.C.

$(Y \times Y)^\sim$ blow up of $Y \times Y$ at $D_1 \times D_1, D_2 \times D_2, \ldots, D_n \times D_n$
and remove proper transf. of $D \times Y \cup Y \times D$.



Difference Artin & Swan
focus on wild ramification

$(\bar{\Gamma}, \Delta_Y)_{(Y \times Y)^\sim}$ indep of $\bar{\Gamma}$.

2. alteration

$$\begin{array}{ccc} W & \subset & Z \\ g \downarrow & & \downarrow s \\ V & \hookrightarrow & Y \end{array} \qquad \frac{1}{[Z:Y]} g_* \left((g\times s)^! \Gamma, \Delta_Z \right)_{(Z\times Z)^\sim}$$

$$\in CH_0(Y \setminus V)_{\mathbb{Q}}$$

indep of Z.

3 limit of cpt'n.

$$\begin{array}{ccc} & & Y' \\ & \nearrow & \downarrow \\ V & \hookrightarrow & Y \end{array} \qquad (\Gamma, \Delta_V) \in \varprojlim CH_0(Y \setminus V)_{\mathbb{Q}} = CH_0(\partial V)_{\mathbb{Q}}$$

Rmk If we assume strong res'n of sing. we don't need to introduce $\mathbb{Q}$.

e.g. dim 2

One can define it as a class supp on "wild ram'n locus".

4 Over O_K.

Suppose we have $V = Y \setminus D$ Y regular. D SNC.

$(Y \times_S Y)^\sim$ log product

$$((\Gamma, \Delta_Y))_{(Y\times_S Y)^\sim} = (-1)^d \Big([\mathrm{Tor}_d^{O_{(Y\times_S Y)^\sim}} (O_\Gamma, O_\Delta)] - [\mathrm{Tor}_{d+1} (\quad)] \Big)$$

is indep for $d \gg 0$, $\in F_0 G(Y_F)$. dim 0-part of Groth gp of coh sheaves

and is indep't of d.

defined even for $\sigma = 1$.

$$((\Gamma_\sigma, \Delta_V)) \in F_0 G(\partial_F V)_{\mathbb{Q}} = \varprojlim F_0 G(Y_F)_{\mathbb{Q}}$$

Swan class

$$Sw_U \mathcal{F} = \frac{1}{|G|} \sum_{\substack{\sigma \in G \\ \neq 1}} f_! ((\Gamma_\sigma, \Delta_V)) \left(\mathrm{Tr}(\sigma : M) - \mathrm{rk}\,\mathcal{F} \right)$$

even assuming res'n

$\neq 0$ only if order of σ is a power of p.

$\in F_0 G(\partial_F U)_{\mathbb{Q}} \leftarrow$ integrality?

Variant total Swan class

$$\overline{Sw}_U \mathcal{F} = \frac{1}{|G|} \sum_{\sigma \in G} f_! ((\Gamma_\sigma, \Delta_V)) \, \mathrm{Tr}(\sigma : M).$$

Theorem (Excision formula)

$U_1 \subset U$ smooth closed subsch $U_0 = U \setminus U_1$

$$\Rightarrow \quad Sw_U \mathcal{F} = Sw_{U_0}(\mathcal{F}|_{U_0}) + Sw_{U_1}(\mathcal{F}|_{U_1}).$$

$$\overline{Sw_U}\mathcal{F} = \quad \overline{} \quad \overline{} \quad -$$

Consequence $Sw_U \mathcal{F}$, $\overline{Sw_U}\mathcal{F}$ can be def. for constructible $\mathcal{F}$.

Sketch of Pf. – Blow up formula $U' \to U$.

– Divisor case

$$Sw_{U'}\mathcal{F} - Sw_U \mathcal{F} = (e-1) \times Sw_{U_1}(\mathcal{F}|_{U_1}).$$

Both cases are deduced from computation of Tor

$U' \to U$. $\quad [\mathrm{Tor}_i^{\mathcal{O}_{U\times_S U}}(\mathcal{O}_{\Delta_U}, \mathcal{O}_{U'\times_S U'})] = [\Delta_{U'}] + \sum_p (-1)^p [\mathbb{P}(N_1 \Omega^p_{E/U_1} \otimes \mathbb{P}(N_2 \mathcal{O}(-1))]$

$$E \times_{U_1} E \subset U' \times_S U'$$

$(U\times_S U)^\sim \to U\times_S U$.

$$[\mathrm{Tor}^{\mathcal{O}_{U\times_S U}}(\mathcal{O}_{\Delta_U}, \mathcal{O}_{(U\times_S U)^\sim})] = [\Delta_U] + [G_{m,U_1}]$$

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2 ℓ-adic Riemann-Roch.

Theorem. $f: U \to T$ morph l.c. of sep sch of f.t/K

$\mathcal{F}$ const ℓ-adic sheaf on U. Then

$$\overline{Sw_T} Rf_! \mathcal{F} = f_! \overline{Sw_U}\mathcal{F}.$$

Cor $T = \operatorname{Spec} K$. $\mathcal{F}$ on U smooth

(1) $Sw_K H_c^*(U_{\bar K}, \mathcal{F}) = \operatorname{rk}\mathcal{F} \cdot Sw_K H_c^*(U_{\bar K}, \mathbb{Q}_\ell) + \deg Sw_U \mathcal{F}$.

(2) $Sw_K H_c^*(U_{\bar K}, \mathbb{Q}_\ell) = -\deg((\Delta_U, \Delta_U))$

~~Thy (the (2))~~ If $X \supset U = X \setminus D$ X regular D S.N.C

$$((\Delta_U, \Delta_U)) = \underset{\substack{\uparrow \\ \text{localized Chern class}}}{c_{\dim X}}(\Omega^1_{X/S}(\log D))_{X_F} \Rightarrow \text{Bloch's conductor formula.}$$