Ramification theory of schemes over a local field

(joint work with K. Kato)

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Abstract

We introduce the Swan class of an ℓ-adic etale sheaf on a variety over a local field. It is a generalization of the classical Swan conductor measuring the wild ramification and is defined as a 0-cycle class supported on the reduction. We establish a Riemann-Roch formula for the Swan class.

Let K be a complete discrete valuation field of characteristic 0. We assume that the residue field F is a perfect field of characteristic $p > 0$.

Let U be a separated scheme of finite type over K and $\mathcal{F}$ be an ℓ-adic sheaf on U where ℓ is a prime different from p. The etale cohomology $H^*_c(U_{\bar{K}}, \mathcal{F})$ with compact support defines an ℓ-adic representation of the absolute Galois group $G_K = \mathrm{Gal}(\bar{K}/K)$. We will give a formula for the alternating sum Sw $H^*_c(U_{\bar{K}}, \mathcal{F})$ of the Swan conductor. In the case where U is smooth over K and $\mathcal{F}$ is a smooth sheaf on U, it takes the form

$$\mathrm{Sw}\ H^*_c(U_{\bar{K}}, \mathcal{F}) - \mathrm{rank}\ \mathcal{F} \times \mathrm{Sw}\ H^*_c(U_{\bar{K}}, \mathbb{Q}_\ell) = \deg\ \mathrm{Sw}\mathcal{F}.$$

We will have a relative version of the formula for an arbitrary sheaf $\mathcal{F}$ and an arbitrary morphism $U \to V$. The general version will be formulated by introducing a map

$$\overline{\mathrm{Sw}}_U : K_0(U, \overline{\mathbb{F}}_\ell) \longrightarrow F_0G(\overline{U}_F)_{\mathbb{Q}}.$$

Here $K_0(U, \overline{\mathbb{F}}_\ell)$ denotes the Grothendieck group of constructible $\overline{\mathbb{F}}_\ell$-sheaves on U and $F_0G(\overline{U}_F)_{\mathbb{Q}}$ denotes the dimension 0-part of the Grothendieck group of coherent modules on the reduction of U whose precise definition will be given later. Note that the reduction modulo ℓ defines a natural map $K_0(U, \overline{\mathbb{Q}}_\ell) \to K_0(U, \overline{\mathbb{F}}_\ell)$. In the case $U = \mathrm{Spec}\ K$, we have

$$K_0(\mathrm{Spec}\ K, \overline{\mathbb{F}}_\ell) = K_0(\mathrm{Rep}_{G_K}(\overline{\mathbb{F}}_\ell)), \quad F_0G(\overline{\mathrm{Spec}\ K}_F)_{\mathbb{Q}} = G(F)_{\mathbb{Q}} = \mathbb{Q}$$

and, for an $\overline{\mathbb{F}}_\ell$-representation V of G_K, we have

$$\overline{\mathrm{Sw}}_{\mathrm{Spec}\ K}(V) = \mathrm{Sw}(V).$$

The main result in this talk is the following.

Theorem 1 *Let $f : U \to V$ be a morphism of separated schemes of finite type over K. Then, the diagram*

$$\begin{array}{ccc} K_0(U, \overline{\mathbb{F}}_\ell) & \xrightarrow{\overline{\mathrm{Sw}}_U} & F_0G(\overline{U}_F)_{\mathbb{Q}} \\ {\scriptstyle f_!}\downarrow & & \downarrow{\scriptstyle f_!} \\ K_0(V, \overline{\mathbb{F}}_\ell) & \xrightarrow{\overline{\mathrm{Sw}}_V} & F_0G(\overline{V}_F)_{\mathbb{Q}} \end{array}$$

is commutative.

First, we define the group $F_0G(\overline{U}_F)_{\mathbb{Q}}$ and the map $\overline{\mathrm{Sw}}_U : K_0(U, \overline{\mathbb{F}}_\ell) \to F_0G(\overline{U}_F)_{\mathbb{Q}}$.

For a noetherian scheme X, the Grothendieck group of the category of coherent O_X-modules is denoted by $G(X)$. Let $F_nG(X) \subset G(X)$ denote the topological filtration generated by the classes of modules of dimension of support at most n.

Let U be a separated scheme of finite type over K. We define $\mathcal{C}_U$ to be the category, whose objects are proper schemes X over the integer ring O_K containing U as a dense open subscheme. A morphism $X' \to X$ in $\mathcal{C}_U$ is a morphism $X' \to X$ over O_K inducing the identity on U. We put

$$F_0G(\overline{U}_F) = \varprojlim_{\mathcal{C}_U} F_0G(X_F). \tag{0.1}$$

The transitions maps are proper push-forwards.

If $U = \mathrm{Spec}\ K$, $\mathrm{Spec}\ O_K$ is the initial object of $\mathcal{C}_{\mathrm{Spec}\ K}$ by the valuative criterion and consequently we have $F_0G(\overline{\mathrm{Spec}\ K_F}) = \mathbb{Z}$.

For a map $f : U \to V$ of separated schemes of finite type over K, the push-forward map

$$f_* : F_0G(\overline{U}_F) \longrightarrow F_0G(\overline{V}_F) \tag{0.2}$$

is defined. In particular, taking $V = \mathrm{Spec}\ K$, the degree map

$$\deg : F_0G(\overline{U}_F) \longrightarrow \mathbb{Z} \tag{0.3}$$

is defined.

For a finite flat map $f : U \to V$ of separated schemes of finite type over K, the flat pull-back map

$$f^* : F_0G(\overline{V}_F) \longrightarrow F_0G(\overline{U}_F) \tag{0.4}$$

is defined by the flattening theorem of Raynaud-Gruson.

The Grothendieck group $K_0(U, \overline{\mathbb{F}}_\ell)$ is generated by the classes of smooth sheaves on smooth subschemes. Thus, we first define $\overline{\mathrm{Sw}}_U(\mathcal{F})$ assuming U is smooth over K and $\mathcal{F}$ is a smooth $\overline{\mathbb{F}}_\ell$-sheaf on U. Let $V \to U$ be a finite etale Galois covering of Galois group G trivializing $\mathcal{F}$.

We put

$$\overline{\mathrm{Sw}}^{\mathrm{naive}}(\mathcal{F}) = \frac{1}{|G|} \sum_{\sigma \in G_{(p)}} -f_*((\Gamma_\sigma, \Delta_V))^{\log}_{V \times_K V} \cdot \mathrm{Tr}^{Br}(\sigma : M) \in F_0G(\overline{U}_F)_{\mathbb{Q}(\zeta_{p^\infty})}.$$

Here $G_{(p)} \subset G$ denotes the set of elements of order a power of p and $\mathrm{Tr}^{Br}(\sigma : M)$ denotes the Brauer trace of the $\overline{\mathbb{F}}_\ell$-representation of G corresponding to $\mathcal{F}$. If $U = \mathrm{Spec}\ K$ and $V = \mathrm{Spec}\ L$, then the term $-f_*((\Gamma_\sigma, \Delta_V))^{\log}_{V\times_K V}$ is the Swan character $\mathrm{Sw}_{L/K}(\sigma)$ and the above formula is nothing but the classical definition of the Swan conductor.

We expect that the naive Swan class $\overline{\mathrm{Sw}}^{\mathrm{naive}}(\mathcal{F}) \in F_0G(\overline{U}_F)_{\mathbb{Q}(\zeta_{p^\infty})}$ in fact lies in $F_0G(\overline{U}_F)_{\mathbb{Q}}$. Since we do not know this in general, we define the Swan class $\overline{\mathrm{Sw}}(\mathcal{F})$ to be the image of $\overline{\mathrm{Sw}}^{\mathrm{naive}}(\mathcal{F})$ by the natural projection $F_0G(\overline{U}_F)_{\mathbb{Q}(\zeta_{p^\infty})} \to F_0G(\overline{U}_F)_{\mathbb{Q}}$ induced by $\varinjlim_n \frac{1}{[\mathbb{Q}(\zeta_{p^n}):\mathbb{Q}]}\mathrm{Tr}_{\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}}$.

A first approximation of $((\Gamma_\sigma, \Delta_V))^{\log}_{V\times_K V}$ is the intersection product $(\Gamma_\sigma, \Delta_Y)_{Y\times_F Y}$ if we *pretend* that V is a smooth variety over F, that V admits a smooth compactification Y and that the action of σ is extended to on Y. This approximation requires the following three modifications.

1. Log blow-up: We further assume that the complement $Y \setminus V = \bigcup_{i=1}^n D_i$ is a divisor with simple normal crossings. Then, we replace $Y \times_F Y$ by the blow-up $(Y \times_F Y)'$ at $D_1 \times D_1, D_2 \times D_2, \ldots, D_n \times D_n$.

2. Alteration: Since we do not know if there exists a smooth compactification Y, we consider a proper surjective generially finite morphism $g : W \to V$ and a smooth compactification $W \subset Z$ such that the complement $Z \setminus W$ is a divisor with simple normal crossings. We consider the intersection product in $(Z \times_F Z)'$ and $\frac{1}{[W:V]}g_*$.

3. Localized intersection product. Since our schemes are defined over O_K but not over F. We need to use a new intersection theory introduced in [1] which is briefly recalled below.

The localized intersection product $((\Gamma_\sigma, \Delta_V))^{\log}_{V\times_K V}$ with the log diagonal is defined using an alteration. We consider a diagram

$$\begin{array}{ccc} W & \longrightarrow & Z \\ {\scriptstyle g}\downarrow & & \\ V & & \downarrow \\ {\scriptstyle f}\downarrow & & \\ U & \longrightarrow & X \end{array} \tag{0.5}$$

of schemes over O_K satisfying the following conditions:

(0.5.1) X is an object of $\mathcal{C}_U$ such that $U \subset X$ is the complement of a Cartier divisor B.

(0.5.2) Z is a proper regular scheme over O_K containing W as the complement of a divisor D with simple normal crossings.

(0.5.3) The proper map $g : W \to V$ is generically finite of constant degree $[W : V]$.

We consider the log self-product $(Z \times_{O_K} Z)^\sim$ and the log diagonal closed immersion $\Delta_Z : Z \to (Z \times_{O_K} Z)^\sim$. Its generic fiber $Z_K \to (Z_K \times_K Z_K)^\sim$ is a regular immersion of codimension d. We give a local description. Assume $Z = \text{Spec } A$ and D is defined by $\prod_{i\in I} t_i$. Then, we have

$$(Z \times_{O_K} Z)^\sim = A \otimes_{O_K} A[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i)\ (i \in I))$$

and the log diagonal map is defined by the map

$$A \otimes_{O_K} A[U_i^{\pm 1}\ (i \in I)]/(t_i \otimes 1 - U_i(1 \otimes t_i)\ (i \in I)) \to A$$

sending $a \otimes 1$ and $1 \otimes a$ to $a \in A$ and U_i to 1 for $i \in I$.

Let d be the dimension of U. We define the logarithmic localized intersection product

$$((\ ,\Delta_Z))^{\log}_{(Z\times_{O_K}Z)^\sim} : F_{d+1}G((Z \times_{O_K} Z)^\sim) \to F_0G(Z_F)$$

by

$$(([\mathcal{F}],\Delta_Z))^{\log}_{(Z\times_{O_K}Z)^\sim} = (-1)^q([\mathcal{T}or_q^{O_{(Z\times_{O_K}Z)^\sim}}(\mathcal{F}, O_Z)] - [\mathcal{T}or_{q+1}^{O_{(Z\times_{O_K}Z)^\sim}}(\mathcal{F}, O_Z)])$$

for a coherent $O_{(Z\times_{O_K}Z)^\sim}$-module $\mathcal{F}$, by taking an arbitrary integer $q > d$.

The localized intersection product $((\Gamma_\sigma, \Delta_V))^{\log}_{V\times_K V} \in F_0G(\overline{V}_F)_{\mathbb{Q}}$ is defined as

$$\frac{1}{[W:V]} g_*((\overline{(g\times g)^*\Gamma_\sigma}, \Delta_Z))^{\log}_{(Z\times_{O_K}Z)^\sim}.$$

by taking a lifting $\overline{(g\times g)^*\Gamma_\sigma} \in F_{d+1}G((Z \times_{O_K} Z)^\sim)$ of $(g\times g)^*\Gamma_\sigma \in F_dG(W \times_K W)$.

We define $\overline{\text{Sw}}(\mathcal{F})$ in the general case. The Grothendieck group $K_0(U, \overline{\mathbb{F}}_\ell)$ is generated by the classes $[\mathcal{F}]$ of smooth sheaves $\mathcal{F}$ on smooth subschemes Z. Their relations are generated by $[\mathcal{F}] = [\mathcal{F}'] + [\mathcal{F}/\mathcal{F}']$ for smooth subsheaves $\mathcal{F}' \subset \mathcal{F}$ and $[\mathcal{F}] = [\mathcal{F}_{Z'}] + [\mathcal{F}_{Z\setminus Z'}]$ for smooth closed subschemes $Z' \subset Z$. Thus the follwing proposition implies that the map $K_0(U, \overline{\mathbb{F}}_\ell) \to F_0G(\overline{U}_F)_{\mathbb{Q}}$ is well-defined.

Proposition 2 (excision) *Let $V \to U$ be a finite etale Galois covering of smooth schemes over K and σ be an element of the Galois group. Let $U' \subset U$ be a smooth closed subscheme and U'' be the complement. We put $V' = U' \times_U V$ and $V'' = U'' \times_U V$. Then we have*

$$((\Gamma_\sigma, \Delta_V))^{\log} = ((\Gamma_{\sigma|_{V'}}, \Delta_{V'}))^{\log} + ((\Gamma_{\sigma|_{V''}}, \Delta_{V''}))^{\log}.$$

We sketch the proof of Theorem 1. We will prove

$$\overline{\text{Sw}}_V Rf_! \mathcal{F} = f_* \overline{\text{Sw}}_U \mathcal{F}.$$

We may put the following additional assumptions by devissage.

- $\overline{\mathbb{F}}_\ell$-sheaf $\mathcal{F}$ on U is smooth.

- The scheme V is smooth over K.
- Either of the following holds.
  (0) $U \to V$ is finite and étale.
  (1) $U \to V$ is a smooth curve. More precisely, there exists a proper smooth curve $X \to V$ of genus g and a divisor $D \subset X$ finite etale over V of degree d such that $U = X \setminus D$ and $2g - 2 + d > 0$.

In the case (0), it is an analogue of the induction formula for the Swan conductor and proved in exactly the same way.

To prove the case (1), we consider a commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{f'} & V' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & V \end{array}$$

of separted smooth schemes of finite type over K where the vertical arrows are finite etale Galois coverings. Let G and G' be the Galois groups respectively. We may further assume that the pull-back of $Rf_!\mathbb{F}_\ell$ to V' is constant. Then, it suffices to show that

$$f_*((\Gamma_\sigma, \Delta_V))^{\log} = \mathrm{Tr}^{\mathrm{Br}}(\sigma' : Rf'_!\mathbb{F}_\ell) \cdot ((\Gamma_{\sigma'}, \Delta_{V'}))^{\log}$$

for an element $\sigma \in G$ of order a power of p.

By alteration, this follows from an associativity formula for localized intersection product and from the log Lefschetz trace formula below.

Theorem 3 *Let L be a complete discrete valuation field and X and X' be a proper and strictly semi-stable scheme purely of relative dimension d over the integer ring O_L. Let $D \subset X$ and $D' \subset X'$ be divisors with simple normal crossings relative to O_L and $U = X_L \setminus D_L$ and $U' = X'_L \setminus D'_L$ be the complements in the generic fiber.*

We consider X and X' as log schemes with the log structure $M_X = j_*O_U^\times \cap O_X$ and $M_{X'} = j'_*O_{U'}^\times \cap O_{X'}$ where $j\colon U \to X$ and $j'\colon U' \to X'$ are the open immersions. Let P be an fs-monoid and $P \to \Gamma(X, \overline{M}_X)$ and $P \to \Gamma(X', \overline{M}_{X'})$ be frames.

Let Γ be a closed subscheme of $U \times_L U'$. Assume that the closure $\bar\Gamma$ of Γ in $(X_L \times_L X'_L)'$ satisfies $\bar\Gamma_L \cap D_L^{(1)\prime} \subset \bar\Gamma_L \cap D_L^{(2)\prime}$.

Let $f\colon X_t \to X'_t$ be a morphism of log schemes compatible with the frames and $\gamma_f\colon X_t \to (X \times_{O_L} X')^\sim$ be the log graph map. Let $[\widetilde\Gamma] \in K((X \times_{O_L} X')^\sim)$ be an element lifting $[O_\Gamma] \in K((X_L \times_L X'_L)^\sim)$ and $\gamma_f^*[\widetilde\Gamma] \in K(X_t)$ be the pull-back.

Let $\Gamma^* \circ f_*\colon H^*_{\log,c}(X_{\bar t}, \mathbb{Q}_\ell) \to H^*_{\log,c}(X_{\bar t}, \mathbb{Q}_\ell)$ denote the composition

$$\begin{array}{ccccc} H^*_{\log,c}(X_{\bar t}, \mathbb{Q}_\ell) & \xrightarrow{f_*} & H^*_{\log,c}(X'_{\bar t}, \mathbb{Q}_\ell) & \longrightarrow & H^*_c(U'_{\bar L}, \mathbb{Q}_\ell) \\ & & & & \downarrow{\scriptstyle \Gamma^*} \\ & & H^*_{\log,c}(X_{\bar t}, \mathbb{Q}_\ell) & \longleftarrow & H^*_c(U_{\bar L}, \mathbb{Q}_\ell). \end{array} \tag{0.6}$$

Then, we have

$$\mathrm{Tr}(\Gamma^* \circ f_* \colon H^*_{\log,c}(X_{\bar{t}}, \mathbb{Q}_\ell)) = \deg \gamma_f^*[\widetilde{\Gamma}] \tag{0.7}$$

References

[1] K. Kato, T. Saito, *On the conductor formula of Bloch*, Publications Mathématiques, IHES 100, (2004), 5-151.

[2] ——, *Ramification theory for varieties over a perfect field*, (preprint) `math.AG/0402010` to appear in Annales of Math.