Characteristic cycle

k perfect field char ≥ 0

X smooth $/k$

Λ finite field char inv. in k

$\mathcal{F}/X$ const. cx of Λ-modules on X_{et}

$\mathrm{Supp}\,\mathcal{F} \subset X$ closed.

stable under $\mathbb{G}_m$-action

$SS\mathcal{F} \subset T^*X$ canonical closed subset of the cotangent b'dle. $\dim T^*X = 2n$

$\| $

$\bigcup C_a$ $\dim C_a = n = \dim X$ defined by <u>Beilinson</u>

$CC\mathcal{F} = \sum m_a C_a$ $m_a \in \mathbb{Z}$. (≥ 0 if $\mathcal{F}$ perverse

$\mathrm{Supp}\, CC\mathcal{F} = SS\mathcal{F}$

Example $U \subset X$ complement of $D \subset X$ div. with SNC.

$j: U \hookrightarrow X$ open imm. $\mathcal{G}/U$ loc. const $\mathcal{F} = j_!\mathcal{G}$.

1. $\dim X = 1$. T^*X line b'dle / curve X

$$CC\mathcal{F} = -\Big(\mathrm{rk}\,\mathcal{G}\,[T^*_X X] + \sum_{x \in D} \mathrm{dt}_x \mathcal{G} \cdot [T^*_x X]\Big)$$

$\mathrm{rk}\,\mathcal{G} + \mathrm{Sw}_x \mathcal{G}$

2. $\mathcal{G}$ tamely ramified along D

$$CC\mathcal{F} = (-1)^n \sum_{I \supseteq \phi} \mathrm{rk}\,\mathcal{G} \cdot T^*_{X_I} X$$

conormal b'dle

$D = \bigcup_{i=1}^{m} D_i$ $X_I = \bigcap_{i \in I} D_i$

Theorem 1 (index formula) If X is proj

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$$ intersection number

If $\dim X = 1$. Grothendieck-Ogg-Shafarevich formula.

Compatibility with proper push forward

Thm 1. is the comp. for $f: X \to \operatorname{Spec} k$.

$f: X \to Y$ proper. (or more generally proper on $\operatorname{supp} \mathcal{F}$)

$$
\begin{array}{ccccc}
T^*X & \leftarrow & X \times_Y T^*Y & \longrightarrow & T^*Y \\
\cup & & \cup & & \cup \\
SS\mathcal{F} & \leftarrow & & \longrightarrow & f_\circ SS\mathcal{F}
\end{array}
$$

regarded as alg. cones. conical closed subset

$m = \dim Y$

$CC\mathcal{F}$ $\qquad f_! CC\mathcal{F} \in CH_m(f_\circ SS\mathcal{F})$

cycle classes

cycle if $\dim f_\circ SS\mathcal{F} = m$.

Satisfied if $\operatorname{char} k = 0$

Conjecture 1. $CC Rf_*\mathcal{F} = f_! CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$?

Thm 1. case $Y = \operatorname{Spec} k$. $\dim Y = 0$.

Assume $\dim Y = 1$

$$CC Rf_*\mathcal{F} = -\left(\dim \chi(X_{\bar\eta}, \mathcal{F}) \cdot T^*_Y Y + \sum_{y \in Y} a_y Rf_*\mathcal{F} \cdot T^*_y Y\right)$$

Artin conductor $\quad a_y Rf_*\mathcal{F} = \chi(X_{\bar\eta}, \mathcal{F}) - \chi(X_{\bar y}, \mathcal{F}) + \operatorname{Sw}_y H^*(X_{\bar\eta}, \mathcal{F})$

Assume $\dim f_\circ SS\mathcal{F} = 1$. Conj 1 is equiv. to

$$-a_y Rf_*\mathcal{F} = \text{Coeff of } T^*_y Y \text{ in } f_! CC\mathcal{F}$$

conductor formula

If $f: X \to Y$ finite morphism of curves.
Conj is the induction formula for conductor.

Proposition 1 Assume $\dim X = 2$, $\dim Y = 1 = \dim f_\circ SS\mathcal{F}$.
$f: X \to Y$ generically smooth. $\Rightarrow$ Conj 1 is true

Idea of Pf. Global argument using the index formula
Sum = Sum. Kill the ramification at the other pts by Epp's theorem.

X singular $X \hookrightarrow M$ imm. to smooth M

$$CC_X \mathcal{F} = [\mathbb{P}CC\mathcal{F}] \in CH_N(X \times_M T^*M)$$

$$\| \qquad$$

$$CH_\bullet(X) = \bigoplus_i CH_i(X)$$

indep of M

$$CC_X : K(X, \Lambda) \to CH_\bullet(X)$$

Functoriality (1) $f : X \to Y$ smooth

$$\begin{array}{ccc} K(Y, \Lambda) & \xrightarrow{CC_Y} & CH_\bullet(Y) \\ {\scriptstyle f^*}\downarrow & & \downarrow {\scriptstyle f^* \cap C_\bullet(T^*_{X/Y})} \\ K(X, \Lambda) & \xrightarrow{CC_X} & CH_\bullet(X) \end{array}$$

but does not satisfy

(2) $f : X \to Y$ proper

$$\begin{array}{ccc} K(X, \Lambda) & \xrightarrow{CC_X} & CH_\bullet(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow {\scriptstyle f_*} \\ K(Y, \Lambda) & \xrightarrow[CC_Y]{} & CH_\bullet(Y) \end{array}$$

Conj by Grothendieck in SGA5 cf. R&S.

OK if $Y = \operatorname{Spec} k$ or char $k = 0$

(3) $X = \operatorname{Spec} k$

$$rk : K(X, \Lambda) \to CH_\bullet(X) = \mathbb{Z}$$

char $k = 0$

$$\begin{array}{ccc} K(X, \Lambda) & \xrightarrow{rk} & F(X) = \{\text{constrble f'ns } X \to \mathbb{Z}\} \\ & {\scriptstyle CC_X} \searrow & \downarrow {\scriptstyle (-1)^\bullet c_X} \\ & & CH_\bullet(X) \end{array}$$

MacPherson Chern class.