Characteristic cycle and singular support of an etale shf [1]

X smooth $/k$ perfect char $p>0$

K constructible complex of Λ-modules on X_{et} $\Lambda/\mathbb{F}_\ell$ finite $\ell \neq p$

Goal. Define $\mathrm{Char}\, K$ as a cycle on the cotangent b'dle T^*X

· Prove · index formula (X proper) for Euler-Poincaré char.

· Milnor formula for vanishing cycles.

Goal has been attained for

- Tamely ramified case. Eulin Yang
- Curves Grothendieck-Ogg-Shafarevich formula
- Surfaces

Higher dimension? First step Define singular support

- Conic closed subset of T^*X
- underlying set of $\mathrm{Char}\, K$
- Should satisfy conditions (SS!) & (SS4)

Thm 1. Assume $S \subset T^*X$ satisfies (SS4).

Then there exists a unique $\mathrm{Char}\, K$ supported on S satisfying the Milnor formula. If X is proper, it also satisfies the index formula.

2. If S satisfies (SS4), it also satisfies (SS!)

Conversely, if $S(\supset T^*_X X)$ satisfies (SS!), it satisfies (SS4)

Thm means that the Goal will be attained if one can construct S satisfying (SS!).

Ramification implies construction of S for surfaces.

1 Classical cases
2. Milnor formula.
3. Conditions (SS!) & (SS4)
4. Points in Proof.

1 $X \supset U = X - D$, $j: U \hookrightarrow X$. $K = j_! \mathcal{F}$ $\mathcal{F}$ l.c.c /U.

- tamely ramified along $D = \bigcup D_i$: simple n.c.d

$$\mathrm{Char}\, K = (-1)^d \,\mathrm{rk}\,\mathcal{F} \cdot \sum_I [T^*_{X_I} X] \qquad d = \dim X$$

conormal b'dle

- Curve

$$\mathrm{Char}\, K = -\Big(\underset{\substack{| \\ \text{0-section}}}{\mathrm{rk}\,\mathcal{F} \cdot [T^*_X X]} + \sum_{x \in D} \underset{\substack{\| \\ \mathrm{rk} + \mathrm{Sw}_x}}{\dim \mathrm{tot}_x(\mathcal{F}_x)} \cdot \underset{\substack{| \\ \text{fiber}}}{[T^*_x X]}\Big)$$

2. $u \in U \xrightarrow{\text{étale}} X$ ($f\,dt =$) df section of T^*X on U

flat $|f$ isolated char pt. $df(U) \cap S$ isolated.

C smooth curve

$$-\dim\mathrm{tot}\, \underset{\substack{| \\ \text{space of van. cycles}}}{\Phi_u(\mathcal{F}, f)} = \underset{\substack{| \\ \text{intersection number}}}{(\mathrm{Char}\, K, df)_{T^*X, u}}$$

3. $S \subset T^*X$ conic closed dim d.

non characteristic morphism (generalization of smooth)

$f: W \to X$, $g: X \to Y$ non char $\Rightarrow$ good cohomological property for K

SS! SS4

. g local acyclicity (smooth $\Rightarrow$ loc. acyclic rel to l.c.c. sheaf)

. f can. isom $f^*K \otimes Rf^!\Lambda \to Rf^!K$ is isom.

f is propagation for K.

(Poincaré duality for smooth morphism)

non characteristic

- $f: W \to X$. $d = \dim X$. $b = \dim W$

(1) $f^*S = W \times_X S$ is finite over T^*W w.r.t.

can. morphism $W \times_X T^*X \to T^*W$

(2) $\forall$ irred cpt of f^*S are of dim b.

- $g: X \to Y$ flat $d = \dim X$ $c = \dim Y$

(1) inv. image of S by $X \times_Y T^*Y \to T^*X$ is

a subset of the 0-section

(2) $\forall y \in Y$ $\forall$ irred cpt of $S \times_Y y$ is of dim $d-c$.

4. 2.
$$\begin{array}{ccc} g: X & \longrightarrow & Y \ \text{(smooth)} \\ i \uparrow \cup & & \rotatebox{90}{$\in$} \\ W & \longmapsto & y \end{array}$$

- i. non char $\Leftrightarrow$ g non char on a nbd of W.
- i propagating $\Leftrightarrow$ g loc. acyclic $\quad$ = .

$\Leftarrow$ SGA $4\frac{1}{2}$ App à Th. finitude (without smooth)

$\Rightarrow$ Reduction to Y. curve, $\Psi = \varinjlim i'^* R j'_*$.

1. $X \subset \mathbb{P}^N$.

local Radon transform.

vanishing cycles over general base scheme

+ (semi) continuity of Swan conductor

$f: W \to X$ non char $\Rightarrow$ $\mathrm{Char}\, f^*K = (-1)^{d-b} f^! \mathrm{Char}\, K$

f. immersion induction on codim.

codim = 1 reduction to $d=2$. global argument

index formula induction on dim using G-O-S