Local Fourier transform and epsilon factors

joint work with Ahmed Abbes

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Abstract

The local epsilon factors appear in the constant term of the functional equation of the L-functions. Laumon proved a formula expressing the local epsilon factor using a local variant of ℓ-adic Fourier transform by a global and arithmetic method and deduced a product formula for the constant term. We will discuss a local and geometric method to prove Laumon's formula, under a certain assumption.

Plan

1. Laumon's formula.
2. local Fourier transform.
3. Witt vectors and Ramification.
4. blow-up.

1 Laumon's formula.

ℓ-adic Fourier transform: k a field of characteristic p, later we will assume $p > 2$:

$$(\ell\text{-adic sheaf on } \mathbf{A}^1_k) \longrightarrow (\ell\text{-adic sheaf on } \mathbf{A}^1_k).$$

Analogue of $\int_{\mathbb{R}} f(x)\psi(xy)dx$. Function-sheaf dictionary.
local version: $K, K^\vee$ local fields of characteristic $p > 0$, $G_K = \mathrm{Gal}(\overline{K}/K)$, $G_{K^\vee} = \mathrm{Gal}(\overline{K}^\vee/K^\vee)$:

$$(\ell\text{-adic representations of } G_K) \longrightarrow (\ell\text{-adic representations of } G_{K^\vee}).$$

L-function: $k = \mathbb{F}_q$ finite field. C curve over k. $\mathcal{F}$ ℓ-adic sheaf on C.

$$L(\mathcal{F}, t) = \prod_{x \in C} \det(1 - \mathrm{Fr}_x t : \mathcal{F}_{\bar{x}})^{-1} \in \overline{\mathbb{Q}_\ell}(t).$$

Functional equation and the product formula:

$$L(\mathcal{F}, t) = \varepsilon(\mathcal{F}) t^{-\chi(C_{\bar{k}}, \mathcal{F})} L(\mathcal{F}^*, (qt)^{-1}).$$

$$\varepsilon(\mathcal{F}) = \prod_{x \in C} \varepsilon_x(\mathcal{F}_x).$$

The local epsilon factors $\varepsilon_x(\mathcal{F}_x)$ play an important role e.g. in the local Langlands correspondence.

Laumon's formula.

$$\det(-\mathrm{Fr} : F_\psi(\mathcal{F})) = \epsilon_x(\mathcal{F}_x). \tag{1.1}$$

Goal: Reprove Laumon's formula, under a certain assumption, by a *local* and *geometric* method, using a new construction from ramification theory.

2 local Fourier transform

$k = \mathbb{F}_q$.

$\mathbf{P}^1_k$ projective line, x inhomogeneous coordinate. T completion at 0. K the fraction field of $\mathcal{O}_T$.

$\mathbf{P}^{1\vee}_k$ the dual projective line, $x^\vee$ inhomogeneous coordinate. $T^\vee$ completion at ∞. $K^\vee$ the fraction field of $\mathcal{O}_{T^\vee}$.

V ℓ-adic representation of the absolute Galois group G_K.

$\mathcal{F}_!$ on T^{et}: zero-extension of the ℓ-adic sheaf corresponding to V.

$\overline{\mathcal{L}}_{\psi_0}(xy)$ on $\mathbf{P}^1 \times \mathbf{P}^{1\vee}$: zero extension of the Artin-Schreier sheaf on $\mathbf{A}^1 \times \mathbf{A}^{1\vee} \subset \mathbf{P}^1 \times \mathbf{P}^{1\vee}$ defined by the equation $X^p - X = xy$ and an additive character $\psi_0 : \mathbb{F}_p \to \overline{\mathbb{Q}_\ell}^\times$.

Local Fourier transform:

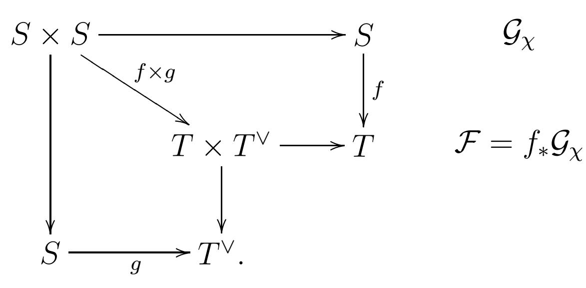
$$F_{\psi_0}(\mathcal{F}_!) = \psi^1(\mathrm{pr}_1^* \mathcal{F}_! \otimes \mathcal{L}_{\psi_0}(xy), \mathrm{pr}_2).$$

The space of nearby cycles with respect to the second projection: ℓ-adic representation of $G_{K^\vee}$.

We consider the case where $V = \mathrm{Ind}_{G_L}^{G_K} L_\chi$ is monomial.

L: a finite separable extension of K. $\chi : G_L \to \overline{\mathbb{Q}_\ell}^\times$ a character.

More precise goal: We assume χ is wildly ramified. We put $S = \mathrm{Spec}\ \mathcal{O}_L$ and let $f : S \to T$ be the map defined by the inclusion $K \subset L$. First, we define a map $g : S \to T^\vee$ and construct a diagram



$F_{\psi_0}(\mathcal{F}_!)$

Then, we compute the pull-back

$$g^*F_{\psi_0}(\mathcal{F}_!) = \psi^1(\mathrm{pr}_1(\mathcal{G}_\chi)_! \otimes \mathcal{L}_{\psi_0}(xy), \mathrm{pr}_2)$$

and prove the following.

Theorem 1 *There exists an isomorphism*

$$F_{\psi_0}(\mathcal{F}) \to g_*(\mathcal{G}_\chi \otimes \mathcal{L}_{\psi_0}(f^*xg^*y) \otimes \mathcal{K}(\frac{1}{2}\frac{f^*dx}{g^*dy}) \otimes \mathcal{Q}) \tag{2.1}$$

under a certain assumption formulated in the next section.

Here the Kummer sheaf $\mathcal{K}(\frac{1}{2}\frac{f^*dx}{g^*dy})$ is defined by the square root of $\frac{1}{2}\frac{f^*dx}{g^*dy} \in L^\times$ and $\mathcal{Q} = H^1_c(\mathbf{A}^1, \mathcal{K}(x) \otimes \mathcal{L}_{\psi_0}(x))$ is the rank 1 representation on which the Frobenius acts as the quadratic Gauss sum $-\sum_{x\in\mathbb{F}_q} \psi_0(\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} x^2)$.

Some partial results have been studied by Fu Lei.

We deduce Laumon's formula (1.1) from the isomorphism (2.1) in Theorem 1. The key ingredients are:

Explicit and standard computation of the local epsilon factors.

Explicit reciprocity law for Artin-Schreier-Witt theory.

Induction formula for the local epsilon factors.

3 Witt vectors and Ramification

Decompose the character $\chi = \chi_t \cdot \chi_w$ where χ_t is tamely ramified and χ_w is defined by a Witt vector $a = (a_0, \ldots, a_m) \in W_{m+1}(L)$ via the Artin-Schreier-Witt theory $W_{m+1}(L) \to H^1(L, \mathbb{Z}/p^{n+1}\mathbb{Z})$ for a fixed embedding $\mathbb{Z}/p^{n+1}\mathbb{Z} \to \overline{\mathbb{Q}_\ell}^\times$. We assume that the Swan conductor $n = \mathrm{Sw}\ \chi = \mathrm{Sw}\ \chi_w > 0$ and that $a \in \mathrm{Fil}^n W_{m+1}(L)$ namely

$$p^{m-i}\mathrm{ord} a_i \leq -n \qquad \text{for } i = 0, \ldots, m.$$

We put

$$F^m da = \sum_{i=0}^{m} a^{p^{m-i}} d\log a_i = \alpha d\log t$$

using a notation for the de Rham-Witt complexes. The assumption implies $\mathrm{ord}\alpha = -n$.

We define $y \in L^\times$ by

$$F^m da + c \cdot db = 0$$

and put

$$d = \mathrm{ord}\frac{d\log b}{d\log t}, \quad d' = \mathrm{ord}\frac{d\log c}{d\log t}.$$

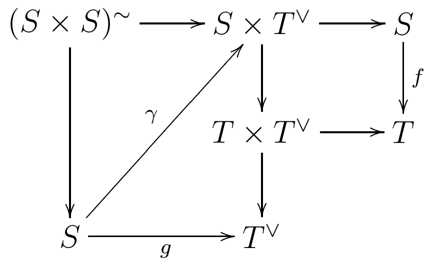
We assume

$$2d + pd' \leq (p-2)n.$$

The assumption means that the wild ramification of χ is deeper than those of f and g.

4 Blow-up

Define a map $g : S \to T^\vee$ by $c \in L^\times$ and consider the diagram (2.1). For simplicity, we assume L is a tamely ramified and totally ramified extension of K with respect to g. Let $(S \times S)'$ denote the blow up of $S \times S$ at the closed point s of the diagonal and $(S \times S)^\sim \subset (S \times S)'$ denote the complement of the proper transforms of $S \times s$ and $s \times S$. Let $\gamma : S \to S \times T^\vee$ be the graph of g. We consider the diagram



Let $\Gamma \subset S \times T^\vee$ be the graph of $g : S \to T^\vee$ and $\Gamma_S^\sim \subset (S \times S)^\sim$ denote the proper transform of the inverse image $\Gamma \times_{T^\vee} S$. Then, by the assumption $d' = 0$, $\Gamma_S^\sim$ is finite etale over S. By the dimension formula for the local Fourier transform, we have $\dim F_{\psi_0}(\mathcal{F}_!) = \deg g$. The cohomology sheaf ψ^1 is supported on the closed fiber $\Gamma_S^\sim \times_S s$.

Let $\delta^\sim : S \to (S \times S)^\sim$ be the log diagonal map. We will define an isomorphism

$$\psi^1(\mathrm{pr}_1^* \mathcal{G}_\chi \otimes \mathcal{L}_{\psi_0}(b_1 c_2))_{\delta^\sim(\bar{s})} \to \mathcal{G}_\chi \otimes \mathcal{L}_{\psi_0}(bc) \otimes \mathcal{K}(\frac{1}{2}\frac{db}{dc}) \otimes \mathcal{Q}. \tag{4.1}$$

This will imply the isomorphism (2.1). We put $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{G}_\chi, \mathrm{pr}_1^* \mathcal{G}_\chi)$ on $\eta \times \eta \subset S \times S$. Then the left hand side is isomorphic to

$$\mathcal{G}_\chi \otimes \mathcal{L}_{\psi_0}(b_2 c_2) \otimes \psi^1(\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2))_{\delta^\sim(\bar{s})}$$

and the isomorphism (4.1) is reduced to

$$\psi^1(\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2))_{\delta^\sim(\bar{s})} \to \mathcal{K}(\frac{1}{2}\frac{db}{dc}) \otimes \mathcal{Q}. \tag{4.2}$$

According to the decomposition $\chi = \chi_t \cdot \chi_w$, we have $\mathcal{G}_\chi = \mathcal{G}_t \otimes \mathcal{G}_w$ and $\mathcal{H}_\chi = \mathcal{H}_t \otimes \mathcal{H}_w$. Since $\mathcal{H}_t$ is extended to a smooth sheaf on $(S \times S)^\sim$ and since the stalk $\mathcal{H}_{t,\delta^\sim(s)}$ is trivial, we may assume $\chi = \chi_w$ is defined by a Witt vector a.

We define an isomorphism (4.2) assuming $n = 2r$ is even. We blow-up r-times the closed point $s \in S \subset (S \times S)^\sim$ in the log diagonal to define $(S \times S)^{(r)} \subset (S \times S)^{[r]} \to (S \times S)^\sim$. Then some elementary computation on Witt vectors shows that $\mathcal{H} \otimes \mathcal{L}_{\psi_0}((b_1 - b_2)c_2)$ is extended to a smooth sheaf on $(S \times S)^{(r)}$. Further, the restriction to the exceptional divisor $\Theta^{(r)} \subset (S \times S)^{(r)}$ is isomorphic to the Artin-Schreier sheaf $\mathcal{L}_{\psi_0}(\frac{1}{2}\beta w^2)$ where $\beta = t^n b' c'$. Thus the assertion follows from the isomorphism

$$H_c^1(\mathbf{A}^1, \mathcal{L}_{\psi_0}(\frac{1}{2}\beta w^2)) \to \mathcal{K}(\frac{1}{2}\frac{db}{dc}) \otimes \mathcal{Q}.$$