Ramification theory for varieties over a perfect field

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1 A Lefschetz trace formula for open varieties

Notations.

X: proper scheme over F.

$U \subset X$: smooth dense open subscheme of dimension d.

$\Gamma \subset U \times U$: a closed subscheme of dimension d.

$p_i : \Gamma \to U$: the composition with the projections $pr_i : U \times U \to U$.

ℓ: a prime number different from the characterstic of F.

Lemma 1.1 *p_2 is proper if and only if*

$$\bar{\Gamma} \cap (D \times X) \subset \bar{\Gamma} \cap (X \times D). \tag{1.1}$$

If $p_2 : \Gamma \to U$ is proper,

$\Gamma^* = pr_{1*} \circ pr_2^* : H^q_c(U, \mathbb{Q}_\ell) \to H^q_c(U, \mathbb{Q}_\ell)$ is defined.

Write $\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Q}_\ell)) = \sum_{q=0}^{2d} (-1)^q \mathrm{Tr}(\Gamma^* : H^q_c(U_{\bar F}, \mathbb{Q}_\ell))$.

Assume

X: smooth

$U \subset X$: the complement of a divisor $D = D_1 \cup \cdots \cup D_m$ with simple normal crossings.

Define

$p : (X \times X)' \to X \times X$: the blow-up at $D_1 \times D_1, \ldots, D_m \times D_m$

$\Delta'_X = X \to (X \times X)'$: the log diagonal.

Theorem 1.2 *Let $\bar{\Gamma}'$ be the closure of Γ in $(X \times X)'$ and assume*

$$\bar{\Gamma}' \cap (D \times X)' \subset \bar{\Gamma}' \cap (X \times D)' \tag{1.2}$$

where $(D \times X)'$ and $(X \times D)'$ are the proper transforms of $D \times X$ and $X \times D$. Then, $p_2 : \Gamma \to U$ is proper and we have

$$\mathrm{Tr}(\Gamma^* : H^*_c(U_{\bar F}, \mathbb{Q}_\ell)) = \deg(\bar{\Gamma}', \Delta'_X)_{(X \times X)'}.$$

Corollary 1.3 (Fujiwara) *Let $\mathbb{F}_q$ be a finite field, U be a smooth and separated scheme of finite type over $\mathbb{F}_q$ and $\Gamma \subset U \times U$ be an algebraic correspondence such that $p_2 : \Gamma \to U$ is proper and p_1 is quasi-finite. Then there exists an integer $n_0 \geq 0$ such that, for every integer $n \geq n_0$, the intersection $Fr_q^n \Gamma \cap \Delta_U$ is finite over F and we have*

$$\mathrm{Tr}(\Gamma^* \circ Fr_q^{*n} : H_c^*(U_{\mathbb{F}_q}, \mathbb{Q}_\ell)) = \deg Fr_q^n \Gamma \cap \Delta_U.$$

Can not replace (1.2) $\bar{\Gamma}' \cap D^{(1)\prime} \subset \bar{\Gamma}' \cap D^{(2)\prime}$ by (1.1) $\bar{\Gamma} \cap D^{(1)} \subset \bar{\Gamma} \cap D^{(2)}$.

Example. $X = \mathbb{P}^1$, $U = \mathbb{A}^1$,

$\Gamma = \{(x, y) \in U \times U | x = y^n\}$ the transpose of the graph of the n-th power map $f : U \to U$.

Then,

$$\mathrm{Tr}(\Gamma^* : H_c^*(U_{\bar{F}}, \mathbb{Q}_\ell)) = \mathrm{Tr}(f_* : H_c^2(U_{\bar{F}}, \mathbb{Q}_\ell)) = 1$$

while

$$(\Gamma, \Delta)_{(X \times X)'} = n.$$

2 Euler characteristic and Swan class of sheaves

(1) Euler characteristic formula.

X: separated scheme of finite type over F.

$U \subset X$: smooth dense open subscheme of dimension d.

$\mathcal{F}$: smooth ℓ-adic sheaf on U. $\ell \neq \mathrm{char}\, F$.

$\chi_c(U_{\bar{F}}, \mathcal{F}) = \sum_{q=0}^{2d} (-1)^q \dim H_c^q(U_{\bar{F}}, \mathcal{F})$.

Goal:

Define

$$\mathrm{Sw}(\mathcal{F}) \in CH_0(X \setminus U) \otimes \mathbb{Q}_\ell^{\mathrm{ur}}$$

and Prove

Theorem 2.1 *If X is proper, we have*

$$\chi_c(U_{\bar{F}}, \mathcal{F}) = \chi_c(U_{\bar{F}}) \cdot \mathrm{rank}\, \mathcal{F} - \deg \mathrm{Sw}(\mathcal{F}). \tag{2.1}$$

For simplicity, assume $\mathcal{F}$ is trivialized by a finite etale Galois covering $f : V \to U$ of Galois group G. $\mathcal{F}$: corresponds to a representation M of G.

Consider a cartesian diagram

$$\begin{array}{ccc} V & \xrightarrow{\subset} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ U & \xrightarrow{\subset} & X \end{array} \tag{2.2}$$

of separated schemes of finite type.

$U \subset X$, $V \subset Y$: dense open subschemes.

(2) Classical case.

X: a smooth curve
Y: the normalization of X in V.
D: $Y \setminus V$.
$(Y \times Y)' \to Y \times Y$: Blow up at $(y, y), y \in D$.
$\Delta_Y \subset (Y \times Y)'$: log diagonal.
$\sigma \in G, \neq 1$,
$\overline{\Gamma_\sigma}$: closure of $\Gamma_\sigma \subset V \times_U V$ in $(Y \times Y)'$.
Define

$$s_{V/U}(\sigma) = -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(D) = \bigoplus_{y\in D} \mathbb{Z},$$

$s_{V/U}(1) = -\sum_{\sigma\neq 1} s_{V/U}(\sigma)$ and

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \sum_{\sigma\in G} s_{V/U}(\sigma)\mathrm{Tr}(\sigma : M) \in CH_0(D). \tag{2.3}$$

Then, we have

Theorem 2.2 (Hasse-Arf) *There exists* $\mathrm{Sw}(\mathcal{F}) \in CH_0(B)$ *satisfying*

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \bar{f}^*\mathrm{Sw}(\mathcal{F}). \tag{2.4}$$

Theorem 2.3 (Grothendieck-Ogg-Shafarevich) *Further if X is proper, we have*

$$\chi_c(U_{\bar{F}}, \mathcal{F}) = \chi_c(U_{\bar{F}}) \cdot \mathrm{rank}\ \mathcal{F} - \deg\ \mathrm{Sw}(\mathcal{F}). \tag{2.1}$$

Note
$V \times_U V = \coprod_{\sigma\in G} \Gamma_\sigma$.
and identify
$CH_d(V \times_U V) = \bigoplus_{\sigma\in G} \mathbb{Z}$.
The key in the classical theory is the map

$$(\ ,\Delta_Y)_{(Y\times Y)'} : CH_d(V \times_U V \setminus \Delta_V) = \bigoplus_{\sigma\in G,\neq 1} \mathbb{Z} \longrightarrow CH_0(D) = \bigoplus_{y\in D} \mathbb{Z}. \tag{2.5}$$

(3) Definition.

In higher dimension, we can not assume resolution but we do have alteration.
Extend the diagram (2.2) to a cartesian diagram

$$\begin{array}{ccccc}
W & \xrightarrow{\subset} & Z & & \\
{\scriptstyle g}\big\downarrow & & {\scriptstyle \bar{g}}\big\downarrow & & \\
V & \xrightarrow{\subset} & Y & \searrow h & \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle \bar{f}}\big\downarrow & & \\
U & \xrightarrow{\subset} & X & \xleftarrow{p} & X'
\end{array}$$

where

$h : X' \to X$: proper and isomorphism on U.

$U \subset X'$: the complement of a divisor B' of X'.

Z: connected and smooth of dimension d.

$W \subset Z$: the complement of a divisor D with simple normal crossings.

$\bar{g} : Z \to Y$: proper surjective and generically finite.

$(Z \times Z)'$: Blow-up of $Z \times Z$ at $D_1 \times D_1, \dots, D_m \times D_m$ where $D_1, \dots, D_m$ are the irreducible components of D.

$\Delta_Z : Z \to (Z \times Z)'$: the log diagonal map.

$(Z \times Z)'$ is smooth and the immersion $Z \to (Z \times Z)'$ a regular immersion of codimension d.

Proposition 2.4 *Let $\sigma \in G, \neq 1$. For $\Gamma' \in Z_d(\overline{W \times_U W \setminus W \times_V W})$ such that $[\Gamma'|_{W \times_U W \setminus W \times_V W}] = (g \times g)^! \Gamma_\sigma$,*

$$\frac{1}{[W : V]} \bar{g}_* (\Gamma', \Delta_Z)_{(Z \times Z)'} \in CH_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

depends only on $U \leftarrow V \subset Y$ and σ.

Definition 2.5 *1. Intersection product with the log diagonal*

$$(2.6) \qquad (\ , \Delta_V)^{\log} : CH_d(U \times_V U \setminus \Delta_V) = \bigoplus_{\sigma \in G, \neq 1} \mathbb{Z} \cdot \sigma \longrightarrow CH_0(Y \setminus V) \otimes_{\mathbb{Z}} \mathbb{Q}$$

by

$$(\Gamma_\sigma, \Delta_V)^{\log} = \frac{1}{[W : V]} \bar{g}_* (\Gamma', \Delta_Z)_{(Z \times Z)'}.$$

2. For $\sigma \in G, \neq 1$,

$$(2.7) \qquad s_{V/U}(\sigma) = -(\Gamma_\sigma, \Delta_V)^{\log}$$

$s_{V/U}(1) = -\sum_{\sigma \in G, \neq 1} s_{V/U}(\sigma)$.

3.

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \sum_{\sigma \in G} s_{V/U}(\sigma) \mathrm{Tr}(\sigma : M) \in CH_0(Y \setminus V) \otimes \mathbb{Q}_\ell.$$

Proposition 2.6 *If $\bar{f} : Y \to X$ is proper*

$$\frac{1}{|G|} \bar{f}_* \mathrm{Sw}_{V/U}(\mathcal{F}) \in CH_0(X \setminus U) \otimes \mathbb{Q}_\ell$$

is independent of $V \subset Y$.

Proof. Proposition 2.4. ■

Definition 2.7

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|}\bar{f}_*\mathrm{Sw}_{V/U}(\mathcal{F}) \in CH_0(X \setminus U) \otimes \mathbb{Q}_\ell.$$

(4) Integrality.

Conjecture 2.8 *1. If X and Y are smooth,* $\mathrm{Sw}_{V/U}(\mathcal{F}) \in CH_0(Y \setminus V) \otimes \mathbb{Q}_\ell$ *is in the image of* $\bar{f}^* : CH_0(X \setminus U) \to CH_0(Y \setminus V) \otimes \mathbb{Q}_\ell$.

2. $\mathrm{Sw}(\mathcal{F}) \in CH_0(X \setminus U) \otimes \mathbb{Q}_\ell$ *is in the image of* $CH_0(X \setminus U) \to CH_0(X \setminus U) \otimes \mathbb{Q}_\ell$.

Theorem 2.9 *1. Conjecture 2.8.1 is true if* $d = \dim U \leq 2$ *and* $\mathrm{rank}\mathcal{F} = 1$.

2. Conjecture 2.8.2 is true if $d = \dim U \leq 2$.

Proof. 1. May assume X is smooth and U is the complement of a divisor $D = \sum_i D_i$ with simple normal crossings. Then, one can define a divisor $D_{\mathcal{F}} = \sum_i \mathrm{sw}_i(\mathcal{F})D_i$. Further after blowing-up, we prove

$$\mathrm{Sw}_{V/U}(\mathcal{F}) = \bar{f}^*(-1)^{d-1}\{c(\Omega_{X/F}(\log D)) \cap (1 - D_{\mathcal{F}})^{-1} \cap [D_{\mathcal{F}}]\}_{\dim 0}.$$

2. By Brauer's theorem and the induction formula for the Swan class, it is reduced to the case where rank $\mathcal{F} = 1$. ■

(5) Proof of Theorem 2.1.

Suffices to show the trace formula for an open variety.

$\mathrm{Tr}(\sigma^* : H_c^*(V_{\bar{F}}, \mathbb{Q}_\ell)) = \sum_{q=0}^{2d}(-1)^q \mathrm{Tr}(\sigma^* : H_c^q(V_{\bar{F}}, \mathbb{Q}_\ell))$.

Theorem 2.10 *Assume Y is proper. For $\sigma \in G$, we have*

$$\deg\ s_{V/U}(\sigma) = \begin{cases} -\mathrm{Tr}(\sigma^* : H_c^*(V_{\bar{F}}, \mathbb{Q}_\ell)) & \text{if } \sigma \neq 1 \\ \chi_c(U_{\bar{F}})[V : U] - \chi_c(V_{\bar{F}}) & \text{if } \sigma = 1. \end{cases}$$

Proof of Theorem 1.2 ⇒ Theorem 2.10. $\overline{W \times_U W} \cap (D \times Y)' = \overline{W \times_U W} \cap (Y \times D)'$. ■

3 Serre's conjecture

Conjecture 3.1 *Let A be a regular local ring with perfect residue field and G be a finite group of automorphisms of A. Assume that, for $\sigma \in G, \neq 1$, $A/(\sigma(a) - a : a \in A)$ is of finite length. Then the function $a_G : G \to \mathbb{Z}$ defined by*

$$a_G(\sigma) = \begin{cases} -\text{length } A/(\sigma(a) - a : a \in A) & \text{if } \sigma \neq 1 \\ -\sum_{\tau \in G, \neq 1} a_G(\tau) & \text{if } \sigma = 1. \end{cases}$$

is a character of G.

Lemma 3.2 *Conjecture 2.8 implies Conjecture 3.1 if A is the local ring at a closed point of a smooth variety over a perfect field.*

Corollary 3.3 ([KSS]) *Conjecture 3.1 is true if A is the local ring at a closed point of a smooth surface over a perfect field.*

4 Proof of Theorem 1.2.

We may assume $F = \bar{F}$. By Poincaré duality and Künneth formula, we identify

$$\bigoplus_q \mathrm{End}\ H_c^q(U, \mathbb{Q}_\ell) = H^{2d}(X \times U, (j \times 1)_! \mathbb{Q}_\ell(d)).$$

Then, we have

$$\mathrm{Tr}(\Gamma^* : H_c^*(U, \mathbb{Q}_\ell)) = \mathrm{Tr}\Delta^*([\Gamma])$$

where

$[\Gamma] \in H^{2d}(X \times U, (j \times 1)_! \mathbb{Q}_\ell(d))$: the cycle class,

$\Delta^* : H^{2d}(X \times U, (j \times 1)_! \mathbb{Q}_\ell(d)) \to H_c^{2d}(U, \mathbb{Q}_\ell(d))$: the pull-back

$j' : (X \times X)' \setminus ((D \times X)' \cup (X \times D)') \to (X \times X)' \setminus (X \times D)'$: open immersion.

$$H^{2d}(X \times U, (j \times 1)_! \mathbb{Q}_\ell(d)) \to H^{2d}((X \times X)' \setminus (X \times D)', j'_! \mathbb{Q}_\ell(d)). \tag{4.1}$$

sends $[\Gamma]$ to $[\tilde{\Gamma}]$ where $\tilde{\Gamma} = \bar{\Gamma}' \setminus \bar{\Gamma}' \cap (X \times D)'$

Points: The assumption implies and $[\tilde{\Gamma}]$ is defined.

The map (4.1) is an isomorphism by Faltings.

By the commutative diagram

$$\begin{array}{ccc} H^{2d}(X \times U, (j \times 1)_! \mathbb{Q}_\ell(d)) & \xrightarrow{\Delta^*} & H_c^{2d}(U, \mathbb{Q}_\ell(d)) \\ \downarrow & & \downarrow \\ H^{2d}((X \times X)' \setminus (X \times D)', (j \times 1)_! \mathbb{Q}_\ell(d)) & \xrightarrow{\Delta'^*} & H^{2d}(X, \mathbb{Q}_\ell(d)), \end{array}$$

$$\mathrm{Tr}\Delta^*([\Gamma]) = \mathrm{Tr}\Delta'^*([\tilde{\Gamma}]) = \deg(\bar{\Gamma}', \Delta'_X)_{(X \times X)'}.$$

■