Ultimate Goal: Define the characteristic cycle of a smooth ℓ-adic sheaf ramified along the boundary as a cycle in the cotangent b'dle.

Partial answer.

Under a certain "non degenerate" condition.

codim of complement ≥ 2.

Consequences.

1. Compatibility with the pull-back by "non-characteristic" morphism. — functoriality of the construction

Characterization by the method of cutting-by-curves

2. Local acyclicity with respect to "non-characteristic" morphism. — reduce to the case rel dim $=1$. ↑ Deligne–Laumon

Characterization?

3. Compute the characteristic class and the Euler number.

Missing.

Compatibility with proper push-forward? $f: X \to$? :: Spec k

~~Earlier~~ Related results

a) Approach using jet b'dles by Deligne.

= Tangent b'dle suffices.

b) rank 1 case by Kato

- higher rk.

c) logarithmic version by S. and Abbes–S.

- cutting-by-curves

~~New~~ Method to link ramification to cotangent b'dle. (2)

1) Blow up at the ramification divisor embedded in the diagonal.

2) Groupoid structure of multiple & self products.

Notation
k perfect field of char $p>0$
X smooth sep scheme of f.t $/k$ $\quad d=\dim X$ $\quad K_i$ local field $\mathrm{Fr}\,\mathcal{O}_{X,\xi_i}$
$U\subset X\supset D$ div. w/ SNC $D_1,\dots,D_m$ irred cpt
$\zeta_p\in\Lambda$ ~~noetherian~~ ring finite over $\mathbb{Z}_\ell$ $\ell\neq p$. condition on ass. rep'n of G_{K_i}
$\mathcal{F}$ smooth sheaf of Λ-modules on U. unique jump ✓ satisfied if

Assumptions Ramification of flat $\mathcal{F}$ along D is iso~~mono~~clinic of slope $R=r_1D_1+\dots+r_mD_m$ for integer $r_i>1$
(simplifying) wildly ram.

- Ramification is "non-degenerate" at multiplicity R
(serious). (nothing particular in codim ≥ 2)

Main construction.

$$\mathrm{Char}\,\mathcal{F}\in Z_d(T^*X)\otimes_{\mathbb{Z}}\mathbb{Z}[\tfrac{1}{p}]$$

linear combination of sub line b'dles defined over a finite covering of irreducible cpt.
$+$ rk $\mathcal{F}$ $\times$ 0-section. $\times(-1)^d$.

Example 1. $\dim X=1$ $\quad D=\{x\}$

$$\mathrm{Char}\,\mathcal{F}=(-1)\big(\underbrace{\mathrm{rk}\,\mathcal{F}\,[T^*_XX]}_{\text{0-section}}+\underbrace{\mathrm{dimtot}_x\mathcal{F}}_{\mathrm{rk}+\mathrm{Sw}}\cdot\underbrace{[T^*_xX]}_{\text{fiber at }x}\big)$$

2 $X = \mathbb{A}^2 \supset D = \operatorname{Spec} k[x, y] \supset U = \operatorname{Spec} k[x^{\pm 1}, y]$

a) $n = 1$ $t^p - t = \frac{1}{x^n}$ $p \nmid n$.

$$\operatorname{Char} \mathcal{F} = [T^*_X X] + (n+1)[T^*_D X]$$

b) $=$ $t^p - t = \frac{y}{x^n}$ $p \mid n$ Lagrangean ~~($p \neq 2$ if p~~

$= [T^*_X X] + \{ n \cdot [dy_{/D}]$ otherwise
$\ \ n\cdot[dy + \sqrt{y}\,dx\ D^{(1/2)}]$ $n = p = 2$ non-Lagrangean

Assume. $\mathcal{F}$ is trivialized by a fin. étale
Ramification of G-torsor Galois covering $V \to U$ of gp G. $F: D^{(1/p)} \to D$

$$R = \sum r_i D_i \subset X \hookrightarrow X \times X$$

$D^{(1/p)} = D \times_{k} k$ (inv. of Frob)

$$\uparrow$$

$P_1^{(R)}$ blow up and remove proper transform of $D \times X \cup X \times D$

$T^{(R)} = TX(-R) \times_X D \hookrightarrow P_1^{(R)} \supset U \times U$

$V \to U$ finite étale G-torsor. G. finite gp

normalization

$$\begin{array}{ccccc} & & Q_1^{(R)} & \hookleftarrow & V \times V / \Delta G \\ & \nearrow & \downarrow & & \downarrow \\ X & \longrightarrow & P_1^{(R)} & \hookleftarrow & U \times U \end{array}$$

Def'n Ram is bdd by R if

$Q_1^{(R)} \to P_1^{(R)}$ is étale on a nbd of $X \subset Q_1^{(R)}$

Groupoid $U \times U \rightrightarrows U$

$$(U \times U) \times_U (U \times U) = U \times U \times U \to U \times U$$

is extended to

$$P_1^{(R)} \times_X P_1^{(R)} \to P_1^{(R)}.$$

$$(V \times V / \Delta G) \times_U (V \times V / \Delta G) = (V \times V \times V)/\Delta G \to V \times V / \Delta G$$

$W_1^{(R)} \subset Q_1^{(R)}$ max open étale over $P_1^{(R)}$

Thm. Bdd by $R \iff W_1^{(R)}$ inherits a gpd str.

$\Updownarrow$

existence of the unit section.

$$E_1^{(R)} = W_1^{(R)} \times_X D \longrightarrow \underbrace{P_1^{(R)} \times_X D}_{=T^{(R)}} = TX(-R) \times_X D$$

étale morphism of smooth gp schem $/D$.

$E_1^{(R)0} \subset E_1^{(R)}$ max open sub gp schme s.t. $E_x^{(R)0} \subset E_x^{(R)}$ is the conn. cpt. containing the unit sectn

Def'n Non degenerate: $E_1^{(R)0} \to T^{(R)}$ is finite.

$$0 \to G^{(R)} \to E^{(R)0} \to T^{(R)} \to 0$$

extension of a vector b'dle by a finite étale group sch

$G^{(R)}$. étale locally ism to $\mathbb{F}_p^n$ for sm n.

Extension is classified by

$$G^{(R)\vee} = \mathrm{Hom}(G^{(R)}, \mathbb{F}_p) \longrightarrow T^{(R)\vee} = \mathrm{Hom}(T^{(R)}, \mathbb{G}_a)$$

defined over a rad. cov. of D.

$$\left(0 \to \mathbb{F}_p \to \mathbb{G}_a \xrightarrow{T \mapsto T^p - T} \mathbb{G}_a \to 0\right)$$

Étale locally $G^{(R)} \subset G$ isoclinic $\Rightarrow P_\chi|_{G^{(R)}} =$ sum of nontrivial char. $\oplus \chi$

$\mathrm{Char}(\mathcal{F}) = (-1)^d \left(\mathrm{rk}\,\mathcal{F}\,[T_X^*X] + \sum_\chi L(\chi)\right)$

1. Compatibility with the pull-back.
   - functoriality of the construction

2. Local acyclicity
   - reduction to rel dim 1 case Deligne - Laumon.

3. Characteristic class.
   - vanishing of $R\bar{p}_!(Rj_*^{(R)} \mathcal{H} \cdot \mathcal{F}_{T^{(R)}})$