

Wild ramification of schemes and sheaves

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Partly joint work with

Ahmed Abbes and Kazuya Kato

Two major problems in ramification theory

- Compute global cohomological invariants
(e.g. Euler number, conductor)
in terms of invariants of ramification
(e.g. Swan class, characteristic class)
- Describe these invariants of ramification
using ramification groups
and characteristic cycle

New inputs (\sim 80's)

- Define invariants of ramification as **0-cycle classes** (Bloch)
- Analogy with irregular singularity of \mathcal{D} -modules. (Deligne, Kato, ...)

New inputs (00's)

- Lefschetz trace formulas for **open** varieties or over a **log point, dvr** etc.
- Kill ramification by **blow-up**, instead of **ramified covering**

PLAN

1. Euler numbers and conductors
2. Characteristic class of an ℓ -adic sheaf
3. Ramification groups of a local field
4. Blow-up to kill ramification
5. Graded pieces of ramification groups
6. Bounding wild ramification
7. Characteristic cycle of an ℓ -adic sheaf

1. EULER NUMBERS AND CONDUCTORS

(a) Generalization of the Grothendieck-Ogg-Shafarevich formula (SGA5)

U/k : separated of finite type, $k = \bar{k}$,

\mathcal{F}/U : \mathbb{Q}_ℓ -sheaf, $\ell \neq \text{char } k$,

$$\chi_c(U, \mathcal{F}) := \sum_{q=0}^{2 \dim U} (-1)^q \dim H_c^q(U, \mathcal{F}).$$

THEOREM 1 (Kato-S.)

$k = \bar{k}$, $\ell \neq \text{char } k > 0$, U/k smooth, \mathcal{F}/U smooth,

Then :

$$\chi_c(U, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \chi_c(U, \mathbb{Q}_\ell) - \text{deg } \text{Sw}_U \mathcal{F}.$$

$\text{Sw}_U \mathcal{F} \in CH_0(X \setminus U)_{\mathbb{Q}}$: Swan class of \mathcal{F} ,

X : compactification of U .

(b) Conductor formula

K : complete discrete valuation field (=cdvf),

$k = \bar{k}$: residue field,

U/K : separated of finite type,

\mathcal{F}/U : \mathbb{Q}_ℓ -sheaf, $\ell \neq \text{char } k$,

$$\text{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F}) := \sum_{q=0}^{2 \dim U} (-1)^q \text{Sw}_K H_c^q(U_{\bar{K}}, \mathcal{F})$$

Sw_K : Swan conductor.

THEOREM 2 (Kato-S.)

K cdvf, $k = \bar{k}$, $\ell \neq \text{char } k > \text{char } K = 0$,

U/K smooth, \mathcal{F}/U smooth.

Then :

$$\text{Sw}_K H_c^*(U_{\bar{K}}, \mathcal{F}) = \text{rank } \mathcal{F} \cdot \text{Sw}_K H_c^*(U_{\bar{K}}, \mathbb{Q}_\ell) - \text{deg Sw}_U \mathcal{F}.$$

$\text{Sw}_U \mathcal{F} \in F_0 G(X_k)_{\mathbb{Q}}$: Swan class of \mathcal{F} ,

X : compactification of U over \mathcal{O}_K .

- Formula for $\text{Sw}_K H_c^*(U_{\bar{K}}, \mathbb{Q}_\ell)$ (Bloch, Kato-S.)
- Relative version (Kato-S.) :

ℓ -adic Grothendieck-Riemann-Roch

$$\overline{\text{Sw}}_V Rf_! \mathcal{F} = f_! \overline{\text{Sw}}_U \mathcal{F}.$$

$f: U \rightarrow V/K$: separated of finite type,

\mathcal{F}/U : \mathbb{Q}_ℓ -sheaf.

2. CHARACTERISTIC CLASS OF AN ℓ -ADIC SHEAF

X/k : separated of finite type,

$\ell \neq \text{char } k$,

\mathcal{F}/X : \mathbb{Q}_ℓ -sheaf

DEFINITION Characteristic class $C(\mathcal{F})$:

$$\begin{array}{ccc}
 1 \in \text{End}(\mathcal{F}) & = & H_X^0(X \times_k X, R\mathcal{H}om(\text{pr}_2^* \mathcal{F}, R\text{pr}_1^! \mathcal{F})) \\
 \downarrow & & \downarrow \delta^* \\
 C(\mathcal{F}) \in H^0(X, K_X) & \xleftarrow{\text{Tr}} & H^0(X, R\mathcal{H}om(\mathcal{F}, \mathcal{F} \otimes K_X))
 \end{array}$$

$\delta : X \rightarrow X \times X$: diagonal, $\text{pr}_i : X \times X \rightarrow X$ projections,

$K_X = Ra^! \mathbb{Q}_\ell$, $a : X \rightarrow \text{Spec } k$,

(= $\mathbb{Q}_\ell(d)[2d]$ if X smooth of dim d),

Tr : Trace map.

THEOREM 3 (Abbes-S.)

U/k smooth,

\mathcal{F}/U smooth, $\ell \neq \text{char } k$,

$j: U \rightarrow X/k$ open immersion.

Then, if \mathcal{F} is “potentially of Kummer type”, :

$$C(j_! \mathcal{F}) = \text{rank } \mathcal{F} \cdot C(j_! \mathbb{Q}_\ell) - \text{cl}(\text{Sw}_U \mathcal{F}).$$

$\text{cl}: CH_0(X \setminus U) \rightarrow H^0(X, K_X)$, cycle class map.

- Theorem 3 \Rightarrow Theorem 1.
- If X proper, Lefschetz trace formula (SGA5)

\Rightarrow

$$\begin{array}{ccc}
 C(\mathcal{F}) \in H^0(X, K_X) & & \\
 \downarrow & & \downarrow \text{Tr}_X \\
 \chi_c(U, \mathcal{F}) \in & & \mathbb{Q}_\ell
 \end{array}$$

GOAL:

(1) Define the characteristic cycle $CC(\mathcal{F})$ as a cycle on the (log) cotangent bundle.

(2) Prove

$$C(j_! \mathcal{F}) = \text{cl}(CC(\mathcal{F})).$$

METHOD:

Blow-up the ramification locus in the diagonal.

3. RAMIFICATION GROUPS OF A LOCAL FIELD

K : complete discrete valuation field,

F : residue field (may be **non-perfect**),

L/K : finite Galois extension,

$G = \text{Gal}(L/K)$,

(a) Lower numbering filtration (classical)

$$G = G_0 \supset G_1 \supset \cdots$$

$$\supset G_i = \text{Ker}(G \rightarrow \text{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i)) \supset \cdots \supset 1.$$

compatible with subgroup

Geometric interpretation.

$$\mathcal{O}_L = \mathcal{O}_K[X_1, \dots, X_n]/(f_1, \dots, f_n).$$

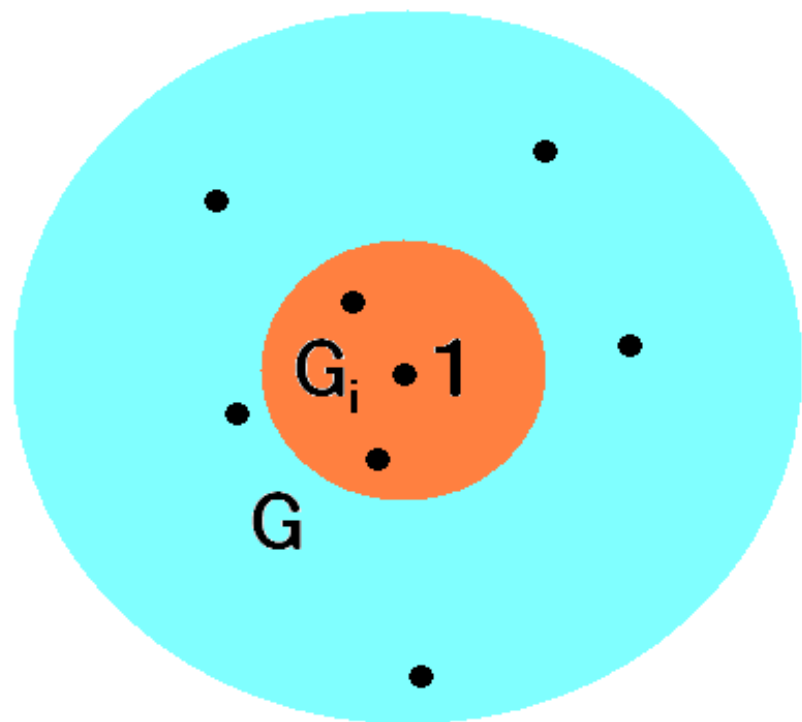
D^n : rigid analytic disk of dimension n over K .

$$f: \begin{array}{ccc} D^n & \longrightarrow & D^n \\ (x_1, \dots, x_n) & \longmapsto & (f_1(x_1, \dots, x_n), \dots, f_n(x_1, \dots, x_n)) \end{array}$$

$$G = \text{Mor}_{\mathcal{O}_K}(\mathcal{O}_L, \overline{K}) = f^{-1}(0)$$

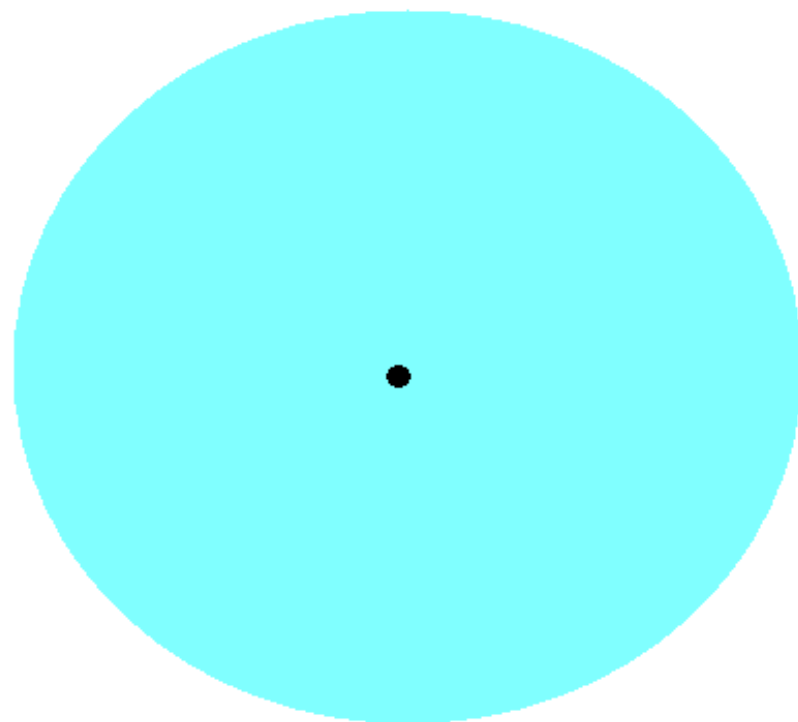
$$\supset G_i = G \cap D^n \left(\frac{i}{e_{L/K}} \right) = \{x \in G \mid d(x, \mathbf{1}) \leq |\pi_L|^i\}.$$

D^n



f
→

D^n



(b) Upper numbering filtration

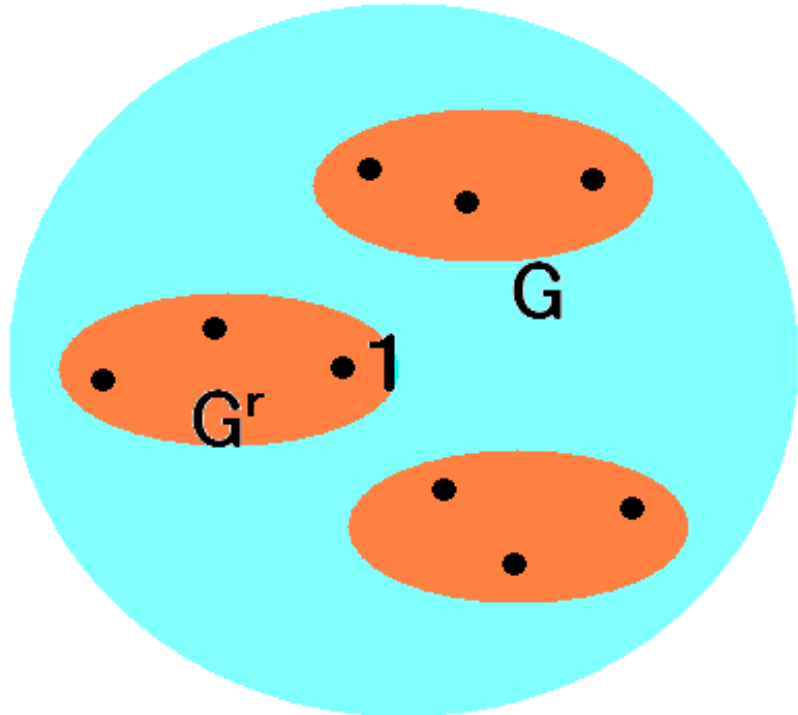
DEFINITION $r \in \mathbb{Q}, r > 0,$

$$G^r := G \cap \left(\begin{array}{l} \text{connected cpt of } f^{-1}(D^n(r)) \\ \text{containing } 1 \in G \end{array} \right).$$

$$f: D^n \rightarrow D^n \supset D^n(r) = \{x \in D^n \mid d(x, 0) \leq |\pi_K|^r\}.$$

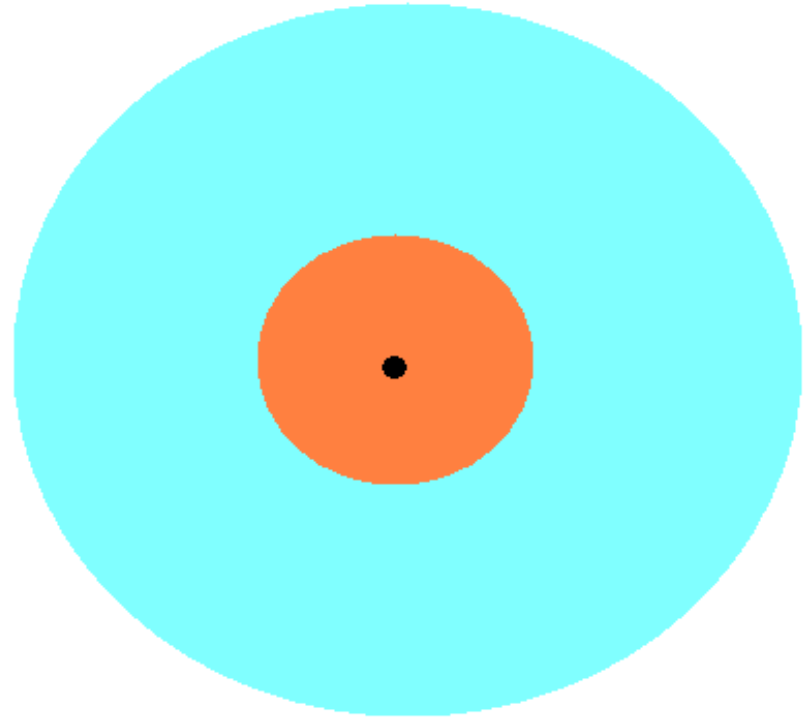
- compatible with quotient
- classical if F perfect

D^n



f
→

D^n



FIRST STEPS :

(1a) Relate **characters** of graded pieces of ramification groups to **differential forms**.

(1b) Define the **characteristic cycle** of an ℓ -adic sheaf using these **differential forms**.

- shrinking the radius (rigid geometry)
= blowing-up (algebraic geometry),
- logarithmic variant G_{\log}^r ,
- globalizing,
- analogy with \mathcal{D} -modules ...



- Graded pieces of ramification groups
- Definition of the **characteristic cycle**

4. BLOW-UP TO KILL RAMIFICATION

$k = \bar{k}$, $\text{char } k = p > 0$,

X/k smooth,

$U = X \setminus D$ complement of a divisor

$D = \cup_i D_i$ simple normal crossings (=sncd).

(a) Blow-up to kill tame ramification

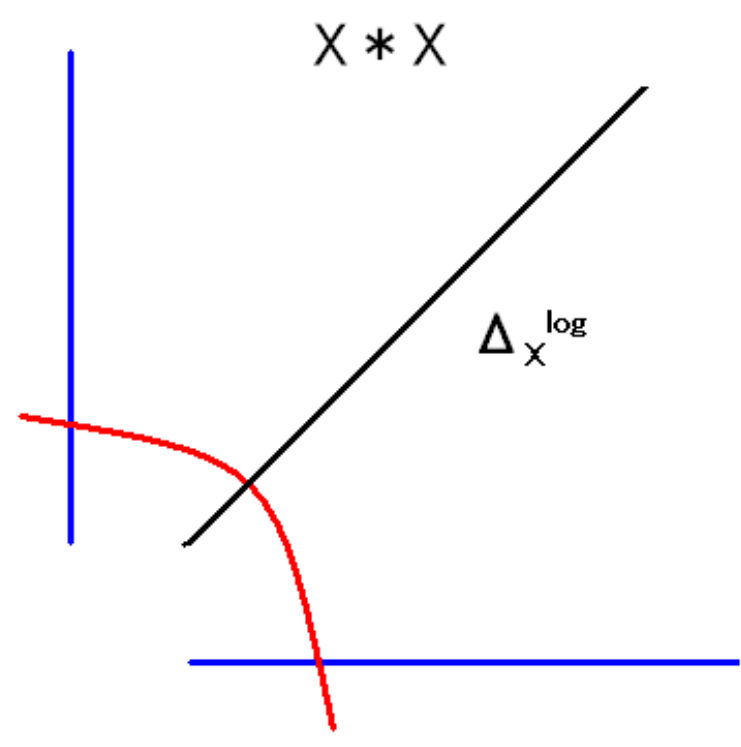
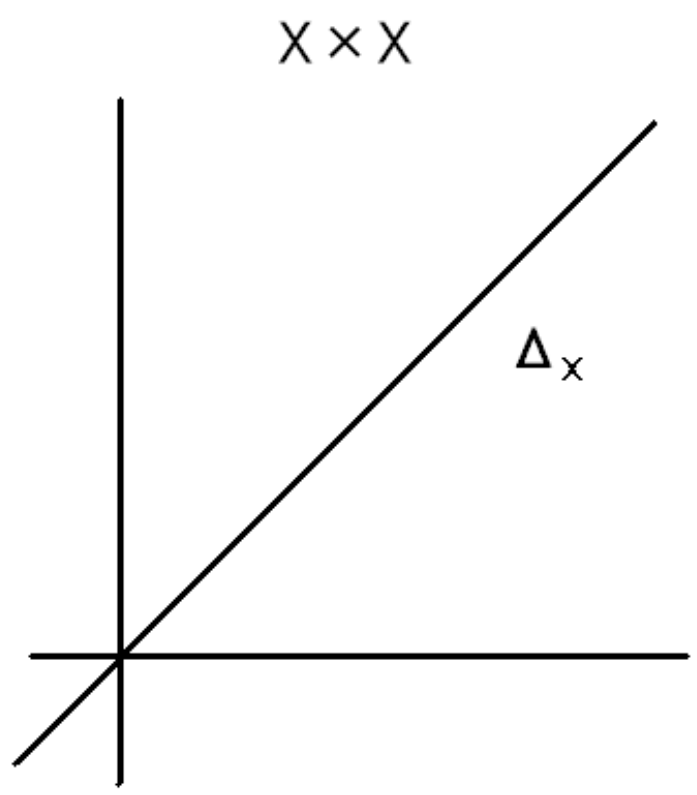
$$X * X \rightarrow X \times X$$

Blow up at all $D_i \times D_i$,

Remove proper transform of $(D \times X) \cup (X \times D)$.

log diagonal $X \rightarrow X * X$.

conormal sheaf $N_{X/X*X} = \Omega_X^1(\log D)$.



EXAMPLE

$$X = \mathbf{A}_k^1 = \text{Spec } k[x] \supset U = \text{Spec } k[x^{\pm 1}]$$

$$\begin{aligned} X \times_k X & \longleftarrow X * X \\ = \text{Spec } k[x, y] & \qquad = \text{Spec } k[x, y, u^{\pm 1}]/(x - uy). \end{aligned}$$

$$\mathcal{F}/U : T^n = x, p \nmid n,$$

$$\Rightarrow \mathcal{H} = \text{Hom}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F})/U \times U : T^n = x/y = u$$

smooth extension on $X * X \supset U \times U$.

(b) Blow-up to kill wild ramification

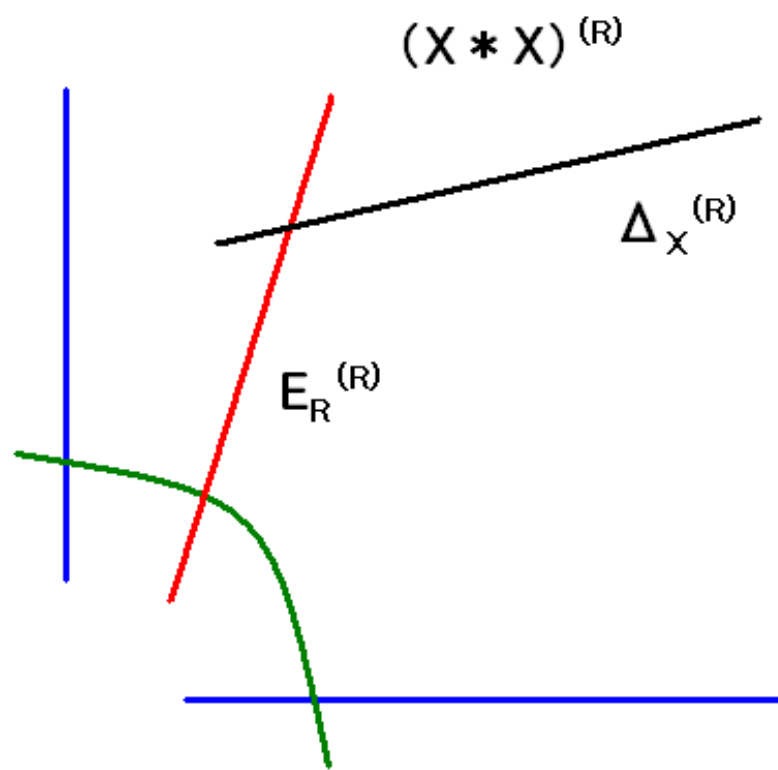
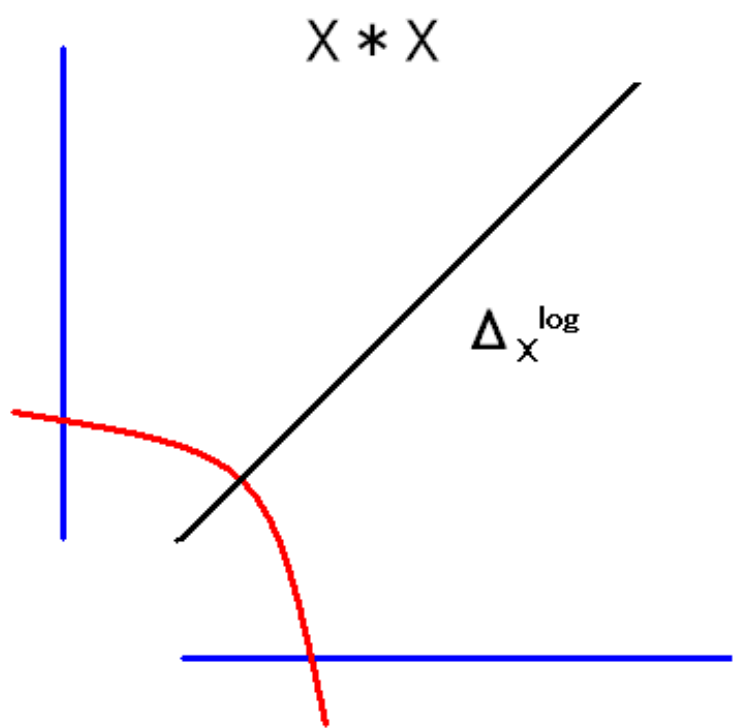
$$R = \sum_i r_i D_i, \quad r_i \in \mathbb{Q}, \quad r_i \geq 0.$$

(Assume $r_i \in \mathbb{N}$ for simplicity.)

$$(X * X)^{(R)} \rightarrow X * X$$

Blow up at $R \subset X$ in the log diagonal $X \rightarrow X * X$,

Remove proper transform of $(X * X) \times_X R$.



EXAMPLE

$$X = \mathbb{A}_k^1 = \text{Spec } k[x] \supset U = \text{Spec } k[x^{\pm 1}],$$

$$\mathcal{F}/U: T^p - T = \frac{1}{x^r}, \quad p \nmid r, \quad r \in \mathbb{N},$$

$$\mathcal{H} = \text{Hom}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F})/U \times U$$

$$R = r(0),$$

$$\begin{aligned} (X * X)^{(R)} &= \text{Spec } k[x, y, t, (1 + ty^r)^{\pm 1}] / (x - (1 + ty^r)y) \\ &\downarrow \\ X * X &= \text{Spec } k[x, y, u^{\pm 1}] / (x - uy). \end{aligned}$$

$$\begin{aligned} \mathcal{H}: \quad T^p - T &= \frac{1}{x^r} - \frac{1}{y^r} \\ &= - (1 + ty^r)^{-r} \left(rt + \binom{r}{2} t^2 y^r + \dots \right) \end{aligned}$$

smooth extension on $(X * X)^{(R)} \supset U \times U$.

lift of the diagonal $\delta^{(R)}: X \rightarrow (X * X)^{(R)}$.

conormal sheaf $N_{X/(X * X)^{(R)}} = \Omega_X^1(\log D)(R)$.

$$\begin{array}{ccccc}
 E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
 \downarrow & & \downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
 R & \xrightarrow{i_R} & X & \xleftarrow{j} & U
 \end{array}$$

$E_R^{(R)} = V(\Omega_X^1(\log D)(R)) \times_X R$: twisted tangent b'dle

$(V(\mathcal{E}) = \text{Spec } S^\bullet \mathcal{E})$

\mathcal{F}/U smooth \mathbb{Q}_ℓ -sheaf. $\ell \neq p = \text{char } k > 0$.

$\mathcal{H} = \text{Hom}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F})$ on $U \times U$.

- $i_R^{(R)*} j_*^{(R)} \mathcal{H}$ on $E_R^{(R)} = V(\Omega_X^1(\log D)(R)) \times_X R$.

$$\begin{array}{ccccc}
 E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
 & & \delta^{(R)} \uparrow & & \uparrow \delta_U \\
 & & X & \xleftarrow{j} & U
 \end{array}$$

- $\delta^{(R)*} j_*^{(R)} \mathcal{H} \rightarrow j_* \delta_U^* \mathcal{H} = j_* \mathcal{E}nd(\mathcal{F})$ base change map

5. GRADED PIECES OF RAMIFICATION GROUPS

$U = X \setminus D$, D sncd,

$K = \text{Frac}(\widehat{\mathcal{O}}_{X,\xi})$: local field at the generic point ξ
of an irreducible component of D .

$F = F(\xi)$: residue fd = function fd of the cpt.

non perfect if $\dim X > 1$

$G_K = \text{Gal}(\bar{K}/K)$, log upper numbering filtration

DEFINITION $r \in \mathbb{Q}$, $r > 0$,

$$\text{Gr}_{\log}^r G_K = G_{K,\log}^r / G_{K,\log}^{r+}$$

$$G_{K,\log}^{r+} = \overline{\bigcup_{s>r} G_{K,\log}^s}$$

PROPOSITION 4 (Abbes-S., S.) $r \in \mathbb{Q}$, $r > 0$,

$\text{Gr}_{\log}^r G_K$ is abelian, killed by p .

\mathcal{F}/U : smooth \mathbb{Q}_ℓ -sheaf,

V : ℓ -adic representation of G_K defined by \mathcal{F}

$$V = \bigoplus_{r \geq 0} V^{(r)} : \text{slope decomposition}$$

$$V^{(r)}|_{\text{Gr}_{K,\log}^r} = \bigoplus_{\chi} V_{\chi} \otimes \chi$$

decomposition by characters of $\text{Gr}_{\log}^r G_K$.

Assume $r > 0$, $V = V^{(r)} = V_\chi \otimes \chi$, $R = rD$, D irred.

THEOREM 5 $\mathcal{H} = \text{Hom}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{F}) / U \times U$

Then : $j_*^{(R)} \mathcal{H}$ is smooth. (=ramification killed)

$\exists!$ linear form $f_\chi \neq 0$ on $E_{\bar{\xi}}^{(R)}$ s.t.

$$(j_*^{(R)} \mathcal{H})|_{E_{\bar{\xi}}^{(R)}} \simeq \text{End}(V_\chi) \otimes \mathcal{L}_\chi$$

\mathcal{L}_χ : sm rk 1 def'd by Artin-Schreier eqn $T^p - T = f_\chi$

$$\begin{array}{ccccccc}
 E_{\bar{\xi}}^{(R)} & \longrightarrow & E_R^{(R)} & \xrightarrow{i_R^{(R)}} & (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \bar{\xi} & \longrightarrow & R = rD & \longrightarrow & X & &
 \end{array}$$

χ : char. of $\mathrm{Gr}_{\log}^r G_K$, $V = V_\chi \otimes \chi$, $\mathcal{H} = \mathrm{Hom}(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F})$

$$(j_*^{(R)} \mathcal{H})|_{E_{\bar{\xi}}^{(R)}} = \mathrm{End}(V_\chi) \otimes \mathcal{L}_\chi, \quad \mathcal{L}_\chi : T^p - T = f_\chi$$

DEFINITION Refined Swan conductor:

linear form $f_\chi \neq 0$ on $E_{\bar{\xi}}^{(R)} = V(\Omega_X^1(\log D)(R)) \times_X \bar{\xi}$

regarded as a differential form

$$\mathrm{rsw}_\chi \in \Omega_X^1(\log D)(R) \otimes F(\bar{\xi}).$$

(Kato : rank 1 case.)

6. BOUNDING WILD RAMIFICATION

$$U = X \setminus D, D = \cup_i D_i \text{ sncd.}$$

\mathcal{F}/U smooth,

$$R = \sum_i r_i D_i, r_i \in \mathbb{Q}, r_i \geq 0,$$

($r_i \in \mathbb{N}, r_i > 0$ for simplicity).

DEFINITION $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_2^* \mathcal{F}, \mathrm{pr}_1^* \mathcal{F}) / U \times U,$

Ramification of \mathcal{F} along D is **bounded by $R+$** :

base change map

$$\delta^{(R)*} j_*^{(R)} \mathcal{H} \longrightarrow j_* \delta_U^* \mathcal{H}$$

is an **isomorphism**.

$$\begin{array}{ccc} (X * X)^{(R)} & \xleftarrow{j^{(R)}} & U \times U \\ \delta^{(R)} \uparrow & & \uparrow \delta_U \\ X & \xleftarrow{j} & U \end{array}$$

ramification of \mathcal{F} along D is bounded by $R+$

$\Rightarrow G_{K_i, \log}^{r_i+}$ acts on V **trivially** $\forall i$. $(R = \sum_i r_i D_i)$

Assume further $V = V^{(r_i)} \forall i$. $(R = \sum_i r_i D_i)$

$V = \bigoplus_{\chi} V_{\chi} \otimes \chi$, χ non-trivial char. of $\text{Gr}_{K_i, \log}^{r_i}$.

rsw_{χ} : linear form on $E_{\bar{\xi}_i}^{(R)}$ = a point of the dual $E_{\bar{\xi}_i}^{\vee(R)}$.

$$\text{rsw}_{\chi} \in E_{\bar{\xi}_i}^{\vee(R)} \rightarrow E_{D_i}^{\vee(R)}, \quad \bar{\xi}_i \rightarrow \xi_i \in D_i.$$

PROPOSITION 6 (Abbes-S.)

$A_{\chi} = \overline{\text{Image}(\text{rsw}_{\chi})} \subset E_{D_i}^{\vee(R)}$ is **finite** over D_i .

7. CHARACTERISTIC CYCLE OF AN ℓ -ADIC SHEAF

$$U = X \setminus D, D = \cup_i D_i \text{ sncd.}$$

\mathcal{F}/U smooth,

$$R = \sum_i r_i D_i, r_i \in \mathbb{Q}, r_i \geq 0,$$

($r_i \in \mathbb{N}, r_i > 0$ for simplicity).

Assume:

- Ramification of \mathcal{F} along D is bdd by $R+$,
- $V = V^{(r_i)} \forall i$, $(R = \sum_i r_i D_i)$
- Cleanliness condition (Kato : rank 1 case):

$$V = \bigoplus_{\chi} V_{\chi} \otimes \chi, \quad A_{\chi} \cap (0\text{-section}) = \emptyset \quad \forall i \quad \forall \chi.$$

$$(A_{\chi} = \overline{\text{Image}(\text{rsw}_{\chi})} \subset \overset{\vee(R)}{E}_{D_i})$$

(Wild ramification is controlled at codimension 1 points.)

$T^*X(\log D) = V(\Omega_X(\log D)^\vee)$: log cotangent bundle,

$L(-R) = V(\mathcal{O}_X(R))$: line bundle,

- $A_\chi \subset \overset{\vee(R)}{E}_{D_i} = V(\Omega_X^1(\log D)(R)^\vee) \times_X D_i$

$\Rightarrow a_\chi: L(-R) \times_X A_\chi \rightarrow T^*X(\log D) \times_X A_\chi$ linear

- $A_\chi \cap (0\text{-section}) = \emptyset \Leftrightarrow a_\chi$ injection

$$(R = \sum_i r_i D_i, V = \bigoplus_{\chi} V_{\chi} \otimes \chi),$$

$$\begin{array}{ccc}
 L(-R) \times_X A_{\chi} \xrightarrow{a_{\chi}} T^*X(\log D) \times_X A_{\chi} & \longrightarrow & T^*X(\log D) \\
 \searrow & & \downarrow \\
 & & A_{\chi} \longrightarrow D_i \longrightarrow X
 \end{array}$$

DEFINITION $d = \dim X$. Characteristic cycle:

$$\begin{aligned}
 CC(\mathcal{F}) &= (-1)^d \left(\text{rank } \mathcal{F} \cdot [0\text{-section}] \right. \\
 &\quad \left. + \sum_i r_i \sum_{\chi} \frac{\dim V_{\chi}}{[A_{\chi} : D_i]} \cdot a_{\chi*} [L(-R) \times_X A_{\chi}] \right) \\
 &\in Z^d(T^*X(\log D)) \otimes \mathbb{Z} \begin{bmatrix} 1 \\ p \end{bmatrix}
 \end{aligned}$$

THEOREM 7

Assume ramification of \mathcal{F} along D is bdd by $R+$,

$V = V^{(r_i)} \forall i$ and $A_\chi \cap (0\text{-section}) = \emptyset \forall i \forall \chi$.

Then:

$$C(j_! \mathcal{F}) = \text{cl}(CC(\mathcal{F}))$$

in $H^{2d}(X, \mathbb{Q}_\ell(d)) = H^{2d}(T^*X(\log D), \mathbb{Q}_\ell(d))$.

COROLLARY

Further, if X proper,

$$\chi_c(U, \mathcal{F}) = (CC(\mathcal{F}), 0\text{-section})_{T^*X(\log D)}.$$

(cf. Dubson-Kashiwara, Laumon, ...

for \mathcal{D} -modules in char. 0)

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