Characteristic Cycle

X smooth $/k$ perfect. Λ finite field char inv. in k.

$\mathcal{F}/X$ const complex of Λ-mod.

Supp $\mathcal{F}$ closed subset of X

$SS\mathcal{F}$ conical closed subset of T^*X cotangent bdle $= \bigcup C_a$ Defined by Beilinson. Stable under $\mathbb{G}_m$-action

C_a irred compt dim $=$ dim X.

$CC\mathcal{F} = \sum m_a C_a$ $m_a \in \mathbb{Z}$. (≥ 0 if $\mathcal{F}$ perverse)

Supp $CC\mathcal{F} = SS\mathcal{F}$.

Examples. $U = X - D$ D div. w. SNC.

$j: U \to X$ open imm $\mathcal{F} = j_! \mathcal{G}[n]$ $n = \dim X$

$\mathcal{G}$ loc const on U

1. $\mathcal{G}$ tamely ramified along D

$CC\mathcal{F} = \text{rk}\,\mathcal{G} \cdot \sum T^*_{X_I} X$ $X_I = \bigcap_{i \in I} D_i$, $D = \bigcup D_i$

including $X_\phi = X$ $T^*_X X =$ 0-section

2. $\dim X = 1$

$CC\mathcal{F} = \text{rk}\,\mathcal{F} \cdot [T^*_X X] + \sum_{x \in D} \text{Sw}_x \mathcal{F} \cdot [T^*_x X]$.

Theorem 1. (index formula) If X proj.

$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$

If $\dim X = 1$. Grothendieck - Ogg - Shafarevich

Thm 1. compatibility with proper pushfwd for $f: X \to S, -k$.

$f: X \to Y$ proper (proper on supp $\mathcal{F}$)

$$\begin{tikzcd} T^*X & X \times_Y T^*Y \arrow[l] \arrow[r] & T^*Y \\ SS\mathcal{F} \arrow[u, phantom, "\cup"] \arrow[rr, dashed] & & f_\circ SS\mathcal{F} \arrow[u, phantom, "\cup"] \\ CC\mathcal{F} \arrow[rr, dashed] & & f_! CC\mathcal{F} \in CH_m(f_\circ SS\mathcal{F}) \end{tikzcd}$$

algebraic correspondence

cycle class

cycle of dim of $f_\circ SS\mathcal{F} = m$.

Conjecture 1. $CC\, Rf_*\mathcal{F} \overset{?}{=} f_! CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$

~~Assume ...~~ $Y = \mathrm{Spec}\, k$ $\quad f_! CC\mathcal{F} = (CC\mathcal{F}, T^*_X X)$.

Assume $\dim Y = 1$.

$$CC\, Rf_*\mathcal{F} = -\left(\dim \chi(X_{\bar\eta}, \mathcal{F}) \cdot T^*_Y Y + \sum_{y \in Y} a_y Rf_*\mathcal{F} \cdot [T^*_y Y]\right)$$

index formula.

$$a_y Rf_*\mathcal{F} = \chi(X_{\bar\eta}, \mathcal{F}) - \chi(X_{\bar y}, \mathcal{F}) + Sw_y H^*(X_{\bar\eta}, \mathcal{F})$$

Artin conductor

Assume further $\dim f_\circ SS\mathcal{F} = 1$. Conj means the cond. formula

$$-a_y Rf_*\mathcal{F} = \text{coeff. in } f_! CC\mathcal{F} \text{ of } [T^*_y Y].$$

<u>Prop 1</u> Assume $\dim X = 2$, $\dim Y = \dim f_\circ SS\mathcal{F} = 1$

$f: X \to Y$ gen. smooth $\Rightarrow$ Conj 1 is true.

Idea of pf. global argument using the index formula

kill the ramification at the other pts by Epp's thm.

MacPherson Chern class

X may be singular. / char $k = 0$

$F(X) = \{\text{const fns } X \to \mathbb{Z}\}$

$\exists !\ c_X : F(X) \to CH_\cdot(X)$ s.t.

(1) $f: X \to Y$ proper $\Rightarrow$

$$\begin{tikzcd} F(X) \arrow[r, "c_X"] \arrow[d, "f_*"'] & CH_\cdot(X) \arrow[d, "f_*"] \\ F(Y) \arrow[r, "c_Y"'] \arrow[ur, phantom, "\circlearrowleft"] & CH_\cdot(Y) \end{tikzcd}$$

(2) f: smooth $\Rightarrow$

$$\begin{tikzcd} F(Y) \arrow[r, "c_Y"] \arrow[d, "f^*"'] & CH_\cdot(Y) \arrow[d, "f^* \cap c_\cdot(T_{X/Y})"] \\ F(X) \arrow[r, "c_X"'] & CH_\cdot(X) \end{tikzcd}$$

(3) $X = \operatorname{Spec} k \Rightarrow c_X : F(X) = \mathbb{Z} \to CH_\cdot(X) = \mathbb{Z}$, $c_X = \mathrm{id}$

$i : X \hookrightarrow M$ smooth

$N = \dim X$

$$\begin{tikzcd} K(X, \Lambda) \arrow[r, "\mathrm{rk}"] \arrow[dr, "CC_X"'] & F(X) \arrow[d, "(-1)^{i} c_X"] \\ ? & CH_\cdot(X) \end{tikzcd}$$

$CH_\cdot(X) = CH_N(\mathbb{P}(X \times_M T^*M \oplus \mathbb{A}^1_X)) \ni [\overline{CC\mathcal{F}}]$

$CC_X : K(X, \Lambda) \to CH_\cdot(X)$ defined without cond. on char k.

Satisfies (2) & (3) but not (1) (conj by Grothendieck in SGA5 unpublished)