Hasse-Arf theorem in higher dim (j. w/ K. Kato)

Plan. 1. Classical H.A. thm
2. Gen'n to higher dim.
3. Rank 1 case
4. Ramification of char.

1. K local fd. (= c.d.v.f w/ perfect res. fd.)
L/K finite Galois ext. $G = \mathrm{Gal}(L/K)$
M. rep'n of G

$$\mathrm{Sw}_K M = \frac{1}{|G|} \sum_{\sigma \in G, \neq 1} s_G(\sigma) (\mathrm{Tr}(\sigma : M) - \dim M)$$

$$s_G(\sigma) = \begin{cases} -\operatorname{ord}_L(\sigma(\pi_L)/\pi_L - 1) & \sigma \in I, \neq 1 \\ 0. & \sigma \notin I \end{cases}$$

H.A - thm $\qquad \mathrm{Sw}_K M \in \mathbb{N}$.

2. $S = \mathrm{Spec}\, \mathcal{O}_K$. U/S. reg flat sep. sch of f.t
$f : V \to U$ finite étale Galois cov. $G = \mathrm{Gal}$.
s.t. $V_K \to U_K$ tamely ramified.
M rep'n of G

$$\mathrm{Sw}_U M = \frac{1}{|G|} \sum_{\sigma \in G, \neq 1} s_{G}(\sigma) (\mathrm{Tr}(\sigma : M) - \dim M)$$

$$\in F_0 G(X_F)_{\mathbb{Q}}$$

$$\begin{array}{ccc} V & \hookrightarrow & Y \\ \downarrow & & \downarrow \bar{f} \\ U & \hookrightarrow & X \ \text{proper}/S \end{array}$$

$$s_G(\sigma) = \bar{f}_*(-((\Gamma_\sigma, \Delta_V))_{(Y \times_S Y)^\sim})$$

~~Assume~~ Y regular $V = Y \setminus D$ cpt of SNCD.
$(Y \times_S Y)^\sim$ log product. $\qquad G \curvearrowright Y$

$$((\ ,\ )) = (-1)^d \left([\mathrm{Tor}_d^{\mathcal{O}_{(Y\times_S Y)^\sim}}(\mathcal{O}_{\bar{\Gamma}}, \mathcal{O}_{\Delta_Y^{\log}})] - [\mathrm{Tor}_{d+1}^{\ldots}] \right)$$

In general alteration.

classical case $Y = \mathcal{O}_L = \mathcal{O}_K[T]/f(T)$ $f(T)$ Eisenstein

$(Y \times_S Y)^{\sim}_{\#} : \mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_L [U^{\pm 1}] / (1 \otimes \tau - \tau \otimes 1 \cdot U)$

$= \mathcal{O}_L [U^{\pm 1}] / f(\tau U) = A$

$f(\tau U) = (U - \sigma(\tau)/\tau) \cdot g_\sigma$

$A \xrightarrow{g_\sigma} A \xrightarrow{U - \sigma(\tau)/\tau} A \to \mathcal{O}_{\Gamma_\sigma} \to 0$

$\otimes_A \mathcal{O}_L$.

$\mathcal{O}_L \to \mathcal{O}_L \quad 0 \quad \mathcal{O}_L \xrightarrow{1 - \sigma(\tau)/\tau} \mathcal{O}_L$

Theorem. (Kato-S.) If $\dim U_F = 1$,

$Sw_U M$ is in Image $(CH_0(X_F) \to F_0G(X_F)_{\mathbb{Q}})$.

Application to Artin char.

A: regular local. G finite gp $\curvearrowright$ A.

st A^G noetherian, A/I_σ finite length for $\sigma \neq 1$

$I_\sigma = (a - \sigma(a) \,;\, a \in A)$

Define $a_G(\sigma) = \begin{cases} -\text{length } A/I_\sigma & \sigma \neq 1 \\ -\sum_{\tau \neq 1} a_G(\tau) & \sigma = 1. \end{cases}$

Conj (Serre) a_G is a char of G.

Thm (Kato) Conj is true if $\dim A \leq 2$.

3. Pf of Thm.

Red'n to rk 1 case: Brauer + Induction formula.

Rank 1 case $M \leftrightarrow \chi \in H^1(U, \mathbb{Q}/\mathbb{Z})$.

Define $c_\chi \in CH_0(X_F)$ and show $c_\chi \mapsto Sw_U M$.

$\dim X = 2$ X regular. $U = X \setminus D$ D. snc d.

- Swan divisor $R_\chi = \sum r_i D_i$.
- χ "clean".

$$c_\chi = \sum_i c_1(\Omega^1_{X/S}(\log D)|_{D_i}) \cap r_i D_i - R^2.$$

- $c_\chi \mapsto Sw_U M$.

red to ord $\chi = p^n$.

$n=1$ Compute them explicity

general induction on n

4. K cdvf res fd may not be perfect.

Fil'n on G_K^{ab} $\Leftrightarrow$ $X_K = H^1(K, \mathbb{Q}/\mathbb{Z}) = H^2(K, \mathbb{Z})$

$$\begin{tikzcd}
\{\ ,\ \}_K : X_K \times K^\times \arrow[r] & Br(K) = H^2(K, \mathbb{G}_m) \\
F \times K^\times \arrow[u, "\theta_a"] \arrow[r] \arrow[ur, phantom, "\Omega"] & \Omega^1_F(\log) = \Omega^1_F \oplus F\, d\log \pi \arrow[u, "\lambda_F\ \text{can map}"']
\end{tikzcd}$$

$T^2 F a$. $\theta_a \uparrow a$

s.t. $(a, b) \longmapsto a\, d\log b$

$$F_r X_K = \{ x \in X_K \mid \{x, 1 + m_K^r \mathcal{O}_L\}_L \subset \mathrm{Im}\, \lambda_{F_L} \text{ for } \forall L/K \}$$

Kummer

$$Gr^F_r X_K \times m_K^r / m_K^{r+1} \longrightarrow \Omega^1_F(\log)$$

$$x \in F_r X_K, \ \mapsto \ \mathrm{rsw}\, x : m_K^r / m_K^{r+1} \to \Omega^1_F(\log)$$

$X \supset U = X \setminus D$ $\quad D_i$ $\quad K_i$: local fd at gen pt

$M \to X$ $\quad$ char of $G_{K_i}^{ab}$ $\quad r_i$ $\quad R = \sum r_i D_i$

$$\mathrm{rsw}_\chi : \mathcal{O}(-R)|_{D_i} \longrightarrow \Omega^1_{X/S}(\log D)|_{D_i}$$

Clean: rsw_χ locally direct summand.

K mixed char $\zeta_p \in K$. $m = p \cdot \mathrm{ord}_K(\zeta_p - 1) = \frac{p}{p-1}\,\mathrm{ord}_K p$

$$\begin{array}{ccc} X_K[p] = H^1(K, \mathbb{Z}/p\mathbb{Z}) & \xleftarrow{\sim} & K^\times/(K^\times)^p = H^1(K, \mu_p) \\ \cup \quad = \mathrm{Fil}^m X_K[p] & & \cup \\ \mathrm{Fil}^r X_K[p] & \simeq & \mathrm{Im}\ 1 + m_K^{m-r} \end{array}$$

$$\begin{array}{ccc} & & a \\ \chi = & \longleftrightarrow & 1 + b \end{array}$$

$\mathrm{rsw}\,\chi \qquad c \longmapsto \frac{1}{\zeta^p} c\, db = \frac{bc}{\zeta^p} \cdot d\log b \qquad r < m$

$\qquad\qquad\quad c \longmapsto \frac{c}{\zeta^p}\, d\log a. \qquad r = m.$