The characteristic class and micro local analysis on of an ℓ-adic étale sheaf (with Ahmed Abbes)

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Plan:

1. Characteristic class.
2. Localization.
3. Rank 1 case.
4. Analogy with Microlocal analysis.

Notation: k field of characteritic $p > 0$.

X separated of finite type over F.

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$ etc. $(\ell \neq p)$

$\mathcal{F}$ Λ-sheaf on X, or more generally, an object in a suitable derived category.

1 Characteristic class.

Let F be a perfect field of characteristic $p > 0$. Let X be a separated scheme of finite type over F and $\mathcal{F}$ be an ℓ-adic sheaf on X. Then the characteristic class

$$C(\mathcal{F}) \in H^0(X, K_X)$$

is defined as follows. Here and in the following $K_X = a^!\Lambda$ where $a : X \to F$. Hence if X is smooth of dimension d, the characteristic class $C(\mathcal{F})$ is defined in $H^{2d}(X, \Lambda(d))$.

We consider

$$\begin{aligned} 1 \in Hom(\mathcal{F}, \mathcal{F}) &= H^0_X(X \times X, R\mathcal{H}om(p_2^*\mathcal{F}, p_1^!\mathcal{F})) \\ &= H^0_X(X \times X, R\mathcal{H}om(p_1^*\mathcal{F}, p_2^!\mathcal{F})). \end{aligned}$$

By the natural pairing, $R\mathcal{H}om(p_2^*\mathcal{F}, p_1^!\mathcal{F}) \otimes R\mathcal{H}om(p_1^*\mathcal{F}, p_2^!\mathcal{F}) \to K_{X\times X}$, their pairing is defined and gives the characteristic class as

$$C(\mathcal{F}) = \langle 1, 1 \rangle = H^0_X(X \times X, K_{X\times X}) = H^0(X, K_X).$$

If X is smooth of dimension d and $\mathcal{F}$ is smooth of rank r, we have $C(\mathcal{F}) = r \cdot (-1)^d c_d(\Omega^1_{X/F})$.

If X is proper, the Lefschetz trace formula in SGA 5 gives

$$\mathrm{Tr} C(\mathcal{F}) = \chi(X_{\bar{F}}, \mathcal{F}).$$

The characteristic class $C(\mathcal{F}) \in H^0(X, K_X)$ may be also regarded as the class of the composition

$$\delta_* \Lambda_X \to \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) \to \delta_* K_X.$$

The first map is the adjoint of $\Lambda_X \to \delta^! \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) = \mathcal{H}om(\mathcal{F}, \mathcal{F})$ and the second map is the adjoint of $\delta^* \mathcal{H}om(pr_2^* \mathcal{F}, pr_1^! \mathcal{F}) = \mathcal{F} \otimes \mathcal{H}om(\mathcal{F}, K_X) \to K_X$.

2 Localization of the characteristic class.

We consider the following case. Assume X is smooth and let $S \subset X$ be a closed subscheme. Let $U = X \setminus S$ be the complement and $j : U \to X$ be the open immersion. We assume $\mathcal{F} = j_! \mathcal{F}_U$ is the zero-extension of a smooth sheaf $\mathcal{F}_U$ on U. Then, we expect that the difference $C(j_! \mathcal{F}) - \mathrm{rank} \mathcal{F} \cdot C(j_! \Lambda)$ may be computed by the ramification of $\mathcal{F}$ along S. Here, we construct a natural lifting of $C(j_! \mathcal{F}) - \mathrm{rank} \mathcal{F} \cdot C(\Lambda)$ in $H^0_S(X, K_X)$.

Let $\mathfrak{X}$ be the formal completion of $X \times X$ with respect to the diagonal $\delta : X \to X \times X$ and $\mathfrak{X}^{\mathrm{rig}}$ be the associated rigid space. Let $\psi : \mathfrak{X}^{\mathrm{rig}} \to X \times X \setminus X$ be the canonical map and $\rho : \mathfrak{X}^{\mathrm{rig}} \to X$ be the specialization. Then, we have a non-commutative diagram

$$\begin{array}{ccc} \mathfrak{X}^{\mathrm{rig}} & \xrightarrow{\psi} & X \times X \setminus X \\ \rho \downarrow & & \downarrow g \\ X & \xrightarrow{\delta} & X \times X \end{array}$$

of associated étale topoi where g denotes the open imersion. The nearby cycle functor $\Psi : D(\mathbb{X}) \to D(\mathfrak{X}^{\mathrm{rig}})$ is defined by $\Psi \mathcal{F} = \psi^* g^* \mathcal{F}$ is defined on $\mathfrak{X}^{\mathrm{rig}}$. The vanishing cycle functor Φ fits in the distinguished triangle

$$\to \rho^* \delta^* \mathcal{F} \to \Psi \mathcal{F} \to \Phi \mathcal{F} \to$$

on $\mathfrak{X}^{\mathrm{rig}}$. If $\mathcal{F} = \delta_* \mathcal{G}$, we have $\Psi \mathcal{F} = 0$ and an isomorphism $\Phi \mathcal{F} \to \rho^* \mathcal{G}[1]$. Fujiwara's theorem gives us a canonical isomorphism $\delta^* g_* \mathcal{F} \to \rho_* \psi^* \mathcal{F}$.

Applying the functor Φ to the composition $\delta_* \Lambda_X \to \mathcal{H}om(pr_2^* j_! \mathcal{F}, pr_1^! j_! \mathcal{F}) \to \delta_* K_X$, we obtain a map $\Phi(\delta_* \Lambda_X) \to \Phi(\delta_* K_X)$. By the isomorphism $\Phi \delta_* \mathcal{G} = \rho^* \mathcal{G}[1]$, it is equivalent to the map $\rho^* \Lambda \to \rho^* K_X$ and by the adjunction further to $\Lambda \to \rho_* \rho^* K_X$. By the assumption that $\mathcal{F}$ is smooth on U, the complex $\mathcal{H}om(pr_2^* j_! \mathcal{F}, pr_1^! j_! \mathcal{F})$ is smooth on $U \times U$. Thus, the restriction of $\Phi \mathcal{H}om(pr_2^* j_! \mathcal{F}, pr_1^! j_! \mathcal{F})$ on U is 0. Hence, the map $\Lambda \to \rho_* \rho^* K_X$ factors through Λ_S and gives an element in $H^0_S(X, \rho_* \rho^* K_X)$.

By the isomorphism $\delta^* g_* \to \rho_* \psi^*$, the target $\rho_* \rho^* K_X$ is identified with $\delta^* g_* \Lambda_{\mathbb{U}} \otimes K_X$. Thus, if X is smooth of dimension d, we have a distinguished triangle $\to \Lambda_X \to K_X \to \rho_* \rho^* K_X \to$ where the class of the first map $\Lambda_X \to K_X$ is the canonical class

$c_X = (-1)^d c_d(\Omega^1_X) \in H^0(X, K_X)$. From the distinguished triangle $\to \Lambda_X \to K_X \to \rho_*\rho^* K_X \to$, we deduce an isomorphism $H^0_S(X, K_X) \to H^0_S(X, \rho_*\rho^*K_X)$. Thus we have obtained a localized class

$$C_S(j_!\mathcal{F}) \in H^0_S(X, K_X).$$

We call it the localized characteristic class.

Theorem 1 *The image of $C_S(j_!\mathcal{F}) \in H^0_S(X, K_X)$ in $H^0(X, K_X)$ is equal to $C(j_!\mathcal{F}) - \mathrm{rank}\mathcal{F} \cdot C(\Lambda)$.*

3 Computation of the characteristic class in rank 1 case.

As a model, we compute the characteristic class of a smooth sheaf of rank 1. Let X be a smooth scheme over a perfect field F and $U \subset X$ be the complement of a divisor D with simple normal crossings. We consider a smooth sheaf $\mathcal{F}$ of rank 1 on U.

First, we recall the Swan divisor $D_{\mathcal{F}}$ and the refined Swan character defined by Kato. Let $D = \bigcup_i D_i$ be the irreducible components and $K_i = \mathrm{Frac}\ \hat{O}_{X,\xi_i}$ be the local field at the generic point ξ_i of D_i. Let $F_i = \kappa(\xi_i)$ be the residue field of K_i. The F_i-vector space $\Omega_{F_i/F}(\log) = \Omega^1_{X/F}(\log D)_{\xi_i} \otimes F_i$ is of dimension d and fits in an exact sequece $0 \to \Omega_{F_i/F} \to \Omega_{F_i}(\log) \to F_i \to 0$ For each D_i, the stalk of $\mathcal{F}$ defines a continuous character $\chi_i : G^{\mathrm{ab}}_{K_i} \to \Lambda^\times$. If its p-part χ'_i has order at most p^{m+1}, it defines an element $H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z})$.

By the Artin-Schreier-Witt theory, we have a natural surjection $W_{m+1}(K_i) \to H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Brylinski defined an increasing filtration

$$F_r W_{m+1}(K_i) = \{(x_0, \dots, x_m) | p^{m-i}\mathrm{ord} x_i \geq -r \text{ for } i = 0, \dots, m\}.$$

Theorem 2 *1. On $H^1(K_i, \mathbb{Z}/p^{m+1}\mathbb{Z}) = Hom(G^{\mathrm{ab}}_{K_i}, \mathbb{Z}/p^{m+1}\mathbb{Z})$ the following three filtrations are equal:*

a. The image of $F_\bullet$.

b. The dual of the filtration defined by Kato.

c. The dual of the logarithmic upper numbering filtration defined by Abbes-Saito.

More precisely, for an integer $r \geq 1$, we have $G^{\mathrm{ab},j}_{K,\log} = G^{\mathrm{ab},r}_{K,\log}$ for $j \in (r-1, r]$ and

$$F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) = Hom(G^{\mathrm{ab}}_{K,\log}/G^{\mathrm{ab},r}_{K,\log}, \mathbb{Z}/p^{m+1}\mathbb{Z}).$$

2. Further the map

$$R_r : Gr^F_r W_{m+1}(K) \to Hom(m^r_K/m^{r+1}_K, \Omega_F(\log))$$

defined by

$$R_r(x_0, \dots, x_m) = x_0^{p^m} d\log x_0 + \cdots + x_m d\log x_m$$

is well defined. It induces an injection

$$\mathrm{rsw}_r : Gr^F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z}) \to Hom(m^r_K/m^{r+1}_K, \Omega_F(\log)).$$

The Swan divisor $D_{\mathcal{F}} = \sum_i r_i D_i$ is defined by Kato by putting r_i to be the minimum integer $r \geq 0$ satisfying $\chi_i \in F_r H^1(K, \mathbb{Z}/p^{m+1}\mathbb{Z})$. Furthermore, he shows there exists a global map $\mathrm{rsw}_{\mathcal{F}} : O(-D_{\mathcal{F}})|_{D_w} \to \Omega^1_X(\log D)|_{D_w}$ whose stalks are rsw_{χ_i} at each i.

Theorem 3 *If $\mathrm{rsw}_{\mathcal{F}} : O(-D_{\mathcal{F}}) \to \Omega^1_X(\log D)$ is locally an isomorphism onto a direct summand, we have*

$$\begin{aligned} C(j_!\mathcal{F}) - C(j_!\Lambda) &= (-1)^{d-1} c_{d-1}(\mathrm{Coker}(\mathrm{rsw}_{\mathcal{F}})) \\ &= (\mathrm{Image}\ \mathrm{rsw}_{\mathcal{F}}, 0\text{-section})_{T^*X(\log)}. \end{aligned}$$

Basic fact in the proof of Theorems 2 and 3 is the following. Let $(X \times X)' \to X \times X$ be the blow-up at every $D_i \times D_i$. Then the diagonal map $X \to X \times X$ is uniquely lifted to the log diagonal map $X \to (X \times X)'$. Let $(X \times X)'' \to (X \times X)'$ be the blow-up at $D_{\mathcal{F}} \subset X$ in the log diagonal. Then, the exceptional divisor of $(X \times X)'' \to (X \times X)'$ is a compactification of an $\mathbb{A}^d$-bundle $E = \mathrm{Spec}\ \mathbf{S}^\bullet(\Omega^1_{X/F}(\log D)(D_{\mathcal{F}})|_{D_w})$ over D_w. Further the smooth sheaf $\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^*\mathcal{F})$ on $U \times U \subset (X \times X)''$ is unramified along E and the restriction on E of the smooth extension is the Artin-Schreier sheaf defined by the linear form $\mathrm{rsw}_{\mathcal{F}}$ on E.

4 Analogy with Microlocal analysis.

Over $\mathbb{C}$, the Riemann-Hilbert correspondence gives an equivalence of categories.

$$(\text{regular holonomic } \mathcal{D}_X\text{-modules}) \to (\text{perverse sheaves of } \mathbb{C}_X\text{-modules}).$$

Let a $\mathcal{D}_X$-module $\mathcal{M}$ be corresponding to $\mathcal{F}$. Then, on the $\mathcal{D}_X$-module side, the characteristic cycle $Char(\mathcal{M})$ is defined as a cycle on the cotangent bundle T^*X as the class of $gr^\bullet(\mathcal{M})$ regarded as an $O_{T^*X} = gr^\bullet(\mathcal{D}_X)$ The cohomology class $[Char(\mathcal{M})] \in H^{2d}(T^*X, \mathbb{Z}(d)) = H^{2d}(X, \mathbb{Z}(d))$ gives the characteristic class $C(\mathcal{F})$. Kashiwara-Schapira define the microsupport $SS(\mathcal{F})$ that is the same as $Char(\mathcal{M})$ directly without resorting the Riemann-Hilbert correspondence, in the following way. They start with $\mathcal{H} = R\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F}, \mathrm{pr}_1^!\mathcal{F})$ on $X \times X$. Then by deforming $X \to X \times X$ to $X \to TX$ and applying the nearby cycle functor to $\mathcal{H}$, they define $\nu hom(\mathcal{F}, \mathcal{F})$. Further applying the Fourier-Sato transform, they obtain $\mu hom(\mathcal{F}, \mathcal{F})$ on T^*X.

Verdier has studied a similar construction in a ℓ-adic setting. However, one can not capture wild ramification in this way.

In rank 1 case, we obtain $\overline{\mathcal{H}}|_E$ on the twisted tangent bundle. By applying the Fourier-Deligne transform, one gets a section $\mathrm{rsw}_{\mathcal{F}}$ of a twisted cotangent bundle, that defines a cycle on the cotangent bundle.