The characteristic class and the Swan class of an ℓ-adic sheaf (with Abbes and with Kato)

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Plan:

0. Outline.
1. Swan class and Grothendieck-Ogg-Shafarevich formula. (with Kato)
2. Characteristic class and its relation with the Swan class. (with Abbes)

Notation: F field of characteritic $p > 0$.

$\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$ etc. $(\ell \neq p)$

X variety over F.

$\mathcal{F}$ Λ-sheaf on X, or more generally, an object in a suitable derived category.

0.1. X variety, $U \subset X$ dense open smooth over F.

$\mathcal{F}$ smooth on U.

The Swan class $\mathrm{Sw}(\mathcal{F})$ is defined in $CH_0(X \setminus U)_{\mathbb{Q}}$. If X is proper,

$$\chi_c(U_{\bar F}, \mathcal{F}) \left(= \sum_{q=0}^{2d} (-1)^q \dim H^q_c(U_{\bar F}, \mathcal{F}) \right) = \mathrm{rank}\mathcal{F} \cdot \chi_c(U_{\bar F}) - \mathrm{deg}\mathrm{Sw}(\mathcal{F}).$$

0.2. The characteristic class $C(\mathcal{F}) \in H^0(X, K_X)$ is implicitly defined in SGA5. In complex geometry, it is defined by Kashiwara-Schapira without mentioning SGA5 explicitly.

$K_X = Ra^!\Lambda, a : X \to \mathrm{Spec}\ F$. If X is smooth of dimension d, $C(\mathcal{F})$ is defined in $H^{2d}(X, \Lambda(d))$. If X is proper,

$$\mathrm{Tr}\ C(\mathcal{F}) = \chi(X_{\bar F}, \mathcal{F}) \left(= \sum_{q=0}^{2d} (-1)^q \dim H^q(X_{\bar F}, \mathcal{F}) \right)$$

(Lefschetz-Verdier trace formula).

Let $j : U \to X$ be the open immersion. Then, the relation

$$C(j_!\mathcal{F}) = \mathrm{rank}\mathcal{F} \cdot C(j_!\Lambda) - \mathrm{cl}\ \mathrm{Sw}(\mathcal{F})$$

in $H^0(X, K_X)$ is verified in many cases. $\mathrm{cl} : CH_0(X) \to H^0(X, K_X)$ cycle class map.

1. $U \subset X$: smooth over F, $\mathcal{F}$ on U smooth.

For simplicity, assume $\mathcal{F}$ is trivialized by a finite Galois covering $V \to U$ of Galois group G. M : representation of G corresponding to $\mathcal{F}$.

Further assume there is a commutative diagram

$$\begin{array}{ccc} Y & \stackrel{\supset}{\longleftarrow} & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \stackrel{\supset}{\longleftarrow} & U \end{array}$$

where $f : Y \to X$ is proper, Y is smooth and V is the complement of a divisor with simple normal crossings. (In general, we consider $\mathcal{F}$ mod ℓ and use the Brauer trace and also consider alteration.)

$\sigma \in G = \mathrm{Gal}(U/V), \sigma \neq 1$.

Figure 1.

Γ_σ : graph of σ.

$(Y \times Y)' \to Y \times Y$: Blow up at $D_1 \times D_1, \ldots, D_m \times D_m$ where $D_1, \ldots, D_m$ are the irreducible components of D.

$\Delta_Y : Y \to (Y \times Y)'$: the log diagonal map.

Figure 2.

$\overline{\Gamma_\sigma}$: closure of $\Gamma_\sigma \subset V \times_U V$ in $(Y \times Y)'$.

tame ramification : no intersection.

wild ramification : non-empty intersection.

Define

$$s_{V/U}(\sigma) = -(\overline{\Gamma_\sigma}, \Delta_Y)_{(Y\times Y)'} \in CH_0(Y - V),$$

$s_{V/U}(1) = -\sum_{\sigma\neq 1} s_{V/U}(\sigma)$ and

$$\mathrm{Sw}(\mathcal{F}) = \frac{1}{|G|}\sum_{\sigma\in G} f_* s_{V/U}(\sigma)\mathrm{Tr}(\sigma : M) \in CH_0(X - U) \otimes \mathbb{Q}. \quad (1)$$

In fact, $\mathrm{Sw}(\mathcal{F})$ is defined as an element of $CH_0(E)_{\mathbb{Q}}$ where $E \subset X - U$ is the wild ramification locus.

Problem: Compute the Swan class in terms of Abbes-Saito filtration. (Partial answer in the rank 1 case.)

We have a generalization of the Grothendieck-Ogg-Shafarevich formula.

Theorem 1 *If X is proper,*

$$\chi_c(U, \mathcal{F}) = \chi_c(U) \cdot \mathrm{rank}\ \mathcal{F} - \deg\ \mathrm{Sw}(\mathcal{F}).$$

Main ingredient of proof. Lefschetz trace formula for an open variety, proved using a log product, cf. Pink-Faltings.

Variant: We may also define $\mathrm{Sw}(\mathcal{F})$ in a mixed characteristic situation. We have a relative version of Theorem 1 that gives a conductor formula with a coefficient sheaf.

2. More generally, the characteristic class is defined for a cohomological correspondence.

X variety over F. $c : C \to X \times X$ closed immersion, $p_i : C \to X$ $(i = 1, 2)$ compositions with the projections.

$\mathcal{F}$ on X, $u : p_2^*\mathcal{F} \to p_1^!\mathcal{F}$ a cohomological correspondence (direction is the inverse of that in SGA 5).

We put $\mathcal{H} = R\mathcal{H}om(pr_2^*\mathcal{F}, pr_1^!\mathcal{F})$. Then, u defines a map $\Lambda_C \to c^!\mathcal{H}$ and hence $u \in H^0_C(X \times X, \mathcal{H})$.

On the other hand, the identity $\mathcal{F} \to \mathcal{F}$ is a cohomological correspondence on the diagonal. It defines a class $1 \in H^0_\Delta(X \times X, \mathcal{H}^*)$ where $\mathcal{H}^* = R\mathcal{H}om(pr_1^*\mathcal{F}, pr_2^!\mathcal{F})$. The canonical pairing $\mathcal{H} \boxtimes \mathcal{H}^* \to K_{X\times X}$ induces the Verdier pairing $\langle\ ,\ \rangle : H^0_C(X \times X, \mathcal{H}) \otimes H^0_\Delta(X \times X, \mathcal{H}^*) \to H^0_{C\cap\Delta}(X \times X, K_X) = H^0(C \cap \Delta, K_{C\cap\Delta})$. The characteristic class $C(\mathcal{F}, C, u) \in H^0_{C\cap X}(X, K_X)$ as $\langle u, 1\rangle$. If X is proper over F, the Lefschetz-Verdier trace formula gives

$$\mathrm{Tr}(u^* : H^*(X_{\bar F}, \mathcal{F})) = \mathrm{Tr}\ C(\mathcal{F}, C, u).$$

Relations.

$$\begin{array}{ccccc} C(j_!\mathcal{F}) \in & H^0(X, K_X) & \xleftarrow{\text{cycle map}} & CH_0(X-U)_{\mathbb{Q}} \ni \mathrm{Sw}(\mathcal{F}) \\ \downarrow & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \deg} \\ \chi_c(U_{\bar F}, \mathcal{F}) \in & \mathbb{Q}_\ell & \supset & \mathbb{Q} \end{array}$$

Conjecture 2 *$U \subset X$: smooth over F, $\mathcal{F}$ smooth $\mathbb{Q}_\ell$-sheaf on U. Then, we have*

$$C(j_!\mathcal{F}) = \mathrm{rank}\ \mathcal{F} \cdot C(j_!\Lambda) - \mathrm{Sw}\mathcal{F} \tag{2}$$

in $H^0(X, K_X)$.

One can prove this under the following assumption. Let $U \subset X$ be an open subscheme. We say a locally constant sheaf $\mathcal{F}$ on U is of Kummer type, if there exists an integer m invertible on X and a finite family of Cartier divisors $D_i, i \in I$ such that $D_i \cap U = \emptyset$ satisfying the following conditions.

For each $x \in X$, there exists a basis t_i of $O(-D_i)$ on a neighborhood W of x such that $U \times_X W[T_i(i \in I)]/(T_i^m - t_i(i \in I))$ trivializes $\mathcal{F}$.

If X is regular and U is a complement of a divisor with simple normal crossings, then tamely ramified implies of Kummer type.

We say an ℓ-adic sheaf $\mathcal{F}$ is potentially of Kummer type if there exists a commutative diagram

$$\begin{array}{ccc} Y & \xleftarrow{\supset} & V \\ {\scriptstyle f}\downarrow & & \downarrow \\ X & \xleftarrow{\supset} & U \end{array}$$

where $f : Y \to X$ is proper, $V \to U$ is finite etale and $f^*\mathcal{F}$ is of Kummer type. If $\dim X \leq 2$ and U is smooth, any ℓ-adic sheaf is potentially of Kummer type.

Theorem 3 *Conjecture 2 is true if $\mathcal{F}$ is potentially of Kummer type.*

Proof is similar to that of Theorem 1.

Assume X is smooth and $D = X - U$ has simple normal crossings. If $\mathcal{F}$ is tamely ramified, we have

$$C(j_!\mathcal{F}) = \text{rank } \mathcal{F} \cdot (-1)^d c_d(\Omega^1_{X/F}(\log D)) \tag{3}$$

in $H^{2d}(X, \Lambda(d))$. In particular,

$$C(j_!\Lambda) = (-1)^d c_d(\Omega^1_{X/F}(\log D)).$$

If $\dim U = 1$ and $\text{rank}\mathcal{F} = 1$, we can prove Theorem 3 integrally.

Theorem 4 *Let X be a smooth curve and $U \subset X$ be a dense open. Let $\mathcal{F}$ be a smooth Λ-sheaf of rank 1. Then, we have*

$$C(j_!\mathcal{F}) = C(j_!\Lambda) - \text{Sw}\mathcal{F} \tag{4}$$

in $H^2(X, \Lambda(1))$.

Sketch of Proof. Assume for simplicity $U = X - \{x\}$. Put $n = \text{Sw}_x\mathcal{F} \geq 0$.

$(X \times X)^{(0)} \to X \times X$ the blow-up at the image of x by the diagonal map $X \to X \times X$.

The diagonal map $X \to X \times X$ is extended to the log diagonal map $X \to (X \times X)^{(0)}$.

We define blow-up $(X \times X)^{(i)} \to (X \times X)^{(i-1)}$ for $i = 1, 2, \ldots, n$ inductively.

$\delta^{(n)} : X \to (X \times X)^{(n)}$: immersion induced by the diagonal

E_i: exceptional divisor.

$(U \times U)^{(n)}$: complement in $(X \times X)^{(n)}$ of the union of the proper transforms of $X \times x$, $x \times X$, and the exceptional divisors E_i for $i = 0, 1 \ldots, n - 1$.

In the commutative diagram

$$\begin{array}{ccc} (X \times X)^{(n)} & \xleftarrow{j^{(n)}} & (U \times U)^{(n)} \\ {\scriptstyle f^{(n)}}\downarrow & & \uparrow{\scriptstyle k^{(n)}} \\ X \times X & \xleftarrow{j} & U \times U, \end{array}$$

the left vertical arrow is the composition of blow-ups and the others are open immersions. The blow-up $(X \times X)^{(n)} \to X \times X$ killes the ramification of $\mathcal{F}$ as in the following Proposition.

Proposition 5 *Put $\mathcal{H} = \mathcal{H}om(pr_2^*\mathcal{F}, pr_1^*\mathcal{F})$. Then, the Λ-sheaf $\mathcal{H}^{(n)} = k_*^{(n)}\mathcal{H}$ is a smooth Λ-sheaf of rank 1 on $(U \times U)^{(n)}$.*

Proof of Proposition. Identify $H^1(K_x, \mathbb{Z}/p^m\mathbb{Z}) = W_m(K_x)/F - 1$ and consider the filtration of Brylinski inducing the filtration by ramification.

Proof of Theorem. The characteristic class $C(j_!\mathcal{F})$ is computed by the map $\Lambda_X \to \delta^{(n)!}\mathcal{H}$ and hence is equal to the intersection product $(X, X)_{(X \times X)^{(n)}}$.

Application of Theorem. Proof of the GOS formula without using the Weil formula. (Brauer induction).