Characteristic cycles of an étale sheaf.

X smooth / $k = \bar{k}$ char $p > 0$ $d = \dim X$.

$\mathcal{F}$ ℓ-adic sheaf on X $\ell \neq p$ (More generally constructible complex)

$Ch\,\mathcal{F}$ conic ~~d~~-cycle on the cotangent bundle T^*X.

(linear combination of irreducible closed subsets of dim d stable under multiplication $\mathbb{Z}[\frac{1}{p}](\mathbb{Z})$-coeffs

Examples, Properties, Motivations, Results, Construction --

Examples. $X \supset U = X - D$. D div. w. SNC. $j: U \to X$

1. $\mathcal{F}$ smooth on U tamely ramified along D

$$Ch\, j_!\mathcal{F} = (-1)^d \operatorname{rk}\mathcal{F} \sum [T^*_{X_I} X].$$ $T^*_{X_I}X$ conormal bdle

$$X_I = \bigcap_{i\in I} D_i \qquad D = \bigcup D_i$$

2. $d=1$ $$Ch\, j_!\mathcal{F} = -(\underbrace{\operatorname{rk}\mathcal{F}\cdot[T^*_X X]}_{\text{0-section}} + \sum_{x\in D} \underbrace{\dim\operatorname{tot}_x \mathcal{F}}_{\text{cond + Sw.}}\cdot[\underbrace{T^*_x X}_{\text{fiber}}])$$

3. $d=2$. $r=1$. $X = \mathbb{A}^2$ $D\ (x=0)$ $\mathcal{F}$ $t^p - t = \begin{cases} \frac{1}{x^n} & p \nmid n \quad \mathrm{I} \\ \frac{y}{x^n} & p \mid n \quad \mathrm{II}\end{cases}$

I $\Rightarrow$ $Ch\, j_!\mathcal{F} = [T^*_X X] + (n+1)[T^*_D X]$

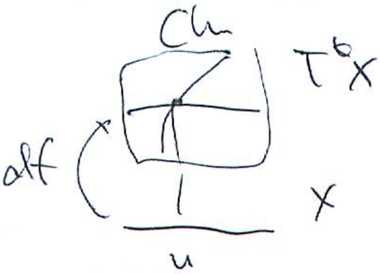
II $\Rightarrow$ $\qquad = [T^*_X X] + n\cdot\langle dy \text{ on } D\rangle$ $p \neq 2$

Properties 1 Index formula. X proper

$$\chi(X,\mathcal{F}) = (Ch\,\mathcal{F}, T^*_X X)_{T^*X}.$$

1. Classical. 2. G-O-S.

2. Milnor formula. $f: X \to C$. $u \in X$ isolated char pt
$v = f(u) \in C$



df dt

proper intersection $\Rightarrow$ intersection multiplicity

$(Ch\,\mathcal{F}, df)_{T^*X,u}$.

$-\dim\operatorname{tot}_v \phi_u(\mathcal{F}, f) = \ \vdots$

vanishing cycles

$\mathcal{F}$ constant $\Rightarrow$ Deligne. SGA 7. tame. E. Yang.

Motivation Analogy

ℓ-adic sheaf in char p — $\mathcal{D}$-mod / cx. mfd.

wild ramification $\leftarrow$ irregular singularity.

? $\leftarrow$ Char cycle / Micro local analysis

$\mathrm{gr}\,M$ $\mathrm{gr}\,\mathcal{D} = \mathcal{O}_{T^*X}$.

Results

Theorem 1. Existence of singular support (= supp of Ch)

$\Rightarrow$ Existence of Char cycle

~~[illegible]~~ characterised by Milnor formula

satisfies Index formula

Rmq S.S. is characterized by local acyclicity

Theorem 2. Outside codim $\geqq 2$. Existence of S.S

Cor X dim $2 \Rightarrow$ Existence of S.S

Main ingredients in construction. (Pf of Thm I.)

(Thm 2. Ramification theory + (Semi) continuity of Swan conductor by Deligne-Laumon)

(i) local version of Radon transform.

(2) Stability of the tot. dim of van. cycles.

(1) X quasi-projective. $X \subset \mathbb{P}$ L pencil

$X \leftarrow X_L \xrightarrow{p_L} L$ Milnor formula for p_L. first step.
blow up

universal family $\mathbb{P}^\vee = \{H \subset \mathbb{P}$ hyp plane$\}$ $\mathbb{H} = \{(x, H) \in \mathbb{P} \times \mathbb{P}^\vee \mid x \in H\}$

$$\begin{tikzcd} X \times_{\mathbb{P}} \mathbb{H} = \{(x,H) \in X \times \mathbb{P}^\vee \mid x \in H\} \arrow[r, "p"] \arrow[d, "q"'] \arrow[dr, phantom, "\square"] & \mathbb{P}^\vee \\ X & \\ X_L \arrow[uu, shift left=20] \arrow[r, "p_L"'] & L \arrow[uu] \end{tikzcd}$$

universal family of hyp plane sections

classically curves

Radon transform. $Rp_* q^* \mathcal{F}$.

local ——— $R\Phi_p\, q^* \mathcal{F}$ nearby cycles functor over general basis

Define $Ch_E \mathcal{F}$ so that M formula holds for p_L.

(Semi) continuity of Swan conductor.