THE CHARACTERISTIC CYCLE AND THE SINGULAR SUPPORT OF AN ETALÉ SHEAF

T. SAITO — NOTES BY LARS KINDLER

Abstract. These are the notes for a series of lectures given by T. Saito at Freie Universität Berlin in July 2015. The typist takes full responsibility for all mistakes and inaccuracies.

Singular support due to A. Beilinson, Characteristic cycle due to T. Saito.

1. Lecture 1

1.1. Introduction.

- $k$ a field of characteristic $p > 0$. Mostly perfect or even algebraically closed.
- $X$ a smooth $k$-scheme, $n = \dim X$. Let $\Lambda$ be a finite extension of $\mathbb{F}_\ell$, $\ell \neq p$.
- $\mathcal{F}$ a constructible complex of $\Lambda$-modules.
- We can take cohomology sheaves $\mathcal{H}^q(\mathcal{F})$; they are constructible and $= 0$ except for finitely many $q$.
- $T^*X$ the cotangent bundle of $X$ associated to $\Omega^1_X$, which is a vector bundle of rank $n$. Thus $T^*X$ has dimension $2n$.
- $C \subseteq T^*X$ a closed conical subset, where conical means: stable under the action of $\mathbb{G}_m$, which naturally acts by multiplication on the vector bundle $T^*X$.
- $T^*X = \text{Spec } S^*(\Omega^1_X)^\vee$ and $C$ is defined by some ideal of $S^*(\Omega^1_X)^\vee$. From this perspective conical means that $C$ is defined by a graded ideal.
- The Singular support of $\mathcal{F}$ is denoted $SS(\mathcal{F}) = C \subseteq T^*X$. It is a closed conical subset of $T^*X$. Moreover, we can write it as union of irreducible components

$$C = \bigcup C_a$$

where $C_a$ is an irreducible component of $\dim C_a = \dim X^1$.
- Today we explain $SS(\mathcal{F})$. Later the characteristic cycle $\text{Char}(\mathcal{F})$.
- $\text{Char}(\mathcal{F}) = \sum_a m_a[C_a]$ with $m_a \in \mathbb{Z}[1/p]$, but it is expected that $m_a \in \mathbb{Z}$.
- The expectation is that the properties of $\mathcal{F}$ are well understood by using $SS(\mathcal{F})$ and $\text{Char}(\mathcal{F})$. Slogan: To understand $\mathcal{F}$ on $X$, we study $SS(\mathcal{F})$ and $\text{Char}(\mathcal{F})$ on $T^*X$. This is analogous to microlocal analysis of $\mathcal{D}_X$-modules on complex manifolds $X$, due to Sato, Kashiwara, etc.

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$SS(\mathcal{F})$ is an invariant of the complex $\mathcal{F}$, and $SS(\mathcal{F}) \subseteq \bigcup q SS(\mathcal{H}^q(\mathcal{F}))$. 

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Example 1.1. \( X \) a curve, i.e., \( n = 1 \). Let \( D \) be a divisor on \( X \) and \( j : U := X \setminus D \hookrightarrow X \) the associated open immersion. Let \( \mathcal{F} := j_! \mathcal{G} \), where \( \mathcal{G} \neq 0 \) is a locally constant sheaf on \( U \). In this case the irreducible components are:

\[
T^*X \supset SS(\mathcal{F}) = \bigcup_{x \in D} \mathcal{T}_x^*X
\]

In fact any conical closed subset of \( T^*X \) has this shape.

In this example,

\[
\text{Char}(\mathcal{F}) = (-1)^{\left( \text{rank} \mathcal{G} \cdot [T^*_X X] + \sum_{x \in D} \text{dimtot}_x \mathcal{F} \cdot [T^*_x X] \right)}
\]

where \( \text{dimtot}_x = \text{dim} + \text{Sw}_x \), with \( \text{Sw}_x \in \mathbb{Z} \) the Swan conductor at \( x \), which is a measure of wild ramification.

On the other hand, if \( \mathcal{F} = j^* \mathcal{G} \), then replace \( \text{dimtot}_x \) by Artin conductor of \( \mathcal{G} \). If \( \mathcal{F} = Rj^* \mathcal{G} \), then \( \text{Char}(j_! \mathcal{G}) = \text{Char}(Rj^* \mathcal{G}) \).

- If \( X \) projective and \( k \) algebraically closed, then \( \chi(X, \mathcal{F}) = \left( \text{Char}(\mathcal{F}) \cdot [T^*_X X] \right) T^*_X X \)

interception number. Have this formula in general, but in the 1-dimensional example from above, this is a reformulation of Grothendieck-Ogg-Shafarevich’s formula.

- Why is there a sign \((-1)^n\)? If \( \mathcal{F} \) is a perverse sheaf (complex), then the coefficients of \( \text{Char}(\mathcal{F}) \) are \( \geq 0 \). In the example above, \( \mathcal{F}[1] \) is perverse. In general, \( \text{Char}(\mathcal{F}[n]) = (-1)^n \text{Char}(\mathcal{F}) \).

1.2. Singular Support (after Beilinson). Want to formulate relations between \( C \subseteq T^*X \) and \( \mathcal{F} \) on \( X \), where \( C \) is a conical subset and \( \mathcal{F} \) a constructible complex on the smooth scheme \( X \) of dimension \( n \).

1.2.1. \( C \)-transversality. Want two definitions of \( C \)-transversality: One for morphisms \( h : W \to X \) into \( X \) and one for morphisms \( f : X \to Y \) from \( X \). Here \( W, Y \) are both smooth \( k \)-schemes, of arbitrary dimension.

Definition 1.2. We say \( f : X \to Y \) is \( C \)-transversal if \( df^{-1}(C) \subseteq X \times_Y T^*_Y Y \).

\[
\begin{array}{ccc}
X \times_Y T^*_Y Y & \xrightarrow{df} & T^*_X X \\
\downarrow & & \downarrow \\
df^{-1}(C) & \rightarrow & C
\end{array}
\]

Example 1.3. (a) \( C = T^*_X X \) the zero-section. Then \( C \)-transversal means that \( df \) is injective, i.e., that \( f \) is smooth.

(b) \( Y = \text{Spec} k \) is a point. Then \( f : Y \to \text{Spec} k \) is \( C \)-transversal for any \( C \).

\( ^2 \)This relies on a deep theorem of Gabber: If \( X \to S \), \( S \) a trait \( \mathcal{F} \) perverse sheaf on \( X \) then \( \Phi \mathcal{F}[-1] \) is perverse.
Definition 1.4. We say \( h : W \to X \) is \( C \)-transversal if
\[
h^*C \cap dh^{-1}(T_W^*W) \subseteq W \times_X T_X^*X
\]
where we use the diagram
\[
\begin{array}{ccc}
h^*C = W \times_X C & \to & T_W^*W \\
\downarrow & & \downarrow \\
W \times_X T_X^*X & \xrightarrow{dh} & T^*W
\end{array}
\]
Moreover, define \( h^0C = dh(h^*C) \subseteq T^*W \). Then \( C \)-transversality implies that \( h^0C \) is closed, and
\[
\begin{array}{ccc}
h^*C & \xrightarrow{\text{finite}} & h^0C \\
\downarrow & & \downarrow \\
W \times_X T_X^*X & \xrightarrow{dh} & T^*W
\end{array}
\]
The terminology used to be non-characteristic (Kashiwara-Schapira).

Example 1.5. (a) If \( h \) is smooth, then \( h \) is \( C \)-transversal for any \( C \), because then \( dh \) is injective and \( h^*C = h^0C \).

Remark 1.6. Being \( C \)-transversal is an open condition on the source of the morphism \( f : X \to Y \).

Need one more definition.

Definition 1.7. Given \( f : W \to Y \) and \( h : W \to X \), we say that the pair \((h, f)\) is \( C \)-transversal if \( h \) is \( C \)-transversal and \( f : W \to Y \) is \( h^0C \) transversal.

Exercise 1.8. (a) Given \( f : W \to Y \) and \( h : W \to X \), then \((h, f)\) is \( C \)-transversal if and only if
\[
(h^*C \times_W (W \times_Y T^*Y)) \cap (\text{inv. image of } T_W^*W) \subseteq 0\text{-section}
\]
where we use the diagram
\[
\begin{array}{ccc}
h^*C \times_W (W \times_Y T^*Y) & \to & T_W^*W \\
\downarrow & & \downarrow \\
(W \times_X T_X^*X) \times_W (W \times_Y T^*Y) & \to & T^*W
\end{array}
\]
(b) If \( f : W \to Y \) is smooth,
\[
\begin{array}{ccc}
h^*C & \to & W \times_Y T^*Y \\
\downarrow & & \downarrow \text{injective because of smoothness} \\
W \times_X T_X^*X & \to & T^*W
\end{array}
\]
Then \((f, h)\) is \( C \)-transversal iff
\[
h^*C \cap (\text{inv. image of } W \times_Y T^*Y) \subseteq 0\text{-section}
\]
1.3. **Local acyclicity.** Given $f : X \to Y$ we have the notion of the *Milnor fiber*. Let $x$ be a geometric point of $X$ and $y := f(x)$, a geometric point of $Y$. Let $Y_y$ be the strict localization of $Y$ at $y$, so $Y_y = \text{Spec} \mathcal{O}_{Y,y}$. Let $z$ be a geometric point of $Y_y$. Notation: $x \mapsto y \leftarrow z$. The Milnor fiber is $X_x \times_{Y_y} z$.

This is not interesting if $z$ maps to $y$, but, e.g., if $z$ maps to the generic point of $Y_y$.

**Definition 1.9.** Let $F$ on $X$ be as in the beginning. We say $f : X \to Y$ is **locally acyclic relatively to $F$** if for all situations $x \mapsto y \leftarrow z$ as above the canonical restriction morphism

$$F_x = R\Gamma(X_x, F) \to R\Gamma(X_x \times_{Y_y} z, F)$$

is an isomorphism.

**Example 1.10.** Let $Y$ be a curve, and $y$ a geometric point over a closed point. Then $Y_y$ only has two points; let $z$ be a geometric point above the generic point of $Y_y$. In this situation we have a distinguished triangle

$$\to F_x \to R\Gamma(X_x \times_{Y_y} z, F) \to \text{vanishing cycles}.$$ 

So local acyclicity in this situation means that there are no nonzero vanishing cycles.

**Definition 1.11.** We say that $f : X \to Y$ is **universally locally acyclic relatively to $F$** if for every $g : Y' \to Y$, $X \times_Y Y' \to Y'$ is locally acyclic relatively to the pullback of $F$.

Enough to just take every smooth $g : Y' \to Y$.

### 2. Lecture 2

Here are some facts about local acyclicity.

**Facts.**

(a) **Local acyclicity of smooth morphisms**, SGA 4) If $f : X \to Y$ is smooth and $F$ on $X$ locally constant (i.e. $H^q(F)$ is locally constant for all $q$), then $f$ is locally acyclic relatively to $F$.

(b) **Generic local acyclicity**, SGA 41/2) Let $F$ be arbitrary and $f : X \to Y$. There exists a dense open subset $V \subseteq Y$, such that $f_V : X \times_Y V \to V$ is universally locally acyclic relatively to $F|_{X \times_Y V}$.

(c) $f : X \to Y$, $g : Y \to Z$ and $F$ on $X$. Suppose that $f$ is (universally) locally acyclic relative to $F$ and $g$ smooth. Then the composition $gf$ is (universally) locally acyclic with respect to $F$. (This is a consequence of (a)).

(d) $f, g, F$ as in (c). Suppose that $gf$ is (universally) locally acyclic relative to $F$ and that $f$ is proper. Then $g$ is (universally) locally acyclic with respect to $Rf_* F$. (This follows from the proper base change theorem).

(e) $F$ is locally constant if and only if $\text{id}_X : X \to X$ is locally acyclic relatively to $F$. In fact, $\text{id}_X$ is locally acyclic with respect to $F$ iff for every specialization $x \leftarrow y$, $F_x \xrightarrow{\sim} F_y$ iff (exercise!) $F$ is locally constant.
2.1. **Micro support.** We combine the notions introduced above. Let $C \subseteq T^*X$ be a closed conical subset and $\mathcal{F}$ a constructible complex of $\Lambda$-modules on $X$.

**Definition 2.1.**

(a) We say $\mathcal{F}$ is **micro supported** on $C$ if for every $C$-transversal pair

$$X \xleftarrow{h} W \xrightarrow{f} Y,$$

the map $f : W \to Y$ is universally locally acyclic relative to $h^* \mathcal{F}$.

(b) We say that $\mathcal{F}$ is **weakly micro supported** on $C$ if the above holds true for pairs

$$X \xleftarrow{h} W \xrightarrow{f} Y$$

where $h$ is an open immersion and $Y$ is a curve ($= \mathbb{A}^1_k$).

**Example 2.2.** $\mathcal{F}$ is locally constant $\iff$ $\mathcal{F}$ is micro supported on the 0-section $C = T^*_X X$.

$\Rightarrow$ is Fact (a).

$\Leftarrow$ is Fact (e): $\text{id}_X$ is transversal to $C = T^*_X X$.

What’s the difference between micro supported and weakly micro supported?

**Lemma 2.3.** Suppose $\mathcal{F}$ is weakly micro supported on $C$ and $C'$. Then $\mathcal{F}$ is weakly micro supported on $C \cap C'$.

**Remark 2.4.**

(a) Suppose $\mathcal{F}$ is (weakly) micro supported on $C$ and let $C'$ be conical closed, such that $C \subseteq C'$. Then $\mathcal{F}$ is (weakly) micro supported on $C'$. The question is: How small can we make $C$?

(b) The statement of the lemma also holds true for micro supported instead of weakly micro supported, but to see this we first have to prove the main theorem.

(c) If $C$ is a minimal (with respect to $\subseteq$) among the conical closed subsets of $T^*X$ on which $\mathcal{F}$ is micro supported, then we say that $\mathcal{F}$ is **tightly supported** on $C$ (a priori there could be many minimal $C$).

(d) On the other hand, for the notion of weakly micro supported, the lemma shows that there is a unique minimal $C$ on which $\mathcal{F}$ is weakly micro supported.

**Definition 2.5.** The smallest conical closed $C \subseteq T^*X$ on which $\mathcal{F}$ is weakly micro supported is called the **singular support** of $\mathcal{F}$ and denoted $SS(\mathcal{F})$.

**Theorem 2.6 (Beilinson).** Every irreducible component of $SS(\mathcal{F})$ is of dimension $\dim X$ and $\mathcal{F}$ is micro supported on $SS(\mathcal{F})$.

This follows from two intermediate theorems.

**Theorem A (Beilinson, Thm. 1.2).** There exists $C \subseteq T^*X$ such that $\mathcal{F}$ is micro supported on $C$ and $\dim C \leq n = \dim X$.

**Theorem B (Beilinson, Thm. 1.3).** Assume that $k$ is perfect and that $\mathcal{F}$ is tightly micro supported on $C$. Then every irreducible component of $C$ is of dimension $n = \dim X$ and $C = SS(\mathcal{F})$.

- Theorem 2.6 follows from Theorem A and Theorem B.
• To prove Theorems A and B we reduce to $\mathbb{P}^n$. To do this one roughly proceeds like this: For Theorem A, take $X \to \mathbb{P}^n$ étale. For Theorem B, take $X \xrightarrow{i} U \xrightarrow{j} \mathbb{P}^n$, $i =$ closed, $j =$ open.

• From now on we assume $X = \mathbb{P}^n$. Here an important tool will be the Radon Transform.

2.2. Radon Transform. Standard reference is Brylinski (Asterisque), and SGA7, Exp. XVII. Let $V$ be an $(n + 1)$-dimensional $k$-vector space and denote by

$$P := \mathbb{P}(V) = \{\text{lines in } V\}$$

the associated projective space. The dual projective space is

$$P^\vee = \mathbb{P}(V^\vee) = \{\text{hyperplanes in } V\} = \{\text{hyperplanes in } P\}.$$

Let $Q \subseteq P \times P^\vee$ be the universal family of hyperplanes, i.e.,

$$Q = \{(x, x^\vee) \in P \times P^\vee | x \in x^\vee\}.$$

We have two projections:

$$Q \xrightarrow{p^\vee} P^\vee \xrightarrow{p} P$$

**Definition 2.7.**

• For $\mathcal{F}$ on $P$, the Radon transform of $\mathcal{F}$ is

$$R(\mathcal{F}) := Rp^\vee_! p^* \mathcal{F}[n - 1].$$

• Given $\mathcal{G}$ on $P^\vee$, we get the inverse (dual) Radon transform

$$R^\vee(\mathcal{G}) := Rp_* p^\vee_* \mathcal{G}[n - 1]$$

These two constructions are almost inverse to each other (i.e., up to a geometrically constant object, but we will not make this precise).

We recall the Legendre transform on $C$. We have the identification

$$Q = \mathbb{P}(T^* P) = (T^* P - T^*_0 P)/\mathbb{G}_m.$$

This identification works as follows: From

$$0 \longrightarrow \Omega^1_P \longrightarrow V^\vee \otimes \mathcal{O}_P(-1) \longrightarrow \mathcal{O}_P \longrightarrow 0$$

we obtain

$$\mathbb{P}(T^* P) \subseteq \mathbb{P}(V^\vee \otimes \mathcal{O}_P(-1)) = \mathbb{P}(V^\vee) \times P = P^\vee \times P.$$

Similarly, we also have an identification $Q = \mathbb{P}(T^* P)$. Let $C \subseteq \mathbb{P}(T^* P)$ be a conical closed subset and consider its projectivization $\mathbb{P}(C)$. We get the
and $C^\vee \subseteq T^*\mathbb{P}^\vee$ is closed and conical and called the Legendre transform of $C$.

2.3. Reformulation of Theorems A and B.

**Definition 2.8.** Let $f : X \to Y$ be a morphism and $\mathcal{F}$ a constructible complex on $X$. Define $E_f(\mathcal{F}) \subseteq X$ to be the closed subset such that its complement $U$ is the largest open subscheme where $f_U : U \to Y$ is universally locally acyclic relative to $\mathcal{F}|_U$ ($U = \emptyset$ possible).

**Theorem A’** (Thm. 1.4, equivalent to Theorem A). For $\mathcal{G}$ on $\mathbb{P}$, $E_{p^\vee}(p^*\mathcal{G})$ is of dimension $\leq n - 1$.

Theorem A’ is equivalent to Theorem A. For $\Leftarrow$, one uses that if $\mathcal{G}$ is micro supported on $C$ with $\dim C \leq n$, then $E_{p^\vee}(p^*\mathcal{G}) \subseteq \mathbb{P}(C)$, with $\dim \mathbb{P}(C) \leq n - 1$.

Let $d \geq 1$ and let

$$i_d : \mathbb{P} \to \mathbb{P} = \mathbb{P}(\Gamma(\mathbb{P}, \mathcal{O}(d))^\vee)$$

be the $d$-th Veronese embedding. We get a diagram

$$\begin{array}{ccc}
\mathbb{P}(C) & \xrightarrow{i_d} & \mathbb{P}(T^*\mathbb{P}) \\
\downarrow & & \downarrow \\
\mathbb{P}(C^\vee) & \xrightarrow{i_d} & \mathbb{P}(T^*\mathbb{P}^\vee)
\end{array}$$

**Theorem B’** (Thm. 1.6, implies Theorem B). Fix $\mathcal{G}$ on $\mathbb{P}$. Assume $d \geq 3$ and let $D \subseteq \mathbb{P}^\vee$ be the complement of the largest open subset $U \subseteq \mathbb{P}$ where $\tilde{R}(i_d, \mathcal{G})$ is locally constant. (Here $\tilde{R}$ is the Radon transform on $\mathbb{P}$.)

(a) $D$ is a divisor, i.e., purely of codimension 1.

(b) For each irreducible component $D_a$ of $D$, there is a unique irreducible closed conical subset $C_a \subseteq T^*\mathbb{P}$ such that $D_a = \tilde{p}^\vee(\mathbb{P}(i_0C_a))$ and $\dim C_a = \dim X$. For the definition of $i_0C_a$, see below. The surjection

$$\tilde{p}^\vee : \mathbb{P}(i_0C_a) \to D_a$$

is generically radicial, i.e., the associated extension of function fields is purely inseparable.

(c) $C = \bigcup C_a \subseteq T^*\mathbb{P}$ is $SS(\mathcal{G})$. 

How to define $i_0C$? If $i : X \hookrightarrow Y$ is a closed immersion with $X, Y$ smooth, and $C \subseteq T^*X$, then $i_0C \subseteq T^*Y$ is defined using the following diagram:

$$
\begin{array}{ccc}
C & \hookrightarrow & X \\
\downarrow & & \downarrow \\
T^*X & \hookrightarrow & X \times_Y T^*Y & \twoheadrightarrow & T^*Y
\end{array}
$$

Then $i_0C$ is defined to be the image in $T^*Y$ of the pullback of $C$ along $X \times_Y T^*Y \to T^*X$. Thus in the situation of Theorem B', (b), $P(i_0C_a) \subseteq P(T^*\tilde{P}) = \tilde{Q}$.

**Remark 2.9.** The fact that $\tilde{p}^\vee : P(i_0C_a) \to D_a$ is generically purely inseparable gives rise to the problem that the coefficients of $\text{Char}(F)$ can (at the moment) only be shown to lie in $\mathbb{Z}[1/p]$ (although they are expected to be integers).

### 3. Lecture 3

**3.1. The Characteristic Cycle.**

- $k$ is a field of characteristic $p > 0$, perfect or even algebraically closed.
- $X/k$ is smooth, $n = \dim X$.
- $F$ a constructible complex on $X$.
- Last time we defined $C = SS(F) \subseteq T^*X$, a closed conical subset, $C = \bigcup_a C_a$, the $C_a$ the irreducible components, $\dim C_a = n$.
- Recall that $F$ is micro supported on $C$ if for every pair of maps $X \xrightarrow{h} W \xrightarrow{f} Y$, where $h$ is $C$-transversal and $f$ is $h^0C$-transversal, $f : W \to Y$ is universally locally acyclic relative to $h^*F$.
- The characteristic cycle will have the form $\text{Char}(F) = \sum_a m_a[C_a]$, $m_a \in \mathbb{Z}[1/p]$.

**3.1.1. Definition of characteristic cycle — Milnor formula.** We slightly generalize the notion of weakly micro supported: Instead of putting a condition on all pairs $X \xleftarrow{j} U \xrightarrow{f} Y$ with $Y$ a curve and $j$ open, we just require $j$ to be étale and $Y$ to be a curve.

**Definition 3.1.** For a fixed closed conical subset $C \subseteq T^*X$, we say that a closed point $u \in U$ is an isolated characteristic point with respect to $C$, if $X \leftarrow U \setminus \{u\} \to Y$ is $C$-transversal.

**Example 3.2.** Let $X \xleftarrow{j} U \xrightarrow{f} Y$ be such that $Y$ is a curve and $j$ is étale. Let $C = T^*_X X$. Then $u$ is an isolated characteristic point if and only if $u$ is an isolated singular point of $f : U \to Y$.

Now assume that $C = SS(F)$. Let $u$ be an isolated characteristic point. We define two invariants. On the “$F$-side”: $f : U \to Y$ is universally locally acyclic relative to $j^*F$ outside $u$. If $k = \bar{k}$ and $v = f(u) \in Y$ (closed point), write $Y_v = \text{Spec}(O_{Y,v}^{sh})$, which is the spectrum of a strictly henselian discrete valuation ring. Let $\tilde{\eta}$ denote a generic geometric point of $Y_v$. 
Recall the definition of universally locally acyclic relative to \( j^* \mathcal{F} \): There is a distinguished triangle
\[
\mathcal{F}_u \to R\Gamma(X_u \times_{X_v} \bar{Y}) \to \Phi_u(j^* \mathcal{F}, f) \to
\]
and locally acyclic means that the first arrow is an isomorphism. \( \Phi_u(j^* \mathcal{F}, f) \) is the stalk of the complex (of \( \Lambda \)-modules) of vanishing cycles. We may assume without loss of generality that \( \Lambda \) is a finite field extension of \( \mathbb{F}_\ell^3 \). Its \( q \)-th cohomology
\[
\Phi^q_u(j^* \mathcal{F}, u)
\]
is a \( \Lambda \)-vector space of finite dimension and which is zero except for finitely many \( q \). It carries a natural continuous action of \( \text{Gal}(\overline{K_v}/K_v) \), where \( K_v = \text{Frac}(\mathcal{O}_{Y,u}) \).

Define
\[
\dim_{\text{tot}} \Phi := \sum_q (-1)^q \dim_{\text{tot}} \Phi^q = \sum_q (-1)^q (\dim(\Phi^q) + \text{Sw}(\Phi^q))
\]
which is an integer by the theorem of Hasse-Arf.

On the "C-side": \( j^* C \subseteq T^* U = U \times_X T^* X \). After shrinking \( Y \) we obtain from \( f : U \to Y \) a map
\[
f : U \to Y \to \mathbb{A}^1_k = \text{Spec} k[t].
\]
This defines \( df := f^* dt \), which is a section of the projection \( T^* U = U \times_X T^* X \to U \). The assumption that \( u \) is an isolated characteristic point means that the intersection of \( j^* C \) and \( df(U) \) consists of at most one isolated closed point (which is essentially independent of the choice of \( t \), as \( C \) is a conic subset). We can take the intersection, because \( \dim j_* C = n \), \( \dim df(U) = n \) and \( \dim T^* U = 2n \). It follows that the intersection number
\[
(j^* \sum_a m_a[C_a], df)_{T^* U, u}
\]
is defined.

**Theorem 3.3 (Milnor Formula).** There exists a unique \( \mathbb{Z}[1/p] \)-linear combination
\[
\text{Char}(\mathcal{F}) = \sum_a m_a C_a
\]
of irreducible components \( C_a \) of \( C = SS(\mathcal{F}) = \bigcup_a C_a \) such that for every pair \( X \xleftarrow{f} U \xrightarrow{f} Y \) as above, with isolated characteristic point \( u \in U \), we have
\[
- \dim_{\text{tot}} \Phi_u(j^* \mathcal{F}, f) = (j^* \text{Char}(\mathcal{F}), df)_{T^* U, u} \tag{\star}
\]

**Example 3.4.**
(a) \( \mathcal{F} = \Lambda \). Then the right hand side of (\star) is length \( \Omega^n_{U/Y,u} \) (Deligne, SGA7 Exp. XVI), and \( \text{Char}(\Lambda) = (-1)^n T^n_X X \).
(b) \( X = \mathbb{A}^2 \),
\[
j : V = \mathbb{A}^2 - D \hookrightarrow \mathbb{A}^2 = \text{Spec } k[x, y],
\]
where \( D \) the \( x \)-axis of \( \mathbb{A}^2 \). Consider the Artin-Schreier equation
\[
\nu^p - t = \frac{u}{y}, \quad p \neq 2, \quad p \mid d.
\]
It defines a cyclic covering \( W \to V \) of degree
\[^3\]To compute the Swan conductor or coefficients of the characteristic cycle we can work on the residue field of \( \Lambda \).
p. Fix a character \( \text{Gal}(W/V) \to \Lambda^\times \), which corresponds to a locally constant sheaf \( G \) of rank 1 on \( V \). Define \( \mathcal{F} = j_! G \). Then
\[
SS(\mathcal{F}) = T_X^* X \cup \langle dy/D \rangle \subseteq T^* X.
\]
and
\[
\text{Char}(F) = [T_X^* X] + d[\langle dy/D \rangle].
\]

Idea of the proof of Theorem 3.3: Follow Deligne! We use a local version of the Radon transform, using vanishing cycles over a general base scheme (Deligne, Laumon, Illusie, Orgogozo). We need to define the multiplicities \( m_a \). In the notations from last lecture, we defined divisors \( D_a \subseteq \mathbb{P}^\prime \), and cut with a general pencil \( L \). This directly gives the coefficients \( m_a \) locally.  

Main point: Show that they are independent of all choices. To this end we use ‘stability of vanishing cycles’. Given

\[
\begin{array}{c}
U \\
\downarrow f \quad \downarrow g \\
Y \quad Y' \\
\end{array}
\]

If \( g, f \) are ‘sufficiently close’, continuity of the Swan conductor (Deligne, Laumon) implies that in this situation \( \dim \text{tot}(-, f) = \dim \text{tot}(-, g) \),

3.1.2. **Functoriality of \( \text{Char}(\mathcal{F}) \) — Index formula.** We would like to have functoriality for maps \( h : W \to X \), and \( f : X \to Y \).

**Definition 3.5.** \( h : W \to X \) is strongly \( C \)-transversal if it is \( C \)-transversal and if \( h^* C := W \times_X C \subseteq W \times_X T^* X \) is equidimensional of dimension \( \dim W \), i.e., every irreducible component of \( h^* C \) has dimension \( \dim W \).

Write \( C = \bigcup_a C_a \), and assume that \( h \) is strongly \( C \)-transversal. Then we can define
\[
\begin{aligned}
\sum_a m_a [C_a] := (-1)^{\dim W - \dim X} \left( \sum_a m_a h^0([C_a]) \right),
\end{aligned}
\]
because we have the diagram

\[
\begin{array}{ccc}
\scriptstyle{C_a} & \downarrow & \scriptstyle{h^* C_a} \\
\scriptstyle{\dim X} & \searrow & \scriptstyle{\dim W} \\
\downarrow & \downarrow & \downarrow \\
T^* X & \leftarrow W \times X T^* X & \to T^* W.
\end{array}
\]

**Theorem 3.6.** If \( h : W \to X \) is strongly \( C \)-transversal for \( C = SS(\mathcal{F}) \), then
\[
\text{Char}(h^* \mathcal{F}) = h^!(\text{Char}(\mathcal{F})).
\]

---

\(^{4}\)The denominators come from the fact that \( p(i_0 C_a) \to D_a \) is purely inseparable, but it is expected that the denominators always cancel
Idea of the proof: We can assume that $W \subseteq X$ is a divisor of $X$, $\dim X = 2$. Then use a global argument originally due to Deligne (and resolution of singularities in dimension 2) and some ramification theory.

**Lemma 3.7.** If $f : X \to Y$ is proper and $C$-transversal for $C = SS(F)$, then $Rf_*F$ is locally constant, i.e., every $R^qf_*F$ is locally constant.

**Theorem 3.8 (Index formula).** Assume that $X$ is projective and $k = \bar{k}$.

Then

$$\chi(X,F) = (\text{Char}(F), T^*_X X)_{T*X}$$

Idea of proof: Induction on $\dim X$. Let $X \leftarrow X' \xrightarrow{p} L$ be a pencil. We compute

$$\chi(X,F) = \chi(X',F') - \chi(Z,F|_Z)$$

where $Z$ is the center of the blow-up $X' \to X$. Use induction hypothesis and Theorem 3.6 to compute $\chi(Z,F|_Z)$.

Using Grothendieck-Ogg-Shafarevich formula, we compute

$$\chi(X',F') = \chi(L) \text{rank}(Rp_*F') - \sum \text{dim tot x } \Phi$$

where $\text{rank}(Rp_*F') = \chi(Y,F|_Y)$ where $Y$ is a generic hyperplane section. Then use induction hypothesis plus Theorem 3.6.

4. **Lecture 4**

4.1. **Equivalent characterization of singular support.** In this section, we define the notion of $F$-transversality. The following table shows how it fits into the story:

<table>
<thead>
<tr>
<th>$\mathcal{F}$-side</th>
<th>quantitative/Char($\mathcal{F}$)</th>
<th>qualitative/SS($\mathcal{F}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$-side</td>
<td>Euler number, Milnor formula</td>
<td>locally acyclic</td>
</tr>
<tr>
<td></td>
<td>Intersection number</td>
<td>$f : X \to Y$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Definition 4.1.** $h : W \to X$ a morphism of smooth schemes over $k$, $\mathcal{F}$ a constructible complex on $X$. We say that $h$ is $\mathcal{F}$-transversal if the canonical morphism

$$h^*\mathcal{F} \otimes L^{Rh_1} \xrightarrow{\Lambda(a)} Rh_1^1 \mathcal{F}$$

is an isomorphism. Here $a$ is an integer depending on $h$.

**Example 4.2.**

(a) If $h$ is smooth, then $h$ is $\mathcal{F}$-transversal for any $\mathcal{F}$ (Poincaré duality).

(b) If $\mathcal{F}$ is locally constant, then any $h$ is $\mathcal{F}$-transversal.

**Theorem 4.3.** Let $\mathcal{F}$ be constructible on $X$, $C \subseteq T^*X$ conical and closed. Then the following conditions are equivalent.

(a) $\mathcal{F}$ is micro supported on $C$.

(b) Every $C$-transversal $h : W \to X$ is $\mathcal{F}$-transversal.

Points of the proof: (a)$\Rightarrow$(b): Easier. Uses smooth base change theorem. (b)$\Rightarrow$(a): Harder. Uses local acyclicity of smooth morphism.
4.2. **Ramification.** We can always find a dense open $U \subseteq X$ such that $\mathcal{F}|_U$ is locally constant. Then

$$SS(\mathcal{F})|_U \subseteq T^*_U U$$

(equality if $\mathcal{F}|_U \neq 0$) and

$$\text{Char}(\mathcal{F})|_U = (-1)^n \text{rank}(\mathcal{F})|_U T^*_U U.$$  

Let $D := X \setminus U$ and assume that it is an irreducible divisor. Let $\xi$ be the generic point of $D$, $F = k(\xi)$ the function field of $D$ and $K = \text{Frac}(\mathcal{O}_{X,\xi})$. $K$ is a henselian discrete valuation field. Let $G_K := \text{Gal}(\overline{K}/K)$. On $G_K$ we have a decreasing filtration $G^r_K$, indexed by $r \in \mathbb{Q}_{>0}$, the ramification filtration of $G_K$. The group $G^1_K$ is the inertia group. For $r \in \mathbb{R}_{>0}$ we define

$$G^r_K := \bigcup_{s \leq r} G^s_K \subseteq G^{-r}_K := \bigcap_{s < r} G^s_K$$

If $r \notin \mathbb{Q}$, then $G^r_K = G^{-r}_K$. If $r \in \mathbb{Q}$ then $G^r_K = G^{-r}_K$. The group $G^{1+}_K$ is also denoted $P$; it is the unique pro-$p$-Sylow subgroup and it is called the **wild inertia group**.

For $r > 1$, $\text{Gr}^r(G_K) := G^r_K/G^{r+}_K$ is abelian and annihilated by $p$. There is a canonical injection

$$\text{Hom}_{\mathbb{F}_p}(\text{Gr}^r(G_K), \mathbb{F}_p) \overset{\text{char}}{\longrightarrow} \text{Hom}(m^r_K/m^{r+}_K, \Omega^1_{X,\xi} \otimes \mathbb{F}),$$

which is called the **characteristic form**. We define

$$\overline{K} \supseteq m^r_K = \{a \in \overline{K} \mid \text{ord } a \geq r\}$$

and

$$\overline{K} \supseteq m^{r+}_K = \{a \in \overline{K} \mid \text{ord } a > r\}$$

where ord is the normalized discrete valuation of $\overline{K}$. The characteristic form links the ramification filtration to the tangent bundle of $X$.

Let $j : U = X \setminus D \hookrightarrow X$ be the open immersion and define $\mathcal{F} = j_! \mathcal{G}$, where $\mathcal{G}$ is locally constant on $U$, hence corresponds to a $\Lambda$-representation $V$ of $\pi_1(U)$. The map $G_K \to \pi_1(U)$ gives rise to the **slope decomposition**

$$V = \bigoplus_{r \geq 1, r \in \mathbb{Q}} V^{(r)}$$

characterized by

$$V^{G^{r+}_K} = \bigoplus_{s \leq r} V^{(s)}.$$  

For example, $V^{(1)}$ is the maximal tame sub-$G_K$-module. In this situation, we define

$$\text{dimtot } V = \sum_{r \in \mathbb{Q}_{\geq 1}} r \dim V^{(r)} \in \mathbb{N}.$$  

This number lies in $\mathbb{N}$: If $X$ is a curve, this is the integrality of the Swan conductor, which follows from Hasse-Arf. In general, we can reduce to the curve case by cutting with curves.

For $r > 1$, and $\zeta_p \in \Lambda^\times$, we compute the character of $V^{(r)}$:

$$V^{(r)} = \bigoplus_{\chi : \text{Gr}^r(G_K) \to \Lambda^\times, \chi \neq 1} \chi^{\oplus \text{m}(\chi)}.$$
Let \( L_\chi := \text{im}(\text{Char}(\chi)) \subseteq T^*X \times_X \xi \). This is a line defined over a finite extension of \( F_\chi \) over \( F \).

Now assume \( V \neq 0 \). The singular support is given by

\[
SS(F)|_{\text{Spec} \mathcal{O}_{X,\xi}} = T^*_X X \cup \bigcup_{r>1} m(\chi) \neq 0 \text{ Image of } L_\chi
\]

Similarly, the characteristic cycle is given by

\[
\text{Char}(F)|_{\text{Spec} \mathcal{O}_{X,\xi}} = (-1)^d \left( \text{rank}(G)[T^*_X X] + \dim V^{(1)} \cdot \pi_{\chi, *}[L_\chi] \cdot m(\chi) \right)
\]

Here \( \pi_\chi : L_\chi \to T^*_X X F \subseteq T^*_X X \text{ Spec } \mathcal{O}_{X,\xi} \) is the map above \( \text{Spec } F_\chi \to \text{Spec } F \):

\[
\begin{array}{ccc}
L_\chi & \xrightarrow{\pi_\chi} & T^*_X X F \\
\downarrow & & \downarrow \\
\text{Spec } F_\chi & \longrightarrow & \text{Spec } F
\end{array}
\]

**Example 4.4.** \( X = \mathbb{A}^2 \), \( D = y\)-axis. \( U = X \setminus D \), \( j : U \hookrightarrow X \), \( \mathcal{G} \) given by \( t^p - t = y/x^d \) and \( \mathcal{F} = j! \mathcal{G}, p/d \). Choose a nontrivial character \( \mathbb{Z}/p\mathbb{Z} \to \Lambda^* \). The case \( p = 2, d = 2 \) is exceptional. Otherwise, we have, \( r = d, \text{Char}(\chi) : x^d \mapsto dy \) and get

\[
SS(F) = T^*_X X \cup \langle dy/D \rangle
\]

and

\[
\text{Char}(\mathcal{F}) = [T^*_X X] + d \langle dy/D \rangle.
\]