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k perfect field of characteristic $p > 0$. X smooth over k. $\mathcal{F}$ constructible sheaf on X. $SS\mathcal{F} = \bigcup_a C_a \subset T^*X$ closed conical subset, C_a irreducible components. $\dim C_a = \dim X$.

$CC\mathcal{F} = \sum_a m_a C_a$. The coefficients m_a are integers and determined by a generalization of the Milnor formula. If $\mathcal{F}$ is a perverse sheaf, we have $m_a > 0$ and the support of $CC\mathcal{F}$ equals the singular support $SS\mathcal{F}$.

The characteristic cycles are characterized by the functoriality together with a normalization. Compatibility with the push-forward to a point gives the index formula. Formulating the functoriality and the Milnor formula requires intersection theory.

2 Intersection product

Let k be a field. Let X be a separated scheme of finite type over k. For an integer $q \geqslant 0$, define a free abelian group $Z^q(X)$ generated by irreducible closed subsets of codimension q.

Then Z^q will define a functor $(\mathrm{Sch}/k) \to$ (abelian groups) from the category of separated schemes of finite type over k contravariant for flat morphisms. For an integer $d \geqslant 0$, define a free abelian group $Z_d(X)$ generated by irreducible closed subsets of dimension d. Then Z_d will define a functor $(\mathrm{Sch}/k) \to$ (abelian groups) covariant for proper morphisms. If X is equi-dimensional of dimension n, we have $Z_d(X) = Z^q(X)$ for $q + d = n$.

For a flat morphism $f\colon X \to Y$ and an irreducible closed subset $V \subset Y$ of codimension q, every irreducible component W of the inverse image $f^{-1}(V)$ is of codimension q in X. For the generic points $x_W \in W$ and $y_V \in V$, the local ring $\mathcal{O}_{X,x_w} \otimes_{\mathcal{O}_{Y,y_V}} k(y_V)$ is of finite length. We define the pull-back $f^![V] \in Z^q(X)$ by

$$f^![V] = \sum_{W \subset f^{-1}(V)} \mathrm{length}(\mathcal{O}_{X,x_w} \otimes_{\mathcal{O}_{Y,y_V}} k(y_V)) \cdot [W].$$

For a proper morphism $f\colon X \to Y$ and an irreducible closed subset $V \subset X$ of codimension d, the image $W = f(V)$ is of codimension $\leqslant d$. If we have an equality, for the generic points $x_V \in V$ and $y_W \in W$, the extension $k(x_V)$ of $k(y_W)$ is of finite degree. We define the pull-back $f_![V] \in Z^d(Y)$ by

$$f_![V] = \begin{cases} [k(x_V) : k(y_W)] \cdot [W] & \text{if } \dim W = d \\ 0 & \text{if otherwise.} \end{cases}$$

To define $f^!$ for every morphism $f\colon X \to Y$ of smooth schemes over k, we introduce a quotient $\mathrm{CH}^q(X)$ of $Z^q(X)$. Since f is decomposed as a closed immersion $(1, f)\colon X \to X \times_k Y$ and the projection $X \times_k Y \to Y$, it suffices to define the pull-back $i^!$ for regular closed immersions.

Let $i\colon X \to Y$ be a closed immersion. We say that an irreducible closed subset $V \subset Y$ of codimension q meets X properly if codimension of $V \cap X \subset X$ is q. Note that we have an inequality $\leqslant q$ in general. In the case where $V \subset Y$ meets X properly, we define $i^![V] = \sum_{W \subset V \cap X} \mathrm{length}(i^*\mathcal{O}_V)_{x_W}[W]$ in the same way as above.

In general, we impose 'homotopy equivalence'. For the immersions $i_0, i_1\colon X \to \mathbf{A}^1_k \times_k X$, we require that $i_0^*, i_1^*\colon \mathrm{CH}^{q+1}(\mathbf{A}^1_k \times_k X) \to \mathrm{CH}^q(X)$ are the same. This implies that for $V \subset \mathbf{A}^1_k \times_k X$ flat over $\mathbf{A}^1_k$, we need to have $i_0^![V] = i_1^![V]$ in $\mathrm{CH}^q(X)$. We make this requirement as a definition:

$$\mathrm{CH}^q(X) = Z^q(X)/\langle i_0^![V] - i_1^![V]; V \subset \mathbf{A}^1_k \times_k X \text{ flat over } \mathbf{A}^1_k\rangle.$$

If $[V] = [V']$ in $\mathrm{CH}^q(X)$, we say V and V' are rationally equivalent.

Lemma 2.0.1.

$$\mathrm{CH}^q(X) = \mathrm{Coker}(\mathrm{div}\colon \bigoplus_{x \in X} k(x)^\times \to Z^q(X)).$$

Proof. By compactifying $\mathbf{A}^1 \subset \mathbf{P}^1$ and changing the coordinate, we have

$$\mathrm{CH}^q(X) = Z^q(X)/\langle i_0^![V] - i_\infty^![V]; V \subset \mathbf{P}^1_k \times_k X \text{ flat over } \mathbf{P}^1_k\rangle.$$

We have $\mathrm{Im}(\mathrm{div}) \subset$. By taking the norm, we get the other inclusion. □

We have a canonical morphism $\mathrm{Pic}(X) \to \mathrm{CH}^1(X)$. If X is smooth, this is an isomorphism.

Proposition 2.0.2. *If E is a vector bundle over X, the projection $p\colon E \to X$ induces an isomorphism $p^*\colon \mathrm{CH}^q(X) \to \mathrm{CH}^q(E)$ for every integer q.*

Let $i\colon X \to Y$ be a closed immersion of smooth schemes over k. Let $\widetilde{\mathbf{A}^1 \times Y}$ be the blow-up at $0\times X \subset \mathbf{A}^1\times Y$ and let $\widetilde{Y} \subset \widetilde{\mathbf{A}^1 \times Y}$ be the complement of the proper transform of $0\times Y$. The dilatation $\widetilde{Y}$ contains $\mathbf{G}_m \times Y$ as an open subset and the complement is the normal bundle T_XY. The immersion $\mathbf{A}^1 \times X \to \mathbf{A}^1 \times Y$ is lifted to $\widetilde{i}\colon \mathbf{A}^1 \times X \to \widetilde{Y}$. We should have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}^q(\widetilde{Y}) \arrow[r, "i_0^!(\text{resp.}i_1^!)"] \arrow[d, "\widetilde{i}^!"'] & \mathrm{CH}^q(T_XY), (\text{resp. } \mathrm{CH}^q(Y)) \arrow[d, "i'^!(\text{resp. } i^!)"] \\ \mathrm{CH}^q(\mathbf{A}^1 \times X) \arrow[r, "i_0^!(\text{resp.}i_1^!)"] & \mathrm{CH}^q(X). \end{tikzcd} \tag{2.1}$$

Since we require $i_0^! = i_1^!$, the two compositions $i'^!i_0^!$ and $i^!i_1^!$ must be equal. Let $V \subset Y$ be a closed subvariety. Let $\widetilde{\mathbf{A}^1 \times V}$ be the blow-up at $0 \times (V \cap X)$ and $\widetilde{V} \subset \widetilde{Y}$ be its intersection with $\widetilde{Y}$. Further set $V_0 = i_0^!\widetilde{V}$. By the commutative diagram (2.1), $i^!(V) = i^!i_1^!(\widetilde{V})$ should be given by $i'^!i_0^!(\widetilde{V}) = i'^!(V_0)$. Since T_XY is a vector bundle over X, the morphism $i'^!\colon \mathrm{CH}^q(T_XY) \to \mathrm{CH}^q(X)$ is the inverse of the pull-back isomorphism $p^!\colon \mathrm{CH}^q(X) \to \mathrm{CH}^q(T_XY)$. Thus, $i^!(V)$ should be defined by the formula

$$i^!(V) = (p^!)^{-1}(V_0).$$

Locally, assume that $Y = \operatorname{Spec} A$ is affine and $X = V(I)$ for an ideal $I \subset A$. Then, we have $\widetilde{Y} = \operatorname{Spec} A[t, I/t], \mathbf{A}^1 \times X = V(I/t)$ and $T_XY = V(t) = \operatorname{Spec} \bigoplus_n I^n/I^{n+1}$. Assume further that V is defined by a quotient ring $\overline{A}$ and let $\overline{I} \subset \overline{A}$ be the image of $I \subset A$. Then, we have $\widetilde{V} = \operatorname{Spec} \overline{A}[t, \overline{I}/t]$ and the intersection $T_XY \cap \widetilde{V}$ is the normal cone $C_{V\cap X}V = \operatorname{Spec} \bigoplus_n \overline{I}^n/\overline{I}^{n+1}$. Thus this construction is called the deformation to normal cone.

refinement.

Let $f\colon X \to Y$ be a morphism of complete intersection. Let $Y' \subset Y$ be a closed subset such that the codimension of $X' = f^{-1}(Y) \subset X$ equals the codimension of $Y' \subset Y$. Then, the pull-back defines a morphism $\mathrm{CH}_{\dim Y - q}(Y') \to \mathrm{CH}_{\dim X - q}(X')$.

Let $f\colon X \to Y$ be a morphism of complete intersection. Let $X' \subset X$ be a closed subset and $Y' = f(X') \subset Y$. Then, the push-forward defines a morphism $\mathrm{CH}_d(X') \to \mathrm{CH}_d(Y')$.

Projective bundle formula.

Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of finite rank and $\mathbf{P} = \operatorname{Proj} S^\bullet\mathcal{E}^\vee$ be the projective space bundle parametrizing sub invertible sheaves in $\mathcal{E}$. Let $\mathcal{L} = \mathcal{O}(1)$ be the tautological invertible sheaf and $h = c_1(\mathcal{L}) \in \mathrm{CH}^1(\mathbf{P})$ be the first Chern class. Then, the canonical morphism

$$\mathrm{CH}^\bullet(X)[h]/(c_\bullet(\mathcal{E})(1+h)^{-1})_{\dim = \operatorname{rank}\mathcal{E}-1} \to \mathrm{CH}^\bullet(\mathbf{P})$$

is an isomorphism. In particular, if $\dim \mathbf{P} = 2\dim X = 2n$, we have an isomorphism

$$\bigoplus_{i=0}^{n} \mathrm{CH}_i(X) \cdot h^i \to \mathrm{CH}_0(\mathbf{P}).$$

3 Axiomatic characterization

Functoriality.

Let X be an equi-dimensional smooth scheme over k and $C \subset T^*X$ be a conical closed subset such that every irreducible component has dimension $\dim X$. Let $A = \sum_a m_a C_a$ be a linear combination of irreducible components of C.

Let $h\colon W \to X$ be a morphism of smooth schemes over k. We say that h is properly C-transversal if h is C-transversal and if every irreducible component of h^*C has dimension $\dim W$. If h is smooth, h is properly C-transversal for every C.

We consider the algebraic correspondence

$$T^*X \xleftarrow{\ b\ } W \times_X T^*X \xrightarrow{\ c\ } T^*W.$$

Assume that h is properly C-transversal. Then, we have morphisms $b^!\colon Z^0(C) \to Z^0(h^*C)$ and $c_!\colon Z^0(h^*C) \to Z^0(h^\circ C)$. Thus, for $A \in Z^0(C)$, we define a linear combination $h^!A$ of irreducible components of $h^\circ C$ by

$$h^!A = (-1)^{\dim X - \dim W} c_! b^! A.$$

The sign is there to have $h^!(CC\Lambda_X) = CCh^*\Lambda_X$.

Let $f\colon X \to Y$ be a proper morphism of smooth schemes over k and consider the algebraic correspondence

$$T^*X \xleftarrow{\ c\ } X \times_Y T^*Y \xrightarrow{\ b\ } T^*Y.$$

Assume that every irreducible component of $f_\circ C$ has dimension $m = \dim Y$. Then, we have morphisms $c^!\colon Z^0(C) \to \mathrm{CH}_m(c^{-1}(C))$ and $c_!\colon \mathrm{CH}_m(c^{-1}(C)) \to Z^0(f_\circ C)$. Thus, for $A \in Z^0(C)$, we define a linear combination $f_!A$ of irreducible components of $f_\circ C$ by

$$f_!A = b_! c^! A.$$

If $Y = \operatorname{Spec} k$, the morphism $c\colon X \to T^*X$ is the 0-section and $c^!\colon Z^n(T^*X) \to \mathrm{CH}_0(X)$ sending A to the intersection product $c^!A = (A, T^*_X X)_{T^*X}$ is induced by the inverse of the isomorphism $\mathrm{CH}^n(X) \to \mathrm{CH}^n(T^*X)$. For the 0-section $A = T^*_X X$, the self-intersection product $(T^*_X X, T^*_X X)_{T^*X}$ is the top Chern class $c_n(\Omega^1_{X/k})$. The push-forward $b_!\colon \mathrm{CH}_0(X) \to \mathrm{CH}_0(\operatorname{Spec} k) = \mathbf{Z}$ for $b\colon X \to \operatorname{Spec} k$ is the degree mapping. Thus we have $f_!A = \deg(A, T^*_X X)_{T^*X}$.

Exercise. Define a morphism $f\colon \mathbf{A}^1 = \operatorname{Spec} k[x] \to \mathbf{A}^1 = \operatorname{Spec} k[y]$ by $x^p - x = y$ and extend this to a morphism $\overline{f}\colon X = \mathbf{P}^1 \to Y = \mathbf{P}^1$. Compute the coefficient n of the fiber $T^*_\infty Y$ in $\overline{f}_![T^*_X X] = p \cdot [T^*_Y Y] + n \cdot [T^*_\infty Y]$.

Theorem 3.0.1. *Let k be a perfect field of characteristic $p \geqslant 0$ and Λ be a finite field of characteristic ℓ invertible in k. Then, there exists a unique way to attach a $\mathbf{Z}$-linear combination $A(\mathcal{F}) = \sum_a m_a C_a$ satisfying the conditions* (1)–(4-2) *below of irreducible components of the singular support $SS\mathcal{F} = C = \bigcup_a C_a \subset T^*X$ to each smooth scheme X over k and each constructible sheaf on X:*

(1) [normalization] *For $X = \operatorname{Spec}\, k$ and $\mathcal{F} = \Lambda$, we have*

$$A(\mathcal{F}) = 1 \cdot T^*_X X.$$

(2) [additivity] *For every distinguished triangle* $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$, *we have*

$$A(\mathcal{F}) = A(\mathcal{F}') + A(\mathcal{F}'').$$

(3) [pull-back] *For every properly* $SS\mathcal{F}$*-transversal morphism* $h\colon W \to X$ *of smooth schemes, we have*

$$A(h^*\mathcal{F}) = h^! A(\mathcal{F}).$$

(4-1) [closed immersion] *For every closed immersion* $i\colon X \to P$ *of smooth schemes, we have*

$$A(i_*\mathcal{F}) = i_! A(\mathcal{F}).$$

(4-2) [Radon transform] *For the Radon transform, we have*

$$A(R\mathcal{F}) = LA(\mathcal{F}).$$

We may replace (4-2) by the following condition:

(4′) [push-forward] For every morphism $f\colon X \to Y$ of projective smooth schemes, if $\dim f_\circ SS\mathcal{F} \leqslant \dim Y$, we have

$$A(f_*\mathcal{F}) = f_! A(\mathcal{F}).$$

Under (3), condition (4-2) is a special case of (4′) for $q^\vee\colon Q \to \mathbf{P}^\vee$.

We expect that we may unify (4-1) and (4′) into the following condition:

(4) [push-forward] For every proper morphism $f\colon X \to Y$ of smooth schemes, if $\dim f_\circ SS\mathcal{F} \leqslant \dim Y$, we have

$$A(f_*\mathcal{F}) = f_! A(\mathcal{F}).$$

Corollary 3.0.2. *Let* $U \subset X$ *be a dense open subset such that the restriction* $\mathcal{F}|_U$ *is locally constant of rank* $\operatorname{rank} \mathcal{F}$. *Then, the coefficient of the* 0*-section* $T^*_X X$ *in* $A(\mathcal{F})$ *equals* $(-1)^{\dim X} \operatorname{rank} \mathcal{F}$.

Proof. Since an étale morphism is properly transversal, by applying (3), we may assume that $\mathcal{F}$ is a constant sheaf. By applying (2), we may assume $\mathcal{F} = \Lambda$. Since a smooth morphism is properly transversal, the assertion follows from (1) and (3) for the morphism to Spec k. □

If $\dim X = 1$, by Corollary 3.0.2, we have

$$A(\mathcal{F}) = -\Big(\operatorname{rank} \mathcal{F} \cdot [T^*_X X] + \sum_{x \in X - U} a_x(\mathcal{F}) \cdot [T^*_x X]\Big) \tag{3.1}$$

for some integer $a_x(\mathcal{F}) \in \mathbf{Z}$ defined for each closed point $x \in X$ where $\mathcal{F}$ is not locally constant.

Corollary 3.0.3. 1. *If* X *is projective and smooth, we have*

$$\chi(X_{\bar{k}}, \mathcal{F}) = (A(\mathcal{F}), T^*_X X)_{T^*X}. \tag{3.2}$$

2. *Further if* $\dim X = 1$, *we have*

$$\chi(X_{\bar{k}}, \mathcal{F}) = \operatorname{rank} \mathcal{F} \cdot \chi(X_{\bar{k}}) - \sum_{x \in X - U} a_x(\mathcal{F}) \cdot \deg_k x. \tag{3.3}$$

The equality (3.3) *determines* $a_x(\mathcal{F})$ *uniquely.*

Proof. 1. By (4-1), we may assume that $X = \mathbf{P}^n$ is a projective space of dimension $n \neq 1$. By (2) and corollary 3.0.2, we have

$$A(R^\vee R\mathcal{F}) - A(\mathcal{F}) = (-1)^n(n-1)\chi(X_{\bar{k}}, \mathcal{F}) \cdot [T^*_{\mathbf{P}}\mathbf{P}]. \tag{3.4}$$

Similarly for A, we have

$$L^\vee LA - A = (-1)^n(n-1)(A, T^*_X X)_{T^*X} \cdot [T^*_{\mathbf{P}}\mathbf{P}]. \tag{3.5}$$

For $A = A(\mathcal{F})$, by applying (4-2) two times, we obtain $A(R^\vee R\mathcal{F}) = L^\vee LA$. Thus, comparing the right hand sides and dividing them by $n - 1 \neq 0$, we deduce (3.2).

2. If $\dim X = 1$, substituting (3.1) in to (3.2), we obtain (3.3). By killing the ramification, $a_x(\mathcal{F})$ is determined uniquely. □

In the proof above, the Grothendieck–Ogg–Shafarevich formula (3.3) is deduced as a special case of the index formula (3.2). In the actual proof of the existence of $A(\mathcal{F})$, conversely we deduce axiom (4-2) from (3.3).

Proof of the uniqueness in Theorem 3.0.1*.* On a dense open subset where $\mathcal{F}$ is locally constant $A(\mathcal{F})$ is determined by Corollary 3.0.2. Further, if $\dim X = 1$, $A(\mathcal{F})$ is determined by (3.1).

By (3), we may assume that X is affine. By (4-1), we may assume that $X = \mathbf{A}^n$ is an affine space. By (3), we may assume that $X = \mathbf{P}^n$ is projective.

Take a closed immersion $i\colon X \to \mathbf{P} = \mathbf{P}^N$ and consider the Radon transform $Ri_*\mathcal{F}$ on the dual $\mathbf{P}^\vee$. Take a Lefschetz pencil $i_L\colon L \to \mathbf{P}^\vee$. Then $A(i_L^* Ri_*\mathcal{F})$ is determined since $\dim L = 1$.

It suffices to remove i_L^*, R and i_* respectively. Since the immersion i_L is properly transversal, $A(Ri_*\mathcal{F})$ is determined by $i_L^! A(Ri_*\mathcal{F}) = A(i_L^* Ri_*\mathcal{F})$ by (3). Since the immersion i is not isomorphism, $A(i_*\mathcal{F})$ is determined by $A(R^\vee Ri_*\mathcal{F}) = A(i_*\mathcal{F}) + **\,[T^*_{\mathbf{P}}\mathbf{P}]$. This is equal to $L^\vee A(Ri_*\mathcal{F})$ (4-2). Since $A(\mathcal{F})$ is determined by $i_! A(\mathcal{F}) = A(i_*\mathcal{F})$ (4-1), the uniqueness follows.

4 Conductor

The key point of the whole theory is the GOS formula (3.3). We compute $a_x(\mathcal{F})$.

The genesis of the Grothendieck–Ogg–Shafarevich formula (3.3) is traced in the exchange of letters of Grothendieck dated on 10.04.63 and of Serre dated April 63 recorded in [3]. In his letter, Grothendieck discusses a plan to generalize the formula obtained earlier by Ogg–Shafarevich in the tame case where the Swan conductor does not appear.

lower ramification groups. Let K be a complete discrete valuation field and L be a finite Galois extension of Galois group G. We define a decreasing filtration $G_i \subset G$ by

$$G_i = \mathrm{Ker}(G \to \mathrm{Aut}(L^\times/(1+\mathfrak{m}_L^i))$$

for $i \geqslant 1$ and for $\sigma \in G - \{1\}$, set

$$i_G(\sigma) = \begin{cases} \max(i : \sigma \in G_i) & \text{if } \sigma \in G_1 \\ 0 & \text{if otherwise.} \end{cases}$$

The subgroup $P = G_1$ is the p-Sylow subgroup of the inertia group $I = \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L))$.

Lemma 4.0.1. *If L is a totally ramified extension and if $s \in L^\times$ satisfies $e_{L/K} \nmid \mathrm{ord}_L s$, then $i_G(\sigma) = \mathrm{ord}_L(\sigma(s)/s - 1)$.*

Proof. Set $G'_i = \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i)^\times) \supset G_i$ and let s' be a uniformizer of L. Then, $G_i = \mathrm{Ker}(G \to \mathrm{Aut}(L^\times/(1+\mathfrak{m}_L^i))$ equals $\{\sigma \in G'_i \mid \mathrm{ord}_L(\sigma(s')/s' - 1) \geqslant i\}$. Since $\mathcal{O}_L/\mathfrak{m}_L^i = (\mathcal{O}_L/\mathfrak{m}_L^i)^\times \cup \mathfrak{m}_L/\mathfrak{m}_L^i$ and since $\mathrm{Ker}(G \to \mathrm{Aut}(\mathfrak{m}_L/\mathfrak{m}_L^i)) = \mathrm{Ker}(G \to \mathrm{Aut}((1+\mathfrak{m}_L)/(1+\mathfrak{m}_L^i))) \supset G'_i$, we have $G'_i = \mathrm{Ker}(G \to \mathrm{Aut}(\mathcal{O}_L/\mathfrak{m}_L^i))$. Since L is totally ramified, the right hand side equals $\{\sigma \in G \mid \mathrm{ord}_L(\sigma(s') - s') \geqslant i\}$. Thus we have $G_i = \{\sigma \in G \mid \mathrm{ord}_L(\sigma(s')/s' - 1) \geqslant i\}$.

Let t be a uniformizer of K. Since we may take a uniformizer s' in the multiplicative subgroup $\langle s, t\rangle \subset L^\times$ generated by s and t, we have $\mathrm{ord}_L(\sigma(s')/s' - 1) \geqslant \mathrm{ord}_L(\sigma(s)/s - 1)$. On the other hand, the equality $G_i = \{\sigma \in G \mid \mathrm{ord}_L(\sigma(s')/s' - 1) \geqslant i\}$ implies $\mathrm{ord}_L(\sigma(s')/s' - 1) \leqslant \mathrm{ord}_L(\sigma(s)/s - 1)$. Thus we have the equality $G_i = \{\sigma \in G \mid \mathrm{ord}_L(\sigma(s)/s - 1) \geqslant i\}$ and $i_G(\sigma) = \mathrm{ord}_L(\sigma(s)/s - 1)$ for $\sigma \in G_1 - \{1\}$. $\square$

Example. Let k be a perfect field of characteristic $p > 0$ and $K = k((t))$ be the field of Laurent power series. Let $x \in K$ be an element satisfying $p \mid n = -\mathrm{ord}_K x > 0$. Let L be a totally ramified Artin–Schreier extension of K defined by the equation $s^p - s = x$. Let $\sigma \in G = \mathrm{Gal}(L/K) \cong \mathbf{F}_p$ be the generator satisfying $\sigma(s) = s + 1$. By Lemma 4.0.1, we have $i_G(\sigma) = \mathrm{ord}_L(\sigma(s)/s - 1) = -\mathrm{ord}_L s = n$.

We define the Swan character $s_G \colon G \to \mathbf{Z}$ satisfying

$$\sum_{\sigma \in G} s_G(\sigma) = 0$$

by

$$s_G(\sigma) = -f_{L/K} \cdot i_G(\sigma)$$

for $\sigma \in G - \{1\}$. The value $s_G(1)$ is given by the length of the $\mathcal{O}_L$-module

$$\mathrm{Hom}_{\mathcal{O}_K}(\mathfrak{m}_L, \mathfrak{m}_K)/\mathcal{O}_L \cdot \mathrm{Tr}_{L/K}$$

if the residue field extension is separable.

Let V be a representation of G on a Λ-vector space of finite dimension. We define the Brauer trace. For $\sigma \in P = G_1$, the subalgebra $\Lambda[\sigma] \subset \mathrm{End}_\Lambda(V)$ is a commutative semi-simple Λ-algebra and is decomposed as a product $\Lambda[\sigma] = \prod_i \Lambda_i$ of finite separable extensions of Λ. Accordingly, V is decomposed as $V = \bigoplus_i V_i$. Define a product $W(\Lambda[\sigma]) = \prod_i W(\Lambda_i)$ of the ring of integers of unramified extensions of $\mathbf{Z}_\ell$ and let $\widetilde{\sigma} = (\widetilde{\sigma}_i)_i$ denote the lifting as a root of 1 of order a power of p. Then, the Brauer trace $\mathrm{Tr}^{\mathrm{Br}}(\sigma \colon V) \in W(\Lambda)$ is defined by

$$\mathrm{Tr}^{\mathrm{Br}}(\sigma \colon V) = \sum_i \dim_{\Lambda_i} V_i \cdot \mathrm{Tr}_{W(\Lambda_i)/W(\Lambda)} \widetilde{\sigma}_i.$$

We define the Swan conductor $\mathrm{Sw}\, V$ by

$$\mathrm{Sw}\, V = \frac{1}{\# G} \sum_{\sigma \in G_1} s_G(\sigma) \mathrm{Tr}^{\mathrm{Br}}(\sigma \colon V).$$

The Swan conductor is a rational number by

$$\mathrm{Sw}\, V = \frac{1}{e_{L/K}} \sum_{i \geqslant 1} \# G_i \cdot (\dim V - \dim V^{G_i}).$$

We have $\mathrm{Sw}\, V \geqslant 0$ and the equality is equivalent to the condition that $P = G_1$ acts trivially on V.

Theorem 4.0.2. *Assume that the residue field extension is separable. Then* $\mathrm{Sw}\, V$ *is an integer. In other words, the Swan character* s_G *is a character of a representation of* G*.*

By the theory of Brauer [6, Partie 3] and the induction formula, the proof of theorem is reduced to the case where $\dim V = 1$, called the Hasse–Arf theorem.

The classical case where the residue field is finite is proved by Hasse [2]. It is generalized to the algebraically closed residue field case by Arf [1]. A more conceptual proof by Serre is given in [5, Chapitre V, §7]. See a comment at the end of the letter of Serre dated on 9 novembre 1958 recorded in [3].

Example. Let $G = \mathrm{Gal}(L/K) \cong \mathbf{F}_p$ be as in the previous example and let V be a representation of defined by a faithful character $G \to \Lambda^\times$. Since $i_G(\sigma) = n$ for the generator $\sigma \in G$ satisfying $\sigma(s) = s + 1$, we have $\mathrm{Sw}\, V = n$.

Let L' be a Galois extension of K containing L as a subfield and assume that the residue field extension is separable. Then the representation V of G may be regarded as a representation of $G' = \mathrm{Gal}(L'/K)$. The Swan conductor $\mathrm{Sw}\, V$ defined for V regarded as a representation of G and that defined for V regarded as a representation of G' are the same.

We formulate the GOS formula. Let k be a perfect field of characteristic $p > 0$. Let X be a proper smooth irreducible curve over k and $U \subset X$ be a dense open subset. Let $\mathcal{F}$ be a locally constant constructible sheaf of Λ-modules on U. For each $x \in X - U$, let K_x denote the fraction field of the completion of the local ring $\mathcal{O}_{X,x}$. Then the pull-back of $\mathcal{F}$ by $\mathrm{Spec}\, K_x \to U$ corresponds to a continuous Λ-representation V_x of $\mathrm{Gal}(\overline{K}_x/K_x)$. This representation factors a finite quotient and the Swan conductor $\mathrm{Sw}\, V_x$ thus defined is independent of the choice of the quotient. Let $\mathrm{Sw}_x\mathcal{F}$ denote $\mathrm{Sw}\, V_x$.

Theorem 4.0.3.

$$\chi_c(U_{\overline{k}}, \mathcal{F}) = \mathrm{rank}\, \mathcal{F} \cdot \chi_c(U_{\overline{k}}) - \sum_{x \in X - U} \mathrm{Sw}_x\mathcal{F} \cdot \deg_k x. \tag{4.1}$$

Example. Let $U = \mathbf{A}^1_k = \mathrm{Spec}\, k[x]$ and $V = \mathbf{A}^1_k = \mathrm{Spec}\, k[t] \to U$ be the étale morphism defined by $t^p - t = x$. Let $G = \mathrm{Gal}(V/U)$ and $\mathcal{F}$ be the locally constant sheaf defined by a faithful character $G \to \Lambda^\times$. Verify the GOS-formula for $\chi_c(U_{\overline{k}}, \mathcal{F})$.

We may assume that k is algebraically closed. Let $V \to U$ be a finite étale Galois covering of Galois group G such that the pull-back $\mathcal{F}_V$ is constant and let W be the integral closure of X in V. Then, the formula follows from the Lefschetz trace formula

$$\sum_{i=0}^{2} (-1)^i \mathrm{Tr}(\sigma \colon H^i(W, \mathbf{Q}_\ell)) = (\Gamma_\sigma, \Delta_W)_{W \times W}$$

and the Riemann–Hurwitz formula

$$\chi(W) = [W : X]\chi(X) - \dim_k \Omega^1_{W/X}.$$

Proposition 4.0.4.

$$a_x(\mathcal{F}) = \dim \mathcal{F}_{\overline{\eta}_x} - \dim \mathcal{F}_{\overline{x}} + \mathrm{Sw}_x\mathcal{F}.$$

Proof. By Corollary 3.0.2 and by the Grothendieck–Ogg–Shafarevich formula (3.3) and (4.1), we have the equality of the sum. By killing the ramification, we obtain the equality. □

5 Milnor formula

For the actual construction of CC, we use the vanishing cycles. The vanishing cycles are the tools to study the difference of the cohomology of the generic fiber and that of the closed fiber.

Let K be a henselian (or complete) discrete valuation field and $\mathcal{O}_K$ be the valuation ring. Let $s, \eta \in S = \operatorname{Spec} \mathcal{O}_K$ denote the closed and the generic point. Let $S^{\mathrm{sh}} = \operatorname{Spec} \mathcal{O}_K^{\mathrm{ur}}$ be the strict localization and let $\overline{s}, \overline{\eta} = \operatorname{Spec} K_{\mathrm{sep}}$ be geometric points of S^{sh} above s, η respectively. Let X be a scheme over S and consider the cartesian diagram

$$\begin{tikzcd} X_{\overline{s}} \arrow[r, "i"] \arrow[d] & X_{S^{\mathrm{sh}}} \arrow[d] & X_{\overline{\eta}} \arrow[l, "\overline{j}"'] \arrow[d] \\ \overline{s} \arrow[r] & S^{\mathrm{sh}} & \overline{\eta}. \arrow[l] \end{tikzcd}$$

For a sheaf $\mathcal{F}$ on X, the complex of nearby cycles $\Psi\mathcal{F}$ is the complex on $X_{\overline{s}}$ defined by $\Psi\mathcal{F} = i^*\overline{j}_*\mathcal{F}|_{X_{\overline{\eta}}}$ equipped with a canonical action of $G_K = \operatorname{Gal}(K_{\mathrm{sep}}/K)$ compatible with the action on $X_{\overline{s}}$. The vanishing cycles $\Phi\mathcal{F}$ fits in a distinguished triangle

$$(5.1) \qquad \longrightarrow \mathcal{F}|_{X_{\overline{s}}} \longrightarrow \Psi\mathcal{F} \longrightarrow \Phi\mathcal{F} \longrightarrow$$

of complexes on $X_{\overline{s}}$ equipped with actions of G_K. The actual construction of the functor Φ as a cone does not work in the derived category and requires to go back to the category of complexes.

Theorem 5.0.1. *Assume that X is of finite type over S and that $\mathcal{F}$ is constructible. Then, $\Psi\mathcal{F}$ and $\Phi\mathcal{F}$ are constructible.*

Example. If $X = S$, the distinguished triangle (5.1) is $\to \mathcal{F}_{\overline{s}} \to \mathcal{F}_{\overline{\eta}} \to \Phi\mathcal{F} \to$. More generally, if X is proper over S, applying the functor $R\Gamma(X_{\overline{s}}, -)$ to (5.1) and replacing $R\Gamma(X_{\overline{s}}, \Psi\mathcal{F})$ by $R\Gamma(X_{\overline{\eta}}, \mathcal{F}|_{X_{\overline{\eta}}})$ by the isomorphism of proper base change, we obtain a distinguished triangle

$$(5.2) \qquad \longrightarrow R\Gamma(X_{\overline{s}}, \mathcal{F}|_{X_{\overline{s}}}) \longrightarrow R\Gamma(X_{\overline{\eta}}, \mathcal{F}|_{X_{\overline{\eta}}}) \longrightarrow R\Gamma(X_{\overline{s}}, \Phi\mathcal{F}) \longrightarrow$$

comparing the cohomology of the two geometric fibers. Further if $\Phi\mathcal{F}$ is supported on a subset $\Sigma \subset X_{\overline{s}}$ finite over $\overline{s}$, the last term is $\bigoplus_{x\in\Sigma} \Phi_x\mathcal{F}$.

Assume that the residue field of K is perfect. For a continuous Λ-representation V of finite degree of the absolute Galois group G_K, define the total dimension to be the sum of the degree and the Swan conductor:

$$\dim\operatorname{tot} V = \dim V + \operatorname{Sw} V.$$

This is additive in V.

Let X be a smooth curve over a perfect field k of characteristic $p > 0$ and $x \in X$ be a closed point. Then the local field K_x at x defined as the fraction field of the completion $\widehat{\mathcal{O}}_{X,x}$ is a complete discrete valuation field and the residue field is a finite extension of k. Let $\mathcal{F}$ be a constructible sheaf on X and consider the identity of the base change to $S = \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$. Then, the space of vanishing cycles $\Phi_x\mathcal{F}$ fits in a distinguished triangle

$$\longrightarrow \mathcal{F}_{\bar{s}} \longrightarrow \mathcal{F}_{\bar{\eta}} \longrightarrow \Phi_x\mathcal{F} \longrightarrow . \tag{5.3}$$

Thus we have

$$\dim\operatorname{tot}\Phi_x\mathcal{F} = \dim\operatorname{tot}\mathcal{F}_{\bar{\eta}_x} - \dim\operatorname{tot}\mathcal{F}_{\bar{x}} = \dim\mathcal{F}_{\bar{\eta}_x} - \dim\mathcal{F}_{\bar{x}} + \operatorname{Sw}\mathcal{F}_{\bar{\eta}_x} = a_x\mathcal{F}.$$

Namely, if $\dim X = 1$, the coefficient of T^*_xX in $A(\mathcal{F})$ is given by the total dimension of the space of vanishing cycles for the identity mapping. This will be generalized as follows.

We say that x is an isolated characteristic point of f if there exists a neighborhood of x, on which f is C-transversal except at x.

Theorem 5.0.2. *Let k be a perfect field of characteristic $p \geqslant 0$ and Λ be a finite field of characteristic ℓ invertible in k. Then, there exists a unique way to attach a $\mathbf{Q}$-linear combination $CC\mathcal{F} = \sum_a m_aC_a$ satisfying the condition below of irreducible components of the singular support $SS\mathcal{F} = \bigcup_a C_a \subset T^*X$ to each smooth scheme X over k and each constructible sheaf on X:*

For every étale morphism $j\colon U \to X$ and every morphism $f\colon U \to \mathbf{A}^1_k$ with isolated characteristic point x, we have the Milnor formula

$$-\dim\operatorname{tot}\Phi_x\mathcal{F} = (j^\circ CC\mathcal{F}, df)_{T^*U,x}. \tag{5.4}$$

The assumption that x is an isolated characteristic point implies that x is an isolated point of the support of $\Phi\mathcal{F}$.

Uniqueness: Suffices to show the existence of a morphism with isolated characteristic point.

Main ingredients of the proof of existence:

continuity of Swan conductor.

nearby cycles over general base scheme.

Proof of Theorem 5.0.2 $\Rightarrow$ *Existence in Theorem* 3.0.1. It suffices to show that $A(\mathcal{F}) = CC\mathcal{F}$ satisfies the integrality, the axioms (1)–(4-2). Theorem 3.0.1 (1) follows from the Milnor formula for the immersion $\operatorname{Spec} k \to \mathbf{A}^1_k$. Theorem 3.0.1 (2) follows from the exactness of the vanishing cycles functor Φ. Theorem 3.0.1 (4-1) follows from the proper base change theorem.

So the essential contents of the proof are:

integrality.

(3) for smooth morphisms.

(3) for immersion.

(4-2) Radon transform.

Theorem 3.0.1 (3) for smooth morphisms follows from the Thom–Sebastiani formula.

Integrality. If $p \neq 2$ or non exceptional case where $p = 2$, we may find a good Lefschetz pencil. In the exceptional case $p = 2$, reduce to non-exceptional case by using pull-back to $X \times \mathbf{A}^1$.

(3) for immersion and (4-2) except for the 0-section. Radon transform.

Remaining: (4-2) for the 0-section.

□

6 Characteristic classes

$\dim X = n$, E vector bundle of rank n over X.

$C = \bigcup C_a \subset E$ closed conical subset.

$A = \sum m_a C_a$ linear combination.

$\overline{C} = \bigcup \overline{C_a} \subset \mathbf{P}(E \oplus \mathbf{A}^1_X)$.

$\overline{A} = \sum m_a \overline{C_a}$.

$[\overline{A}] \in \mathrm{CH}_n(\mathbf{P}(E \oplus \mathbf{A}^1_X)) = \mathrm{CH}_\bullet(X) = \bigoplus_{q=0}^n \mathrm{CH}_q(X)$.

Applying this to $E = T^*X$ and $A = A(\mathcal{F})$, we obtain a morphism

$$cc_X \colon K(X, \Lambda) \to \mathrm{CH}_\bullet(X) \tag{6.1}$$

sending a class $[\mathcal{F}]$ to $cc_X(\mathcal{F}) = [\overline{CC\mathcal{F}}]$ by additivity (2).

If characteristic 0, this is defined by MacPherson. compatibility with push-forward. characteristic 0 OK but $p > 0$ not.

We prove (4-2) for the coefficient of the 0-section by proving the commutativity of the diagram

$$A(n) \qquad \begin{array}{ccc} K(\mathbf{P}, \Lambda) & \xrightarrow{cc} & \mathrm{CH}_\bullet(\mathbf{P}) \\ {\scriptstyle R}\big\downarrow & & \big\downarrow{\scriptstyle L} \\ K(\mathbf{P}^\vee, \Lambda) & \xrightarrow{cc} & \mathrm{CH}_\bullet(\mathbf{P}^\vee), \end{array}$$

that is a consequence of (4-2), and the index formula

$$B(n) \qquad \begin{tikzcd} K(\mathbf{P}, \Lambda) \arrow[r, "cc"] \arrow[dr, "\chi"'] & \mathrm{CH}_\bullet(\mathbf{P}) = \mathrm{CH}_n(\mathbf{P}(T^*\mathbf{P} \oplus \mathbf{A}^1_{\mathbf{P}})) \arrow[d, "(\ , T^*_{\mathbf{P}}\mathbf{P})_{\mathbf{P}(T^*\mathbf{P} \oplus \mathbf{A}^1_{\mathbf{P}})}"] \\ & \mathbf{Z} \end{tikzcd}$$

by induction on n for $\mathbf{P} = \mathbf{P}^n$. More precisely, we prove implications $B(n-1) \Rightarrow A(n) \Rightarrow B(n)$.

They are proved for small n as follows.

$A(0)$: We have $R = L = 0$ since $Q = \varnothing$.

$B(0)$: The normalization (1).

$A(1)$: We have $\mathbf{P} = \mathbf{P}^\vee$ and $R = 1$ and $L = 1$.

$B(1)$: GOS formula for $\mathbf{P}^1$.

$B(n-1) \Rightarrow A(n)$: Commutativity is known except for the coefficient of the 0-section. By Corollary 3.0.2 and $B(n-1)$, the coefficient of the 0-section in $CCR\mathcal{F}$ equals rank $R\mathcal{F} = \chi(H, \mathcal{F}|_H) = (CCh^*\mathcal{F}, T^*_H H)_{T^*H}$ for the immersion $h \colon H \to \mathbf{P}$ of a generic hyperplane, by proper base change theorem. The right hand side equals $(h^! CC\mathcal{F}, T^*_H H)_{T^*H}$ by (3) for immersion and the coefficient of the 0-section in $LCC\mathcal{F}$.

$A(n) \Rightarrow B(n)$ for $n \neq 1$: The coefficient of the 0-section is given by

$$p_0 \colon \mathrm{CH}^n(\mathbf{P}(T^*\mathbf{P}^\vee \oplus \mathbf{A}^1_{\mathbf{P}^\vee})) = \mathrm{CH}^\bullet(\mathbf{P}^\vee) \to \mathrm{CH}^0(\mathbf{P}^\vee) = \mathbf{Z}.$$

By (3.4) and $A(n)$, we have $(-1)^n(n-1)\chi = p_0(L^\vee L - 1)cc$. Since $n \neq 1$, this implies that $d = (-1)^n p_0(L^\vee L - 1)/(n-1) \colon \mathrm{CH}_\bullet(\mathbf{P}) \to \frac{1}{n-1}\mathbf{Z}$ satisfies $\chi = d \circ cc$. It suffices to show that d is given by the intersection product $(\ , T^*_{\mathbf{P}}\mathbf{P})_{\mathbf{P}(T^*\mathbf{P} \oplus \mathbf{A}^1_{\mathbf{P}})}$ with the 0-section.

It suffices to show the equalities $d[L] = (L, T^*_{\mathbf{P}}\mathbf{P})_{\mathbf{P}(T^*\mathbf{P}\oplus\mathbf{A}^1_{\mathbf{P}})}$ for generators of $\mathrm{CH}_\bullet(\mathbf{P})$. The Chow group $\mathrm{CH}_\bullet(\mathbf{P}) = \bigoplus_{q=0}^n \mathbf{Z}$ is generated by the classes of linear subspaces L and hence by $\overline{CC}i_{L*}\Lambda$. The assertion follows from $d[L] = \chi(L_{\bar{k}}, \Lambda) = (-1)^{\dim L}(T^*_L L, T^*_L L)_{T^*L} = (-1)^{\dim L}(T^*_L\mathbf{P}, T^*_{\mathbf{P}}\mathbf{P})_{T^*\mathbf{P}} = (CCi_{L*}, T^*_{\mathbf{P}}\mathbf{P})_{T^*\mathbf{P}}$.

Composing with p_0, $(A(n))$ gives (4-2) for the coefficient of the 0-section.

7 (3) for immersion and (4-2)

To show (3) in the case where h is an immersion, we first consider the case where X is an projective space $\mathbf{P}^n$.

Lemma 7.0.1. *Let $h\colon W \to P = \mathbf{P}^n$ be an immersion and*

$$\begin{tikzcd} W \arrow[d, "h"'] & W \times_P Q \arrow[l, "p_W"'] \arrow[r, "p_W^\vee"] \arrow[d] & P^\vee \\ P & Q \arrow[l, "p"'] & \end{tikzcd}$$

be the cartesian diagram. Let $\mathcal{G}$ be a constructible complex on $P^\vee$ micro supported on $C^\vee$ and assume that h is properly C-transversal for $C = L^\vee C^\vee$. Then, we have

$$\mathbf{P}(CCRp_{W*}p_W^{\vee*}\mathcal{G}) = \mathbf{P}(p_{W!}p_W^{\vee!}CC\mathcal{G}).$$

Proof. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $p_{W!}p_W^{\vee!}CC\mathcal{G}$ satisfies the Milnor formula for $Rp_{W*}p_W^{\vee*}\mathcal{G}$ and for *smooth* morphisms $f\colon W \to Y$ to a curve defined locally on W. Since h is C-transversal, $p_W^\vee\colon Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is $C^\vee$-transversal by Lemma **??**.2 and $p_W^*\mathcal{G}$ is micro supported on $p_W^\vee C^\vee$. Since $p_W^\vee\colon Q \times_{\mathbf{P}} W \to \mathbf{P}^\vee$ is smooth outside $\mathbf{P}(C_W)$, we have $CCp_W^{\vee*}\mathcal{G} = p_W^{\vee\circ}CC\mathcal{G}$ outside $\mathbf{P}(C_W)$ as (3) is already proved for smooth morphisms.

Assume that f is smooth and has only isolated characteristic point. Then, by Lemma **??**.2, the composition fp_W is $p^\vee C$-transversal outside the inverse images of the characteristic points. Further it is $p^\vee C$-transversal outside of finitely many closed points in the inverse images by Lemma **??**.3 and these points are not contained in $\mathbf{P}(C_W)$ by Lemma **??**.1. Hence the assertion follows.

Lemma 7.0.1 implies also $\mathbf{P}(CCh^*\mathcal{F}) = \mathbf{P}(h^!CC\mathcal{F})$. Since the coefficient of the 0-section is determined by the generic rank as in (**??**), we deduce (3) in the case $X = \mathbf{P}$. In the general case, since the assertion is local, we may assume that there exists an open subscheme $U \subset \mathbf{P}$ and a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, "j"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "i"] \\ V \arrow[r, "g"] & U \subset P \end{tikzcd}$$

where $i\colon X \to U$ and $g\colon V \to U$ are closed immersions of smooth subschemes meeting transversely. Then, since h is properly C-transversal, g is properly $i_\circ C$-transversal. Hence the case where $X = \mathbf{P}$ implies $CCg^*i_*\mathcal{F} = g^!CCi_*\mathcal{F} = g^!i_!CC\mathcal{F}$. This implies $j_!CCh^*\mathcal{F} = CCj_*h^*\mathcal{F} = j_!h^!CC\mathcal{F}$ and (**??**).

We show (5). The case $W = \mathbf{P}$ in Lemma 7.0.1 means the projectivization

$$\mathbf{P}(CCR\mathcal{F}) = \mathbf{P}(LCC\mathcal{F}) \tag{7.1}$$

of (5). Hence it remains to show that the coefficients of the 0-section in $CCR\mathcal{F} = LCC\mathcal{F}$ are the same. Similarly as in the proof of Corollary **??**, this is equivalent to the index formula (**??**) for $X = \mathbf{P}^n$. To prove this, we introduce the characteristic class.

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