Contents

1 Introduction

Weil conjectures.

Exercise. Compute $Z(\mathbf{P}^n_{\mathbf{F}_p}, t)$ and show $Z(\mathbf{P}^n_{\mathbf{F}_p}, t) = (-1)^{n+1}p^{(n+1)/2}t^{n+1}Z(\mathbf{P}^n_{\mathbf{F}_p}, 1/p^n t)$

$Z(X, t) = \det(1 - Ft\colon H^q(X_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell))^{(-1)^{q+1}}$.

6 operations.

micro local.

$\mathcal{F}$: an étale sheaf on a smooth scheme X over a field k.

November: Existence of $SS\mathcal{F} = \bigcup C_a \subset T^*X$.

Dimension $\dim C_a = \dim X$ for every irreducible component.

February: $CC\mathcal{F} = \sum_a m_a C_a$. Index formula X projective $\chi(X_{\overline{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$.

2 Étale topology

X scheme. étale morphism. locally of finite presentation, flat and $\Omega^1_{U/X} = 0$. The last condition is equivalent to; for every $u \in U$ and $x = f(u) \in X$, the local ring $\mathcal{O}_{U,u} \otimes_{\mathcal{O}_{X,x}} k(x)$ is a finite separable extension of $k(x)$.

Exercise. Give an example of étale morphism that is not neither an open immersion or finite extension or their base change.

$X_{\text{ét}}$ étale site of X.

Objects: étale morphisms $U \to X$.

Morphisms: (étale) morphisms $U \to V$ over X.

Presheaves: contravariant functors $\mathcal{F}\colon X_{\text{ét}} \to$ (Abelian groups).

Sheaf condition: For every family $(U_i \to U)_{i\in I}$ such that $U = \bigcup_{i\in I} \mathrm{Im(U_i \to U)}$, the morphism $\mathcal{F}(U) \to \mathrm{Ker}(\prod_{i\in I} \mathcal{F}(U_i) \to \prod_{i,j\in I} \mathcal{F}(U_i \times_U U_j))$ is an isomorphism.

$\mathcal{O}_X$ is a sheaf. More generally, quasi-coherent $\mathcal{O}_X$-modules are sheaves. Abelian category $\mathrm{Sh}/X_{\text{ét}}$. Enough injectives. cohomology.

Locally constant constructible if and only if representable by finite étale scheme.

stalk at geometric point $\mathcal{F}_x = \varinjlim \mathcal{F}(U)$. $\mathcal{O}_{X,x}$ a strictly local ring. Hensel's lemma + residue field is separably closed.

conservative. specialization.

Example: $X = \mathrm{Spec}\, K$. $G_K = \mathrm{Gal}(\overline{K}/K)$ the absolute Galois group. Equivalences of categories $X_{\text{ét}} \to$ (discrete sets with continuous G_K-action) sending U to $U(\overline{K})$, $\mathrm{Sh}/X_{\text{ét}} \to$ (G_K-modules) sending $\mathcal{F}$ to $M = \mathcal{F}(\overline{K}) = \varinjlim \mathcal{F}(L)$. $H^q(X,\mathcal{F}) = H^q(G_K, M)$ Galois cohomology.

Construction: G finite group. M representation of G on a Λ-vector space of finite dimension. Étale G-torsor over X is a finite étale scheme W over X with G-action on X such that $W \to X$ is surjective and that $(\mathrm{pr}_2, \mu)\colon G \times W \to W \times_X W$ is an isomorphism. Then the presheaf $\mathcal{F}$ defined by $\mathcal{F}(U) = \{G$-equivariant locally constant functions $W \times_X U \to M\}$ is a locally constant constructible sheaf of Λ-modules. The restriction $\mathcal{F}|_W$ is the constant sheaf M.

Exercise. k field of characteristic $p > 0$. $G = \mathbf{F}_p$. $W = \mathbf{A}^1_k = \mathrm{Spec}\, k[t] \to X = \mathbf{A}^1_k = \mathrm{Spec}\, k[x]$. Action of $i \in \mathbf{F}_p = G$ on W by sending t to $t + i$.

1. Show that W is an étale G-torsor over X.

2. Construct a locally constant sheaf $\mathcal{L}$ of Λ-modules of rank 1 on X by taking an injection $\mathbf{F}_p \to \Lambda^\times$.

Relative. $f\colon X \to Y$ morphism of schemes. $f^*\colon Y_{\text{ét}} \to X_{\text{ét}}$. direct image $f_* : \mathrm{Sh}/X_{\text{ét}} \to \mathrm{Sh}/Y_{\text{ét}}$. If f is finite, f_* is exact. If f is universally homeomorphism, f_* is an isomorphism.

Exercise. Decompose $f_*\Lambda$ above as a $\Lambda[G] = \prod \Lambda$-module to obtain $\mathcal{L}$.

adjoint $f^* : \mathrm{Sh}/Y_{\text{ét}} \to \mathrm{Sh}/X_{\text{ét}}$.

higher direct image $R^q f_* : \mathrm{Sh}/X_{\text{ét}} \to \mathrm{Sh}/Y_{\text{ét}}$. $(R^q f_* \mathcal{F})_x = H^q(Y \times_X X_x, \mathcal{F})$, $X_x = \mathrm{Spec}\, \mathcal{O}_{X,x}$.

Exercise. 1 $j\colon U \to X$ open immersion, j^* is the restriction.

2. i closed immersion. $i^! = i^* \mathcal{H}om(i_* \mathbf{Z}, -)$.

$j\colon U \to X$ open immersion, $i\colon Y \to X$ closed immersion of the complement. $j_! = \mathrm{Ker}(j_* \to i_* i^* j_*)$.

Constructible: There exists a finite stratification $X = \coprod X_i$ by locally closed subsets such that the restrictions $\mathcal{F}|_{X_i}$ are locally constant constructible. support the complement of the largest open subset U such that $\mathcal{F}|_U = 0$.

Exercise. If we replace $\mathbf{A}^1$ by $\mathbf{P}^1$, we obtain a constructible sheaf but not locally constant sheaf.

specialization $\mathcal{F}_s \leftarrow \Gamma(X_x, \mathcal{F}) \to \mathcal{F}_t$.

Lemma 2.0.1. *Let $\mathcal{F}$ be a constructible sheaf. Then the following conditions are equivalent.*

$\mathcal{F}$ is locally constant.

Every specialization $\mathcal{F}_s \to \mathcal{F}_t$ is an isomorphism.

derived category $D(X_{\text{ét}})$. Category of complexes. invert quasi-isomorphisms. $Rf_*\colon D^+(Y_{\text{ét}}) \to D^+(X_{\text{ét}})$. $f^*\colon D(X_{\text{ét}}) \to D(Y_{\text{ét}})$.

$D^b_c(X,\Lambda)$. ℓ invertible on X, Λ finite extension of $\mathbf{F}_\ell$. $\mathcal{H}^q(\mathcal{F})$ constructible for every q, is 0 except finitely many q. support the complement of the largest open subset U such that $\mathcal{H}^q\mathcal{F}|_U = 0$ for every q. locally constant $\mathcal{H}^q\mathcal{F}$ is locally constant for every q.

Lemma 2.0.2. *Assume that $X = \operatorname{Spec}\mathcal{O}_K$. Then the following conditions are equivalent.*
(1) *$\mathcal{F}$ is locally constant.*
(2) *$\mathcal{F} \otimes Rj_*\Lambda \to Rj_*j^*\mathcal{F}$ is an isomorphism.*

Exercise. K strictly henselian discrete valuation field. $H^1(K,M) = M_I(-1)$.
f separated of finite type. $Rf_! = R\overline{f}_* \circ j_!$.
6 operations. $Rf_*, f^*, Rf_!, Rf^!, \otimes, R\mathcal{H}om$ preserve D^b_c.
X smooth over a field k. $\mathcal{F}$ sheaf = object of $D^b_c(X,\Lambda)$.
$C = SS\mathcal{F} \subset T^*X$. closed conical subset.
support of $\mathcal{F}$ closed subset of X.
upgrade
$\mathcal{F}$ is micro supported on C. properties of $\mathcal{F}$ is controlled by properties of C.
Singular support: smallest C on which $\mathcal{F}$ is micro supported.

3 $\mathcal{F}$-transversality

$h\colon W \to X$ separated morphism of finite type. $\mathcal{F}$ sheaf on X.
canonical morphism

$$c_{h,\mathcal{F}}\colon h^*\mathcal{F} \otimes h^!\Lambda \to h^!\mathcal{F}.$$

by adjunction $h_!(h^*\mathcal{F} \otimes h^!\Lambda) \to \mathcal{F}$. Projection formula $\mathcal{F} \otimes h_!h^!\Lambda \to h_!(h^*\mathcal{F} \otimes h^!\Lambda)$. j open immersion, the inverse of the adjoint of the isomorphism $j^*\mathcal{F} \otimes \mathcal{G} \to j^*(\mathcal{F} \otimes j_!\mathcal{G})$. h proper, adjoint of $1 \otimes \mathrm{adj}\colon h^*\mathcal{F} \otimes h^*Rh_*\mathcal{G} \to h^*\mathcal{F} \otimes \mathcal{G}$. isomorphism, by proper base change, fiberwise $\mathcal{F} \otimes R\Gamma(W,\mathcal{G}) \to R\Gamma(X, \mathcal{F} \otimes \mathcal{G})$.
Thus $c_{h,\mathcal{F}}$ is the adjoint of $1 \otimes \mathrm{adj}$.

Definition 3.0.1. *$\mathcal{F}$-transversal.*

later $\mathcal{F}$ perverse sheaf, h $\mathcal{F}$-transversal, then $h^*\mathcal{F}$ perverse sheaf.

Lemma 3.0.2. *Assume that h is $\mathcal{F}$-transversal.*
1. *Then the following are equivalent.*
(1) *hg is $\mathcal{F}$-transversal.*
(2) *g is $h^*\mathcal{F}$-transversal.*
2. *$f\colon X' \to X$ smooth. Then $h'\colon W' \to X'$ is $f^*\mathcal{F}$-transversal.*

Lemma 3.0.3. *Let $h\colon W \to X$ be a closed immersion and let $j\colon U = X - W \to X$ be the open immersion of the complement. Then, the following conditions are equivalent:*
h is $\mathcal{F}$-transversal.
The morphism $\mathcal{F} \otimes Rj_*\Lambda \to Rj_*j^*\mathcal{F}$ is an isomorphism.

Lemma 3.0.4. *$i\colon X \to X \times \mathbf{A}^1$.*
i is $i_*\mathcal{F}$-transversal.
$\mathcal{F} = 0$.

Proposition 3.0.5. 1. *If $\mathcal{F}$ is locally constant, every morphism h is $\mathcal{F}$-transversal.*

2. *If h is smooth, for every sheaf $\mathcal{F}$, h is $\mathcal{F}$-transversal.*

2. Poincaré duality.

Conversely

Proposition 3.0.6. *Assume that X is regular. Assume that for every morphism h of regular schemes of finite type, h is $\mathcal{F}$-transversal. Then $\mathcal{F}$ is locally constant.*

Let $S \subset T \subset X$ be the closure of the images of s and t. Let T' be the normalization of the blow up of T at S and let $D \subset T'$ be the exceptional divisor. By replacing T' by the localization at the generic point of D, it is reduced to dvr.

Smooth base change.

Proposition 3.0.7. *Let*

$$\begin{tikzcd} X' & W' \arrow[l, "h'"'] \arrow[d, "g'"] \\ X \arrow[from=u, "g"'] & W \arrow[l, "h"'] \end{tikzcd}$$

be a cartesian diagram. Assume that $g'^*h^!\Lambda \to h^!\Lambda$ is an isomorphism.

1. *Assume that h' is $\mathcal{F}'$-transversal. Then the following conditions are equivalent:*

(1) *h is $g_*\mathcal{F}'$-transversal*

(2) *$h^*g_*\mathcal{F}' \to g'_*h'^*\mathcal{F}'$ is an isomorphism.*

2. *Assume that g is finite. Then the following conditions are equivalent:*

(1) *h is $g_*\mathcal{F}'$-transversal*

(2) *h' is $\mathcal{F}'$-transversal.*

The condition (1) is satisfied if h is smooth. The condition (2) is satisfied if g is proper on the support of $\mathcal{F}'$.

Proof. 1. We consider the commutative diagram

$$\begin{array}{ccc} h^*g_*\mathcal{F}' \otimes h^!\Lambda & \longrightarrow & h^!g_*\mathcal{F}' \\ \downarrow & & \downarrow \\ g'_*(h'^*\mathcal{F}' \otimes h'^!\Lambda) & \longrightarrow & g'_*h'^!\mathcal{F}'. \end{array}$$

By proper base change theorem, the right vertical arrow is an isomorphism. By the assumption, the lower horizontal arrow is an isomorphism. Further by the assumption, the left vertical arrow is an isomorphism if and only if (2) holds.

2. The vertical arrows are isomorphisms. □

Lemma 3.0.8. *Let*

$$\begin{tikzcd} X \arrow[d, "f"'] & W \arrow[l, "h"'] \arrow[d, "f'"] \\ X' & W' \arrow[l, "h'"'] \end{tikzcd}$$

be a cartesian diagram of smooth schemes. Assume that the base change morphism $f'^*h'^!\Lambda \to h^!\Lambda$ is an isomorphism and that f is proper.

1. *If h is $\mathcal{F}$-transversal, then h' is $f_*\mathcal{F}$-transversal.*

2. *Assume that there exists a closed subset $E \subset W$ outside of which h is $\mathcal{F}$-transversal and that h' is $f_*\mathcal{F}$-transversal. Then h is $\mathcal{F}$-transversal.*

Proof. 1. We have a commutative diagram

$$\begin{array}{ccc} h'^* f_*\mathcal{F} \otimes h'^! \Lambda & \xrightarrow{c_{f_*\mathcal{F},h}} & h'^! f_*\mathcal{F} \\ \downarrow & & \downarrow \\ f'_*(h^*\mathcal{F} \otimes h^!\Lambda) & \xrightarrow{f'_* c_{\mathcal{F},h}} & f'_* h^! \mathcal{F} \end{array}$$

and the vertical arrows are isomorphisms.

2. By the assumption $c_{\mathcal{F},h}$ is an isomorphism on the complement $W - E$ finite over W'. Hence if $f'_* c_{\mathcal{F},h}$ is an isomorphism then $c_{\mathcal{F},h}$ is an isomorphism. □

Definition 3.0.9. *$(h, f): W \to X \times Y$ is $\mathcal{F}$-acyclic if for any $\mathcal{G}$ on Y, (h, f) is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.*

We say that $(h, f): W \to X \times Y$ is universally $\mathcal{F}$-acyclic if for every smooth morphism $Y' \to Y$, the pair (h', f') of morphisms $h': W' = W \times_Y Y' \to W \to X$ and $f': W' \to Y'$ is $\mathcal{F}$-acyclic.

We say $f: X \to Y$ is $\mathcal{F}$-acyclic if $(1, f)$ is $\mathcal{F}$-acyclic. We say $f: X \to Y$ is universally $\mathcal{F}$-acyclic if for every smooth morphism $Y' \to Y$, the base chane $f': X' \to Y'$ is $\mathcal{F}$-acyclic.

h is $\mathcal{F}$-transversal if and only if $(h, \mathrm{can}): W \to X \times \operatorname{Spec} k$ is $\mathcal{F}$-acyclic.

f is universally $\mathcal{F}$-acyclic if and only if $(1, f)$ is universally $\mathcal{F}$-acyclic.

Example. 1. If $(h, f): W \to X \times Y$ is smooth, (h, f) is $\mathcal{F}$-acyclic for every $\mathcal{F}$.

2. If $\mathcal{F}$ is locally constant and if $f: W \to Y$ is smooth, then (h, f) is $\mathcal{F}$-acyclic.

Lemma 3.0.10. 1. *Assume h is universally $\mathcal{F}$-transversal. Then,*

(hg, f) is $\mathcal{F}$-acyclic.

(g, f) is $h^*\mathcal{F}$-acyclic.

2. *Assume g is proper on the support of $\mathcal{F}'$. If (h', fg') is $\mathcal{F}'$-acyclic, then (h, f) is $g_*\mathcal{F}'$-acyclic.*

3. *Assume (h, f) is $\mathcal{F}$-acyclic and $f': Y \to Y'$ is smooth. Then, $(h, f'f)$ is $\mathcal{F}$-acyclic*

Proof. 1. $W \times Y \to X \times Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal. Apply Lemma 3.0.2 to $V \to W \times Y \to X \times Y$.

2. If $W' \to X' \times Y$ is $\mathcal{F}' \boxtimes \mathcal{G}$-transversal, then $W \to X \times Y$ is $g_*\mathcal{F} \boxtimes \mathcal{G}$-transversal.

3. $X \times Y \to X \times Y'$ is $\mathcal{F} \boxtimes \mathcal{G}'$-transversal. Apply Lemma 3.0.2 to $W \to X \times Y \to X \times Y'$. □

Construction. Assume $g: X' \to X$ proper. $fg: X' \to Y$ is smooth. $\mathcal{F}$ direct summand of $g_*\mathcal{F}$. Then, $(1_X, f)$ is $\mathcal{F}$-acyclic.

Exercise. 1. Compute the normalization X of $\mathbf{P}^1 \times \mathbf{A}^1$ in the finite étale covering of $\mathbf{A}^2$ defined by $t^p - t = x^p y$ and show that $\mathrm{pr}_1: X \to \mathbf{A}^1$ is smooth.

2. Deduce from 1 that $(j \times 1)_! \mathcal{L}(x^p y)$ on $\mathbf{P}^1 \times \mathbf{A}^1$ is $(1, \mathrm{pr}_1)$-acyclic.

3. Show that $(j \times 1)_! \mathcal{L}(xy)$ on $\mathbf{P}^1 \times \mathbf{A}^1$ is $(1, \mathrm{pr}_1)$-acyclic.

Lemma 3.0.11. *Assume that $(1, f)$ is universally $\mathcal{F}$-acyclic. Then, for any morphism $g: Y' \to Y$ and any sheaf $\mathcal{G}'$ on Y', the morphism $\mathcal{F} \otimes f^* g_* \mathcal{G}' \to g'_*(g'^*\mathcal{F} \otimes f'^*\mathcal{G}')$ is an isomorphism.*

Proof. By decomposing $g = hi$ as the composition a smooth morphism h and a closed immersion, we may consider the 2 cases separately. If g is proper, the assertion follows from the projection formula and the proper base change theorem. Assume g is smooth and consider the cartesian diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \times Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X \times Y \end{array}$$

and the commutative diagram

$$\begin{array}{ccc} \mathcal{F} \otimes f^* g_* \mathcal{G}' \otimes \gamma^! \Lambda & \longrightarrow & \gamma^!(\mathcal{F} \boxtimes g_* \mathcal{G}') \\ \downarrow & & \downarrow \\ g'_*(g'^* \mathcal{F} \otimes f'^* \mathcal{G}' \otimes \gamma'^! \Lambda) & \longrightarrow & g'_* \gamma'^! (\mathcal{F} \boxtimes \mathcal{G}'). \end{array}$$

Since f is assumed universally $\mathcal{F}$-acyclic, the horizontal arrows are isomorphisms. By pbc, the right vertical arrow is an isomorphism. Hence the right vertical arrow is an isomorphism. □

4 C-transversality

C closed conical subset is determined by its base and the projectivization.

Definition 4.0.1. *h is C-transversal if the intersection with the kernel is a subset of the 0-section.*

Example. Let $Z \subset X$ be a closed subscheme smooth over k and let $C = T^*_Z X$. Then, h is C-transversal means that h is transversal to the immersion $i\colon Z \to X$. Namely, $Z \times_X W$ is smooth over k and the codimensions are the same. This implies that h is $i_*\Lambda$-transversal.

Lemma 4.0.2. *If h is C-transversal and $C' \subset C$, then h is C'-transversal.*

Lemma 4.0.3. $\{w \in W \mid C_w \cap \subset 0\}$ *is an open subset.*

$h^\circ C$.

Proof. 1. Projectivization

2. Graded ring. □

Exercise. Let $A \to B$ be a morphism of graded rings and $J \subset B$ be the graded ideal. Let $S \subset B$ be a set of homogeneous elements. Show that the following conditions are equivalent:

(1) The A-module B/J is generated by the image of S.

(2) The $A/A_{\geqslant 1}$-module $B/(J + A_{\geqslant 1}B)$ is generated by the image of S.

(2)⇒(1): By induction on n, the $A/A_{\geqslant n}$-module $B/(J + A_{\geqslant n}B)$ is generated by the image of S. Since $A_{\geqslant n}B \subset B_{\geqslant n}$, the $A/A_{\geqslant n}$-module $B/(J + B_{\geqslant n})$ is also generated by the image of S. Hence it suffices to apply the following lemma to $M = B/J$.

Lemma 4.0.4. *Let A be a graded ring and M be a graded A-module. Let S be a subset of M consisting of homogeneous elements. Then the following are equivalent:*

(1) *The A-module M is generated by S.*

(2) *For every $n \geqq 0$, the $A/A_{\geqslant n}$-module $M/M_{\geqslant n}$ is generated by the image of S.*

By replacing M by the quotient generated by S, we may assume that $S = \varnothing$. Then the assertion is clear.

Proposition 4.0.5. 1. *The following conditions are equivalent.*
(1) *h is T^*X-transversal.*
(2) *h is smooth.*
2. *If C is a subset of the 0-section, then every h is C-transversal.*

Proposition 4.0.6. 1. *Assume h is C-transversal. The following conditions are equivalent.*
(1) *hg is C-transversal.*
(2) *g is $h^\circ C$-transversal.*
2. *Assume that f is proper on the support of C'. The following conditions are equivalent.*
(1) *h' is C'-transversal.*
(2) *h is $f_\circ C'$-transversal.*

Definition 4.0.7. *We say (h, f) is C-acyclic if (h, f) is $C \times T^*Y$-transversal.*

Example 1. If (h, f) is smooth, (h, f) is C-acyclic for every C.
2. (h, f) is $T^*_X X$-acyclic if and only if f is smooth.

Lemma 4.0.8. *Assume that (h, f) is C-acyclic and let $B \subset C$ be the base of C.*
1. *f is smooth on a neighborhood $W_1 \subset W$ of the inverse image $h^{-1}(B)$.*
2. *Let $Y' \to Y$ be a morphism of smooth schemes over k and let $W_1' = W_1 \times_Y Y' \subset W' = W \times_Y Y'$. Let (h_1', f_1') be the restrictions of the composition $h' = h\mathrm{pr}_1 \colon W' \to X$ and the base change $f' \colon W' \to Y'$. Then (h_1', f_1') is C-acyclic.*

Lemma 4.0.9. 1. *The following conditions are equivalent.*
(1) *(h, f) is T^*X-acyclic.*
(2) *(h, f) is smooth.*
2. *The following conditions are equivalent.*
(1) *Every (h, f) such that f is smooth is C-acyclic.*
(2) *C is a subset of the 0-section.*

The condition that $T^*Y \to T^*W$ is injective means that the intersection of $\mathrm{Ker}(T^*X \times T^*Y \to T^*W)$ with $T^*_X X \times T^*Y$ is a subset of the 0-section.

For $(h, f) = (1, 1)$, the kernel $\mathrm{Ker}(T^*X \times T^*X \to T^*X)$ is the diagonal and its intersection with $C \times T^*X$ is identified with C.

Lemma 4.0.10. 1. *Assume h is C-transversal.*
(1) *(hg, f) is C-acyclic.*
(2) *(g, f) is $h^\circ C$-acyclic.*
2. *Assume g is proper on the support of C'.*
(1) *(h', fg') is C'-acyclic.*
(2) *(h, f) is $f_\circ C'$-acyclic.*

5 Micro support

Definition 5.0.1. *$\mathcal{F}$ is ms on C if every C-acyclic (h, f) is $\mathcal{F}$-acyclic.*

Since C-acyclic implies universally C-acyclic, we may replace $\mathcal{F}$-acyclic by universally $\mathcal{F}$-acyclic.

Proposition 5.0.2. 1. *Every $\mathcal{F}$ is micro supported on T^*X.*

2. *$\mathcal{F}$ is micro supported on $\varnothing$*
$\mathcal{F} = 0$.
3. *$\mathcal{F}$ is micro supported on $T^*_X X$*
$\mathcal{F}$ is locally constant.

1. (h, f) T^*X-acyclic means (h, f) is smooth.

2. Every (h, f) is $\varnothing$-acyclic. $f = 0\colon X = W \to Y = \mathbf{A}^1_k$. $\mathcal{G} = i_*\Lambda$. $\mathcal{F} \otimes Ri^!\Lambda \to Ri^!i_*\mathcal{F} = \mathcal{F}$

3. (h, f) is $T^*_X X$-acyclic if and only if f is smooth. Example at the end of 3.

Conversely, for $Y = \operatorname{Spec} k$, every h is $T^*_X X$-acyclic. Hence every h is $\mathcal{F}$-transversal. This means $\mathcal{F}$ is locally constant.

Alternatively, the diagonal $\delta\colon X \to X \times X$ is $\mathcal{F} \boxtimes D_X\mathcal{F}$ transversal. This implies that $\mathcal{F} \otimes D_X\mathcal{F} \to R\mathcal{H}om(\mathcal{F}, \mathcal{F})$ is an isomorphism. Then $\mathcal{F}$ is locally acyclic by [Lu-Zheng Theorem 2.16].

More alternatively, for any morphism $g\colon V \to U$ over X of smooth schemes over k, $\gamma\colon U \to X \times U$ is $\mathcal{F} \boxtimes g_*\Lambda$-transversal and $\gamma'\colon V \to X \times V$ is $\mathcal{F} \boxtimes \Lambda$-transversal. Hence by Proposition 3.0.7.1, $\mathcal{F} \otimes g_*\Lambda \to g_*g^*\Lambda$ is an isomorphism. This implies that $\mathcal{F}$ is locally acyclic.

Proposition 5.0.3. 1. *If $\mathcal{F}$ is micro supported on C and if h is C-transversal, then $h^*\mathcal{F}$ is micro supported on $h^\circ C$.*

2. *If $\mathcal{F}'$ is micro supported on C' and if g is proper on the base of C, then $g_*\mathcal{F}$ is micro supported on $g_\circ C$.*

Corollary 5.0.4. *If $\mathcal{F}$ is micro supported on C, the support of $\mathcal{F}$ is a subset of the base of C.*

$j^*\mathcal{F}$ is micro supported on $\varnothing$.

Lemma 5.0.5. 1. *Assume that $\mathcal{F}$ is micro supported on C and that $\mathcal{F}|_U$ is micro supported on C'. Then $\mathcal{F}$ is micro supported on $C|_{X - U} \cup \overline{C}'$.*

2. *$\mathcal{F}$ is micro supported on C if and only if $\mathcal{F}|_{U_i}$ is micro supported on $C|_{U_i}$.*

3. *Let $\to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be a distinguished triangle and suppose that $\mathcal{F}'$ and $\mathcal{F}''$ are micro supported on C' and on C'' respectively. Then $\mathcal{F}$ is micro supported on $C = C' \cup C''$.*

1. open

Lemma 5.0.6. *Let $i\colon X \to P$ be a closed immersion*

1. *Assume that $\mathcal{F}$ is micro supported on $C \subset T^*X$. Then, $i_*\mathcal{F}$ is micro supported on $i_\circ C \subset T^*P$.*

2. *Assume that $i_*\mathcal{F}$ is micro supported on $C_P \subset T^*P|_X \subset T^*P$. Let $C \subset T^*X$ be the closure of the image of C_P by the surjection $T^*P|_X \to T^*X$. Then, $\mathcal{F}$ is micro supported on C.*

3. *Assume that $i_*\mathcal{F}$ is micro supported on $C_P \subset T^*P|_X \subset T^*P$. Let s be a section of the surjection $T^*P|_X \to T^*X$. Then, $\mathcal{F}$ is micro supported on $s^{-1}C_P \subset T^*X$.*

Proof. 1. Let $h\colon V \to P$ and $f\colon V \to Y$ be morphisms of smooth schemes over k such that (h, f) is $i_\circ C$-acyclic. Then, $X \to P$ and $V \to P$ is transversal on a neighborhood of the basis of C. Hence shrinking V, we may assume that $W = V \times_P X$ is smooth. Further the pair (h', f') of the base change $h'\colon W \to X$ and the composition $f'\colon W \to Y$ is C-acyclic and the assertion follows.

2. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that (h, f) is C-acyclic. We show that (h, f) is $\mathcal{F}$-acyclic. By replacing P and X by $P \times W$ and $X \times W$, we may assume that h is an immersion. Then locally on $P \times Y$, there exists a transversal cartesian diagram

$$\begin{array}{ccc} X \times Y & \longleftarrow & W \\ \big\downarrow & & \big\downarrow \\ P \times Y & \longleftarrow & V \end{array}$$

of smooth schemes over k. Then the pair of $V \to P$ and $V \to Y$ is C_P-acyclic and the assertion follows.

3. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k such that (h, f) is $s^{-1}C_P$-acyclic. We show that (h, f) is $\mathcal{F}$-acyclic. As in the proof of 2, we may assume that $h\colon W \to X$ is an immersion. Then, we may take V as loc. cit. further satisfying that $T^*_V(P\times Y)|_W \subset T^*(P\times Y)|_W$ equals the image of $T^*_W(X\times Y) \subset T^*(X\times Y)|_W$ by the section $s\colon T^*X \to T^*P|_X$ times 1_{T^*Y}. Then the pair of $V \to P$ and $V \to Y$ is C_P-acyclic and the assertion follows. $\square$

Lemma 5.0.7. *Let k' be a finite separable extension of k.*

1. *If $\mathcal{F}$ is micro supported on C, then $\mathcal{F}_{k'}$ is micro supported on $C_{k'}$.*
2. *If $\mathcal{F}_{k'}$ is micro supported on C', then $\mathcal{F}$ is micro supported on $p_\circ C'$.*

Proof. 1. Let (h', f') be a pair of morphisms of smooth schemes over k'. Then, (h', f') is a pair of morphisms of smooth schemes over k, $\mathcal{F}_{k'}$ is micro supported on $p^\circ C = C_{k'}$.

2. Let (h, f) be a pair of morphisms of smooth schemes over k. Then, its base change (h', f') is a pair of morphisms of smooth schemes over k'. If (h, f) is $p_\circ C'$-acyclic, then (h', f') is C'-acyclic. If (h', f') is $\mathcal{F}_{k'}$-acyclic, then (h, f) is $\mathcal{F}$-acyclic. Hence $\mathcal{F}$ is micro supported on $p_\circ C'$. $\square$

6 Singular support

Definition 6.0.1. *C is the singular support $SS\mathcal{F}$ of $\mathcal{F}$ if $\mathcal{F}$ is micro supported on C' and is equivalent to $C' \supset C$.*

Lemma 6.0.2. *The following conditions are equivalent:*

(1) *$SS\mathcal{F}$ exists.*

(2) *If $\mathcal{F}$ is micro supported on C_1 and on C_2, then $\mathcal{F}$ is micro supported on $C_1 \cap C_2$.*

(3) *$\mathcal{F}$ is micro supported on $C_0 = \bigcap C$ where $C \subset T^*X$ runs closed conical subsets on which $\mathcal{F}$ is micro supported.*

Proof. (1)⇒(2): If $SS\mathcal{F} \subset C_1$ and $SS\mathcal{F} \subset C_2$, then $SS\mathcal{F} \subset C_1 \cap C_2$.

(2)⇒(3): Since $T^*X - C_0$ is quasi-compact and $\mathcal{F}$ is micro supported on T^*X, there exists finitely many closed conical subsets $C_1, \ldots, C_n, n \geqslant 1$ on which $\mathcal{F}$ is micro supported satisfying $C_0 = C_1 \cap \cdots \cap C_n$.

(3)⇒(1): $C_0 = SS\mathcal{F}$. □

Theorem 6.0.3. 1. *$SS\mathcal{F}$ exists.*

2. *Every irreducible component of $SS\mathcal{F}$ has the same dimension as X.*

Lemma 6.0.4. 1. *Assume that $SS\mathcal{F}$ exists. Then, $SS(\mathcal{F}|_U) = (SS\mathcal{F})|_U$.*

2. *If $SS\mathcal{F}|_{U_i}$ exists for every i, then $\bigcup SS\mathcal{F}|_{U_i}$ is $SS\mathcal{F}$.*

1. $\mathcal{F}|_U$ micro supported on C'. Then, $\mathcal{F}$ is micro supported on $T^*X|_{X-U} \cup \overline{C}' \supset SS\mathcal{F}$.

Lemma 6.0.5. *Let k' be a finite Galois extension of k. If $SS\mathcal{F}$ and $SS\mathcal{F}_{k'}$ exist, then we have $SS\mathcal{F}_{k'} = p^*SS\mathcal{F}$.*

Proof. Let $C = SS\mathcal{F}$ and $C' = SS\mathcal{F}_{k'}$. Then, by Lemma 5.0.7, we have $C \subset p_\circ C'$ and $C' \subset p^\circ C \subset p^\circ p_\circ C'$. Since C' is G-stable, we have $p^\circ p_\circ C' = C'$ and the assertion follows. □

Proposition 6.0.6. *Assume that $SSi_*\mathcal{F}$ exists.*

1. *$SS\mathcal{F}$ exists.*

2. *If k is infinite, we have $SSi_*\mathcal{F} = i_\circ SS\mathcal{F}$.*

3. *If k is finite, assume further that $SSi_{k'*}\mathcal{F}_{k'}$ exists for any finite extension k' of k. Then we have $SSi_*\mathcal{F} = i_\circ SS\mathcal{F}$.*

Proof. 1. Let $C_P = SSi_*\mathcal{F} \subset T^*P|_X \subset T^*P$ and define $C \subset T^*X$ to be the closure of the image of C_P by the surjection $T^*P|_X \to T^*X$ as in Lemma 5.0.6.2. Then, $\mathcal{F}$ is micro supported on C by loc. cit. We show that C is the smallest. Assume that $\mathcal{F}$ is micro supported on $C' \subset T^*X$. Then, $i_*\mathcal{F}$ is micro supported on $i_\circ C' \subset T^*P$ by Lemma 5.0.6.1. Since C_P is the smallest, we have $C_P \subset i_\circ C'$. This implies $C \subset C'$ and hence C is the smallest.

2. If s is a section of the surjection $T^*P|_X \to T^*X$, $\mathcal{F}$ is micro supported on $s^{-1}C_P \subset T^*X$ by Lemma 5.0.6.3. Since C is the smallest, we have $C \subset s^{-1}C_P$. The other inclusion is obvious and we have $C = s^{-1}C_P$. If k is infinite, there are sufficiently many sections locally on X and the equality implies $i_\circ C = C_P$.

3. If k is finite, there are sufficiently many sections locally on X over finite extensions of k. Hence using Lemma 5.0.7, we also have $i_\circ C = C_P$. □

By Lemma, we may assume that X is affine. By Lemma, we may assume that X is $\mathbf{A}^n_k$. By Lemma, we may assume that X is $\mathbf{P}^n_k$.

Advantage of $\mathbf{P}^n_k$. Radon transform. The universal family Q of hyperplanes is $\mathbf{P}(T^*\mathbf{P}^n) \supset \mathbf{P}(C)$.

Let $j\colon U \to X$ be a smooth morphism and consider morphisms

$$T^*X \xleftarrow{j_*} T^*X \times_X U \xrightarrow{dj} T^*U.$$

For a closed conical subset $C \subset T^*U$, we define a closed conical subset $j_!C \subset T^*X$ by

$$j_!C = T^*X - (j_*(T^*X \times_X U - dj^{-1}(C))).$$

Proposition 6.0.7. *ASSUME that SS exists. Let $j\colon U \to X$ be a smooth surjection of smooth schemes over k and assume that $j^*\mathcal{F}$ is micro supported on $C \subset T^*U$. Then, $\mathcal{F}$ is micro supported on $j_!C$.*

Proof. First, we prove the case where j is étale. Let $h\colon W \to X, f\colon W \to Y$ be a pair of morphisms of smooth schemes over k such that (h, f) is $j_!C$-acyclic. Let $x \in X$ be any geometric point and show that (h, f) is $\mathcal{F}$-acyclic on an étale neighborhood of x. Let $n \geqslant 1$ be the degree of the fiber $U \times_X x = \{x_1, \ldots, x_n\}$ at x. Let $U_n = U \times_X \cdots \times_X U$ be the n-fold fiber product and $j_n\colon U_n \to X$ and $\mathrm{pr}_i\colon U_n \to U, i = 1, \ldots, n$ be the projections. Since SS exists, $j_n^*\mathcal{F}$ is micro supported on the intersection $C_n = \bigcap_{i=1}^n \mathrm{pr}_i^\circ C$.

Let $h_n\colon W_n = W \times_X U_n \to U_n$ be the base change of h and $f_n\colon W_n \to Y$ be the composition with f. Since (h, f) is $j_!C$-acyclic, the pair (h_n, f_n) is $j_n^\circ j_!C$-acyclic. At the geometric point $u = (u_1, \ldots, u_n)$, the fiber $C_{n,u} = \bigcap_{i=1}^n C_{u_i}$ equals $(j_n^\circ j_!C)_u$. Since the C-transversality is an open condition, (h_n, f_n) is C_n-acyclic on a Zariski neighborhood of u. Since $j_n^*\mathcal{F}$ is micro supported on C_n, the pair (h_n, f_n) is $j_n^*\mathcal{F}$-acyclic on the neighborhood of u as required.

We show the general case. Since $j^*\mathcal{F}$ is micro supported on $j^\circ T^*X$ and since SS exists, by replacing C by the intersection with $j^\circ T^*X$, we may assume that $C \subset j^\circ T^*X$. Let $x \in X$ be a geometric point. We show that there exists a subscheme $i\colon V \to W$ such that the composition $s\colon V \to X$ is étale, $V_x \neq \varnothing$ that $s_!i^\circ C = j_!C$ at x. For any point $\omega \in (T^*X - j_!C)_x$ in the complement, the inverse image $(T^*X \times_X U - dj^{-1}(C)) \times_{T^*X} \omega$ is a non-empty open subset. Since, $(T^*X - j_!C)_x$ is quasi-compact, there exist geometric points $u_1, \ldots, u_n$ of $U \times_X x$ above closed points with residue fields separable over $k(x)$ such that $(j_!C)_x = \bigcap_{i=1}^n C_{u_i}$. We take a subscheme $i\colon U \to W$ étale over X containing $u_1, \ldots, u_n$. Then the inclusion $j_!C \subset s_!i^\circ C$ is an equality at x.

Let $h\colon W \to X, f\colon W \to Y$ be a pair of morphisms of smooth schemes over k such that (h, f) is $j_!C$-acyclic. Let $x \in X$ be any geometric point and show that (h, f) is $\mathcal{F}$-acyclic on a Zariski neighborhood of x. Since $s_!i^\circ C = j_!C$ at x and since the C-transversality is an open condition, the pair (h, f) is $s_!i^\circ C$-acyclic on a neighborhood of x. Since $i^*\mathcal{F}$ is micro supported on $i^\circ C$ and since the assertion is already proved for an étale morphism, $\mathcal{F}$ is micro supported on $s_!i^\circ C$ on a neighborhood of x. Hence (h, f) is $\mathcal{F}$-acyclic on the neighborhood of x as required. □

Corollary 6.0.8. *Let $h\colon W \to X$ be a smooth morphism of smooth schemes over k. Then, we have $SSh^*\mathcal{F} = h^\circ SS\mathcal{F}$.*

Proof. Since $h^*\mathcal{F}$ is micro supported on $h^\circ SS\mathcal{F}$, it suffices to show that, for any closed conical subset $C \subset T^*W$ on which $h^*\mathcal{F}$ is micro supported, we have an inclusion $h^*SS\mathcal{F} \subset C$. Since h is an open mapping, by replacing X by the image of h, we may assume that h is a surjection. By Proposition, we have $SS\mathcal{F} \subset h_!C$. This implies $h^\circ SS\mathcal{F} \subset h^\circ h_!C \subset C$. □

7 Legendre transform and Radon transform

Let V be a k vector space of dimension $n+1$ and $\mathbf{P} = \mathbf{P}(V)$ be the projective space of dimension n parametrizing lines in V. The dual projective space $\mathbf{P}^\vee = \mathbf{P}(V^\vee)$ is the moduli space of hyperplanes in $\mathbf{P}$.

By the exact sequence $0 \to \Omega^1_{\mathbf{P}/k}(1) \to \mathcal{O}_{\mathbf{P}} \otimes V^\vee \to \mathcal{O}_{\mathbf{P}}(1) \to 0$ of locally free $\mathcal{O}_{\mathbf{P}}$-modules, we define a closed subscheme $Q = \mathbf{P}(T^*\mathbf{P}) \subset \mathbf{P} \times \mathbf{P}(V^\vee) = \mathbf{P} \times \mathbf{P}^\vee$ of codimension

1. This equals the universal family of hyperplanes since it is defined by the tautological section $\Gamma(\mathbf{P} \times \mathbf{P}^\vee, \mathcal{O}(1) \boxtimes \mathcal{O}(1)) = V^\vee \otimes V$ corresponding to the identity $1 \in \mathrm{End}(V)$. Let $p\colon Q \to \mathbf{P}$ and $p^\vee\colon Q \to \mathbf{P}^\vee$ be the restrictions of the projections $\mathbf{P} \times \mathbf{P}^\vee \to \mathbf{P}$ and $\mathbf{P} \times \mathbf{P}^\vee \to \mathbf{P}^\vee$. By symmetry, $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ is identified with $\mathbf{P}(T^*\mathbf{P}^\vee)$.

The conormal bundle $L_Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q$ is a line bundle. Since $1 \in \mathrm{End}(V) = V^\vee \otimes V$ regarded as a global section of $\mathcal{O}(1) \boxtimes \mathcal{O}(1)$ is the bilinear form defining $Q \subset \mathbf{P} \times \mathbf{P}^\vee$, the morphism $N_{Q/(\mathbf{P}\times\mathbf{P}^\vee)} \to \Omega^1_{(\mathbf{P}\times\mathbf{P}^\vee)/\mathbf{P}^\vee} \otimes_{\mathcal{O}_{\mathbf{P}\times\mathbf{P}^\vee}} \mathcal{O}_Q = \Omega^1_{\mathbf{P}/k} \otimes_{\mathcal{O}_{\mathbf{P}}} \mathcal{O}_Q$ defines a tautological sub invertible sheaf on $Q = \mathbf{P}(T^*\mathbf{P})$. In other words, the tautological sub line bundle $L \subset T^*\mathbf{P} \times_{\mathbf{P}} Q$ is the image of L_Q by the first projection $\mathrm{pr}_1\colon (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*\mathbf{P} \times_{\mathbf{P}} Q$. By symmetry, the image by the second projection equals the tautological sub line bundle $L^\vee$ on $Q = \mathbf{P}(T^*\mathbf{P}^\vee)$.

Since the conormal bundle L_Q is the kernel of the surjection $(T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*Q$, the intersection $p^\circ T^*\mathbf{P} \cap p^{\vee\circ} T^*\mathbf{P}^\vee \subset T^*Q$ equals the image of the tautological bundle $L \subset T^*\mathbf{P} \times_{\mathbf{P}} Q$. By symmetry, the intersection also equals the image of the tautological bundle $L^\vee \subset T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q$.

Let $C \subset T^*\mathbf{P}$ denote a closed conical subset. We define the Legendre transform $C^\vee \subset T^*\mathbf{P}^\vee$ to be $p^\vee_\circ p^\circ C$. We consider projectivizations $\mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P})$ and $\mathbf{P}(C^\vee) \subset \mathbf{P}(T^*\mathbf{P}^\vee)$ as closed subsets of Q.

Proposition 7.0.1. *Let $C \subset T^*\mathbf{P}$ be a closed conical subset. Let $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ be the projectivization. Let $L_Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q$ be the conormal line bundle.*

1. *The projectivization $E = \mathbf{P}(C) \subset Q$ is the complement of the largest open subset where $(p, p^\vee)$ is C-acyclic.*
2. *The Legendre transform $C^\vee$ equals the image of the union of $L|_E \subset p^\circ T^*\mathbf{P} \cap p^{\vee\circ} T^*\mathbf{P}^\vee \subset T^*Q$ and its base. We have $\mathbf{P}(C) = \mathbf{P}(C^\vee)$.*
3. *We have $C^{\vee\vee} \subset C^+$.*

Proof. 1. The kernel $\mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*Q)$ equals the conormal bundle L_Q and the first projection induces an isomorphism $L_Q \to L$. By this isomorphism, the intersection $(C \times T^*\mathbf{P}^\vee)|_Q \cap L_Q$ is identified with $p^*C \cap L$. Hence $(p, p^\vee)$ is C-acyclic on $U \subset Q$ if and only if the restriction $(p^*C \cap L)|_U$ is a subset of the 0-section. Since the projectivization $E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P})$ equals $\mathbf{P}(p^*C \cap L) \subset \mathbf{P}(L) = Q$, the assertion follows.

2. The Legendre transform $C^\vee$ is the image of the intersection $p^\circ C \cap p^{\vee\circ} T^*\mathbf{P}^\vee \subset T^*Q$ by $p^{\vee\circ} T^*\mathbf{P}^\vee \to T^*\mathbf{P}^\vee$. We identify the intersection $p^\circ T^*\mathbf{P} \cap p^{\vee\circ} T^*\mathbf{P}^\vee \subset T^*Q$ with the tautological line bundle $L \subset T^*\mathbf{P} \times_{\mathbf{P}} Q$. Then, the intersection $p^\circ C \cap p^{\vee\circ} T^*\mathbf{P}^\vee \subset T^*Q$ is identified with $p^*C \cap L$. Since the projectivization $E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P})$ equals $\mathbf{P}(p^*C \cap L) \subset \mathbf{P}(L) = Q$, the closed conical subset $p^*C \cap L$ equals $L|_E$ up to the base.

Since $L|_E$ is identified with $L^\vee|_E$ inside T^*Q, we have $\mathbf{P}(C^\vee) = \mathbf{P}(L^\vee|_E) = E \subset \mathbf{P}(T^*\mathbf{P}^\vee) = Q$.

3. By 2 and symmetry, we have $\mathbf{P}(C) = \mathbf{P}(C^\vee) = \mathbf{P}(C^{\vee\vee})$. Hence we have $C^{\vee\vee} \subset C^+$. □

We define the naive Radon transform $R\mathcal{F}$ to be $Rp^\vee_* p^*\mathcal{F}$ and the naive inverse Radon transform $R^\vee\mathcal{G}$ to be $Rp_* p^{\vee *}\mathcal{G}$.

Lemma 7.0.2. *Assume that $\mathcal{F}$ is micro supported on C.*

1. *The Radon transform $R\mathcal{F}$ is micro supported on $C^\vee$.*
2. *$R^\vee R\mathcal{F}$ is micro supported on C^+.*

Proof. 1. $R\mathcal{F} = p^\vee_* p^* \mathcal{F}$ is micro supported on $C^\vee = p^\vee_\circ p^\circ C$.

2. By 1, $R^\vee R\mathcal{F}$ is micro supported on $C^{\vee\vee} \subset C^+$. $\square$

By computing $R^\vee R\mathcal{F}$, we prove more refined assertion.

Lemma 7.0.3. *We consider the commutative diagram*

$$\begin{tikzcd}
Q \times_{\mathbf{P}^\vee} Q \arrow[r, "i"] \arrow[dr, "p\times p"'] & \mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P} \arrow[d, "\mathrm{pr}_{13}"] & \\
& \mathbf{P} \times \mathbf{P} & \mathbf{P} \arrow[l, "\delta_{\mathbf{P}}"']
\end{tikzcd}$$

where $\delta_{\mathbf{P}} \colon \mathbf{P} \to \mathbf{P} \times \mathbf{P}$ is the diagonal immersion. Then the closed immersion i induces isomorphisms

$$R^q(p \times p)_* \Lambda_{Q\times_{\mathbf{P}^\vee} Q} \to \begin{cases} \Lambda(-i)[-2i] & \text{if } q = 2i, 0 \leqslant i \leqslant n-2, \\ \delta_{\mathbf{P}*}\Lambda(-(n-1))[-2(n-1)] & \text{if } q = 2(n-1) \end{cases} \tag{7.1}$$

and $R^q(p \times p)_* \Lambda_{Q\times_{\mathbf{P}^\vee} Q} = 0$ otherwise.

Proof. The immersion i induces morphisms

$$R^q\mathrm{pr}_{13*}\Lambda_{\mathbf{P}\times\mathbf{P}^\vee\times\mathbf{P}} \to R^q(p \times p)_* \Lambda_{Q\times_{\mathbf{P}^\vee} Q} \tag{7.2}$$

and we have isomorphisms $R^q\mathrm{pr}_{13*}\Lambda_{\mathbf{P}\times\mathbf{P}^\vee\times\mathbf{P}} \to \Lambda(-i)[-2i]$ for $q = 2i, 0 \leqslant i \leqslant n$ and $R^q\mathrm{pr}_{13*}\Lambda_{\mathbf{P}\times\mathbf{P}^\vee\times\mathbf{P}} = 0$ otherwise. The restriction of the closed immersion $i \colon Q \times_{\mathbf{P}^\vee} Q \to \mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P}$ on the diagonal $\mathbf{P} \subset \mathbf{P} \times \mathbf{P}$ is the sub $\mathbf{P}^{n-1}$-bundle $Q \subset \mathbf{P} \times \mathbf{P}^\vee$. On the complement $\mathbf{P} \times \mathbf{P} - \mathbf{P}$, it is a sub $\mathbf{P}^{n-2}$-bundle. Hence (7.2) is an isomorphism for $q \neq 2(n-1)$ and induces an isomorphism $\delta_{\mathbf{P}*}R^{2(n-1)}p_*\Lambda_Q \to R^{2(n-1)}(p \times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee} Q}$. $\square$

We consider the diagram

$$\begin{tikzcd}
\mathbf{P} & \mathbf{P} \times \mathbf{P} \arrow[l, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"'] & Q \times_{\mathbf{P}^\vee} Q \arrow[l, "p\times p"'] \\
& \mathbf{P}. &
\end{tikzcd} \tag{7.3}$$

Proposition 7.0.4. 1. *We have a canonical isomorphism*

$$R^\vee R\mathcal{F} \to R\mathrm{pr}_{2*}\big(\mathrm{pr}_1^*\mathcal{F} \otimes R(p \times p)_* \Lambda_{Q\times_{\mathbf{P}^\vee} Q}\big). \tag{7.4}$$

2. *The isomorphism* (7.4) *induces a distinguished triangle*

$$\to \bigoplus_{q=0}^{n-2} R\Gamma(\mathbf{P}_{\bar{k}}, \mathcal{F})(-q)[-2q] \to R^\vee R\mathcal{F} \to \mathcal{F}(-(n-1))[-2(n-1)] \to .$$

Proof. 1. By the cartesian diagram

$$\begin{tikzcd}
\mathbf{P} & Q \arrow[l, "p"'] \arrow[d, "p^\vee"'] & Q \times_{\mathbf{P}^\vee} Q \arrow[l, "\mathrm{pr}_1"'] \arrow[d, "\mathrm{pr}_2"] \\
& \mathbf{P}^\vee & Q \arrow[l, "p^\vee"'] \arrow[d, "p"] \\
& & \mathbf{P},
\end{tikzcd}$$

we have $R^\vee R\mathcal{F} = Rp_*p^{\vee *}Rp^\vee_*p^*\mathcal{F}$. By the proper base change theorem, we have a canonical isomorphism $Rp_*p^{\vee *}Rp^\vee_*p^*\mathcal{F} \to R(p\circ \mathrm{pr}_2)_*(p\circ \mathrm{pr}_1)^*\mathcal{F}$. In the notation of (7.3), the latter is identified with $R(\mathrm{pr}_2\circ(p\times p))_*(\mathrm{pr}_1\circ(p\times p))^*\mathcal{F}$. This is identified with $R\mathrm{pr}_{2*}(\mathrm{pr}_{1*}\mathcal{F}\otimes R(p\times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q})$ by the projection formula.

2. By the isomorphisms (7.1) and (7.4), we have a distinguished triangle

$$\to \tau_{\leqq 2(n-2)}R\Gamma(\mathbf{P}^\vee_{\bar k},\Lambda)\otimes\Lambda_{\mathbf{P}\times\mathbf{P}} \to R(p\times p)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q} \to \Lambda_{\mathbf{P}}(n-1)[2(n-1)] \to .$$

Proposition 7.0.5. *For $\mathcal{F}$ on $\mathbf{P}$ and $C\subset T^*\mathbf{P}$, we have implications* (1)⇒(2)⇒(3)⇒(4).

(1) *$\mathcal{F}$ is micro supported on C.*

(2) *$(p,p^\vee)$ is universally $\mathcal{F}$-acyclic outside $E=\mathbf{P}(C)$.*

(3) *$R\mathcal{F}$ is micro supported on $C^{\vee+}$.*

(4) *$\mathcal{F}$ is micro supported on C^+.*

Proof. (1)⇒(2): The pair $(p,p^\vee)$ of $p\colon Q\to\mathbf{P}$ and $p^\vee\colon Q\to\mathbf{P}^\vee$ is C-acyclic outside $E=\mathbf{P}(C)$ by Proposition 7.0.1.1. Hence (1) implies that $p^\vee$ is universally $p^*\mathcal{F}$-acyclic outside E.

(2)⇒(3): Assume that a pair of morphisms $h\colon W\to\mathbf{P}, f\colon W\to Y$ is C^+-acyclic and show that (h,f) is $R\mathcal{F}$-acyclic. We consider the commutative diagram

$$\begin{tikzcd}
\mathbf{P} & Q \arrow[l, "p"'] \arrow[d, "p^\vee"'] & Q_W \arrow[l, "h'"'] \arrow[d, "p^\vee_W"'] \arrow[dr, "f'"] & \\
& \mathbf{P}^\vee & W \arrow[l, "h"'] \arrow[r, "f"] & Y
\end{tikzcd}$$

with cartesian square. Since $p^\vee$ is proper, it suffices to show that (h',f') is $p^*\mathcal{F}$-acyclic by Lemma 3.0.9.2. Since p is smooth, it suffices to show that (ph',f') is $\mathcal{F}$-acyclic by Lemma 3.0.9.1.

By (2), $(ph',p^\vee_W)$ is $\mathcal{F}$-acyclic on the complement of E_W. Since (h,f) is $C^{\vee+}$-acyclic and $C^{\vee+}$ contains the 0-section, the morphism $f\colon W\to Y$ is smooth. Hence (ph',f') is $\mathcal{F}$-acyclic on the complement of E_W by Lemma 3.0.9.3.

By the description of $C^\vee$ in Proposition 7.0.1.3 and by the open condition Lemma 4.0.3, the $C^\vee$-acyclicity of (h,f) implies the $T^*\mathbf{P}$-acyclicity of (ph',f) on a neighborhood U of $E_W\subset Q_W$. Since $\mathcal{F}$ is micro supported on $T^*\mathbf{P}$ by Proposition 5.0.2.1, (ph',f) is $\mathcal{F}$-acyclic as required.

(3)⇒(4) By (3) and (1)⇒(3), $R^\vee R\mathcal{F}$ is micro supported on $(C^{\vee+})^{\vee+}=C^+$. By the distinguished triangle in Proposition 7.0.4.2, $\mathcal{F}$ is also micro supported on C^+. □

We prove Theorem 6.0.3 for $X=\mathbf{P}$.

Corollary 7.0.6. *Let $\mathcal{F}$ be a sheaf on $\mathbf{P}$. Let $E\subset Q=T^*\mathbf{P}$ be the complement of the largest open subset on which $(p,p^\vee)$ is universally $\mathcal{F}$-acyclic. Then the closed conical subset $C\subset T^*\mathbf{P}$ corresponding to $B=\operatorname{supp}\mathcal{F}$ and E is the singular support of $\mathcal{F}$.*

Proof. By Proposition 7.0.5 (2)⇒(4), $\mathcal{F}$ is micro supported on C^+. Hence $\mathcal{F}$ is micro supported on $C=C^+|_B$.

Assume that $\mathcal{F}$ is micro supported on C'. Then, by Proposition 7.0.5 (1)⇒(2), we have $\mathbf{P}(C')\supset E=\mathbf{P}(C)$ since E is the smallest. Since the base of C' contains $B=\operatorname{supp}\mathcal{F}$ as a subset, we have $C'\supset C$. □

8 Upper bound of dimension

Let $L \subset \mathbf{P} = \mathbf{P}_S$ be a sub $\mathbf{P}^1$-bundle. Let $\mathbf{P}_L \to L$ be the moduli space of lines in the $\mathbf{P}^n$-bundle $\mathbf{P} \times_S L$ over L passing through the points of $L \subset \mathbf{P} \times_S L$ and let $\widetilde{\mathbf{P}}_L \subset \mathbf{P}_S \times_S \mathbf{P}_L$ be the universal $\mathbf{P}^1$-bundle over $\mathbf{P}_L$. If $\mathbf{P}_S = \mathbf{P}(E_S)$ for a vector bundle E_S over S of rank $n+1$ and if $V \subset E_S \times_S \mathbf{P}_S$ denotes the universal sub line bundle, then $\mathbf{P}_L$ is the $\mathbf{P}^{n-1}$-bundle $\mathbf{P}((E_S \times_S \mathbf{P}_S)/V) \times_{\mathbf{P}_S} L$ over L.

The dual $\mathbf{P}^{n-1}$-bundle $\mathbf{P}^\vee_L$ over L is the moduli space of hyperplanes in the $\mathbf{P}^n$-bundle $\mathbf{P} \times_S L$ over L passing through the points of $L \subset \mathbf{P} \times_S L$. The coincidence variety $\overline{Q}_L \subset \mathbf{P}_L \times_L \mathbf{P}^\vee_L$ consists of pairs of a line and a hyperplane including the line. We consider the cartesian diagram

$$\begin{array}{ccccccc}
\widetilde{\mathbf{P}}_L & \longleftarrow & \widetilde{Q}_L & & Q_L & \longrightarrow & Q \\
{\scriptstyle r}\downarrow & & \downarrow{\scriptstyle \overline{r}} & & {\scriptstyle p^\vee_L}\downarrow & & \downarrow{\scriptstyle p^\vee} \\
\mathbf{P}_L & \xleftarrow{\overline{p}} & \overline{Q}_L & \xrightarrow{\overline{p}^\vee} & \mathbf{P}^\vee_L & \longrightarrow & \mathbf{P}^\vee
\end{array}$$

where $\mathbf{P}^\vee_L \to \mathbf{P}^\vee$ is the tautological morphism. The morphism $\mathbf{P}^\vee_L \to \mathbf{P}^\vee$ induces an open immersion on the open subset $\mathbf{P}^{\vee\circ}_L \subset \mathbf{P}^\vee_L$ consisting of hyperplanes not containing the line L. The image in $\mathbf{P}^\vee$ consists of hyperplanes meeting L transversally.

Let $s\colon \mathbf{P}_L \to \widetilde{\mathbf{P}}_L$ be the section defined by the intersection $\widetilde{\mathbf{P}}_L \cap (L \times_S \mathbf{P}_L) \subset \mathbf{P}_S \times_S \mathbf{P}_L$ and $\overline{s}\colon \overline{Q}_L \to \widetilde{Q}_L$ be the base change. Similarly, let $s^\vee\colon \mathbf{P}^\vee_L \to Q_L$ be the section defined by the intersection $Q_L \cap (L \times_S \mathbf{P}^\vee_L) \subset \mathbf{P}_S \times_S \mathbf{P}^\vee_L$. Since we have an inclusion $\widetilde{Q}_L \subset Q_L \times_{\mathbf{P}^\vee_L} \overline{Q}_L$ of lines in hyperplanes over $\overline{Q}_L$, we have a canonical morphism

$$\widetilde{r}\colon \widetilde{Q}_L \to Q_L.$$

Since the fiber of $\widetilde{Q}_L \to \mathbf{P}^\vee_L$ is the universal family of lines in the hyperplane passing through the point of L, the morphism $\widetilde{r}\colon \widetilde{Q}_L \to Q_L$ is the blow-up of the $\mathbf{P}^{n-1}$-bundle Q_L over $\mathbf{P}^\vee_L$ at the section $s^\vee\colon \mathbf{P}^\vee_L \to Q_L$. The exceptional divisor in $\widetilde{Q}_L$ is the image of the section $\overline{s}\colon \overline{Q}_L \to \widetilde{Q}_L$. The morphism $\widetilde{r}\colon \widetilde{Q}_L \to Q_L$ induces an isomorphism on the complement $\widetilde{Q}_L - s^\vee(\mathbf{P}^\vee_L) \to Q_L - \overline{s}(\overline{Q}_L)$ of the sections.

Proposition 8.0.1. *Let $D \subset \mathbf{P} = \mathbf{P}_S$ be the complement of the largest open subset on which $\mathcal{F}$ is locally constant. On the inverse image of the complement $L^\circ = L - (L \cap D)$, we have the following.*

1. *The morphism $\widetilde{r}\colon \widetilde{Q}_L \to Q_L$ induces a bijection*

$$E(\mathcal{F}_{Q_{L^\circ}}, p^\vee_{L^\circ}\colon Q_{L^\circ} \to \mathbf{P}^\vee_{L^\circ}) \to E(\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \overline{p}^\vee\overline{r}\colon \widetilde{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ}).$$

2. *The $\mathbf{P}^1$-bundle $\overline{r}\colon \widetilde{Q}_L \to \overline{Q}_L$ induces a finite surjection*

$$E(\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \overline{p}^\vee\overline{r}\colon \widetilde{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ}) \to E(\overline{r}_*\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \overline{p}^\vee\colon \overline{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ}).$$

Proof. 1. On the inverse image of L°, the pull-backs of $\mathcal{F}$ are locally constant on neighborhoods of sections $s^\vee\colon \mathbf{P}^\vee_{L^\circ} \to Q_{L^\circ}$ and $\overline{s}\colon \overline{Q}_{L^\circ} \to \widetilde{Q}_{L^\circ}$. Hence, the intersections of the sections with $E(\mathcal{F}_{Q_{L^\circ}}, Q_{L^\circ} \to \mathbf{P}^\vee_{L^\circ})$ and $E(\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \widetilde{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ})$ are empty.

2. Since $\overline{r}\colon \widetilde{Q}_L \to \overline{Q}_L$ is a $\mathbf{P}^1$-bundle and the closed subset $E(\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \widetilde{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ})$ does not meet the section $\overline{s}\colon \overline{Q}_{L^\circ} \to \widetilde{Q}_{L^\circ}$, the closed subset $E(\mathcal{F}_{\widetilde{Q}_{L^\circ}}, \widetilde{Q}_{L^\circ} \to \mathbf{P}^\vee_{L^\circ})$ is finite over $\overline{Q}_{L^\circ}$. □

Since $\mathbf{P}_L$ is a $\mathbf{P}^{n-1}$ bundle over L, the assertion follows by induction.

Lemma 8.0.2. *Let $x_0, \ldots, x_n \in \mathbf{P}^n$ be $n+1$ points not contained in any hyperplane and let L_{ij} be the line containing x_i and x_j for $0 \leqslant i < j \leqslant n$. Then a hyperplane $H \subset \mathbf{P}^n$ is uniquely determined by the subset $I = \{i \mid x_i \in H, i = 0, \ldots, n\} \subset I = \{0, \ldots, n\}$ and the intersections $H \cap L_{ij}$ for $\{i, j\} \cap I = \varnothing$.*

Lemma 8.0.3 ([2, 3.10])**.** *Let $s\colon \overline{Q}_{T^\circ} \to \widetilde{Q}_{T^\circ}$ and s' be the tautological sections of r and $p^\vee$ respectively.*

1. *$q\colon \widetilde{Q}_{T^\circ} \to Q_{T^\circ}$ induces an isomorphism on the complements of the images of the sections s, s'. The images of the sections are disjoint with the inverse images of $D \subset \mathbf{P}$.*
2. *r is finite on the inverse image $\pi^{-1}(D)$.*

1. Since $t \notin D$, they are disjoint. A line is uniquely determined by the points $t \neq x$.
2. Since $\widetilde{Q}$ is a line bundle over $\overline{Q}$, the assertion follows from 1.

It suffices to show that there exists an open subset $U \subset \mathbf{P}^S$ such that the complement $\mathbf{P}^S - U$ is generically finite and that $\dim E \cap p^{\vee -1}(U) \leqslant \dim \mathbf{P}_S - 1$.

To prove this, by Lemma8.0.4, it suffices to show the following:

Let $T \subset \mathbf{P}$ be a line not contained in D. Let $\mathbf{P}^\vee_T$ be the open subset of $\mathbf{P}^\vee$ consisting of hyperplanes meeting transversally with T. Then, there exists an open subset U_T such that the inverse image $\mathbf{P}^\vee_{U_T}$ by the morphism $\mathbf{P}^\vee_T \to T$ sending a hyperplane to the intersection satisfies $\dim E \cap p^{\vee -1}(\mathbf{P}^\vee_{U_T})$ and that the complement $T - U_T$ is generically finite.

9 Perverse sheaves and Radon transform

Theorem 9.0.1. 1. *The subcategory* $\mathrm{Perv}(X) \subset D^b_c(X, \Lambda)$ *is an abelian category. Every object of* $\mathrm{Perv}(X)$ *is of finite length.*

2. *Let $\mathcal{F}$ be a simple perverse sheaf. Then, its support $Z = \operatorname{supp} \mathcal{F}$ is irreducible. If $\dim Z = d$, there exist a dense open subset $U \subset Z$, a simple locally constant sheaf $\mathcal{L}$ on U and an isomorphism $\mathcal{F}|_U = \mathcal{L}[d]$.*

Lemma 9.0.2. *Let X be a connected smooth scheme over k.*

1. *Assume that $\mathcal{F}$ is a perverse sheaf on X. If $\mathcal{F}$ is geometrically constant, every constituant of $\mathcal{F}$ is geometrically constant.*
2. *The following conditions are equivalent:*

(1) *$\mathcal{F}$ is locally constant.*
(2) *Every constituant of $\mathcal{F}$ is locally constant.*

Proof. 1. $\mathcal{F}$ is a successive extension of irreducible locally constant sheaves $\mathcal{F}_i$. Since $\mathcal{F}$ is geometrically constant, every $\mathcal{F}_i$ is geometrically constant.

2. (1)⇒(2): Constituants of $\mathcal{F}$ as locally constant sheaves corresponds to constituants of $\mathcal{F}$ as perverse sheaves.

(2)⇒(1): For every q, ${}^{\mathrm{p}}\mathcal{H}^q\mathcal{F}$ is a successive extension of locally constant sheaves and hence is itself locally constant. Hence $\mathcal{H}^q\mathcal{F}$ is locally constant for every q. □

Theorem 9.0.3. *Let f be an affine morphism. Then, f_* is right* $\mathtt{t}$*-exact and $f_!$ is left* $\mathtt{t}$*-exact.*

Let $U = \mathbf{P} \times \mathbf{P}^\vee - Q$ be the complement and $u\colon U \to \mathbf{P}$ and $u^\vee\colon U \to \mathbf{P}^\vee$ be the projections. Define a functor $R_!\colon D^b_c(\mathbf{P}, \Lambda) \to D^b_c(\mathbf{P}^\vee, \Lambda)$ by $R_! = u^\vee_! u^*$. For $\mathcal{F} \in D^b_c(\mathbf{P}, \Lambda)$, we have a distinguished triangle

$$R_!\mathcal{F} \to p^{\vee *}p_*\mathcal{F} \to R\mathcal{F} \to .$$

With an appropriate shift, this defines

$$\mathcal{R}\mathcal{F} \to \mathcal{R}_!\mathcal{F} \to p^{\vee *}p_*\mathcal{F}[n] \to .$$

Dually, we have

$$p^{\vee *}p_*\mathcal{F}[n] \to \mathcal{R}_*\mathcal{F} \to \mathcal{R}\mathcal{F} \to .$$

By Theorem 9.0.3, for $\mathcal{F} \in \mathrm{Perv}(\mathbf{P}, \Lambda)$, we have $\mathcal{R}_!\mathcal{F} \in {}^{\mathrm{p}}D^{\geqslant 0}(\mathbf{P}^\vee, \Lambda)$, $\mathcal{R}_*\mathcal{F} \in {}^{\mathrm{p}}D^{\leqslant 0}(\mathbf{P}^\vee, \Lambda)$, and $p^{\vee *}p_*\mathcal{F}[n] \in {}^{\mathrm{p}}D^{[-2n,0]}(\mathbf{P}^\vee, \Lambda)$

Proposition 9.0.4. *Let $\mathcal{F}$ be a sheaf on $\mathbf{P}$ and let $\mathcal{R}\mathcal{F}$ be the Radon transform.*

1. *Assume that $\mathcal{F}$ is a perverse sheaf. For $q \neq 0$, the perverse sheaf ${}^{\mathrm{p}}\mathcal{H}^q\mathcal{R}\mathcal{F}$ is geometrically constant. Further if $\mathcal{F}$ is geometrically constant, every ${}^{\mathrm{p}}\mathcal{H}^q\mathcal{R}\mathcal{F}$ is geometrically constant.*
2. *Assume that $\mathcal{F}$ is a simple perverse sheaf and is not geometrically constant. Then, the perverse sheaf ${}^{\mathrm{p}}\mathcal{H}^0\mathcal{R}\mathcal{F}$ has a unique constituant $\mathcal{R}\mathcal{F}^0$ not geometrically constant.*
3. *Let $\mathcal{F}_i, i = 1, \ldots, n$ be the constituants of $\mathcal{F}$ not geometrically constant. Then, the constituants of $\mathcal{R}\mathcal{F}$ not geometrically constant are $\mathcal{R}\mathcal{F}^0_i, i = 1, \ldots, n$.*

2. Let $\mathcal{G}_i, i = 1, \ldots, m$ be the non-geometrically constant constituants of $\mathcal{R}\mathcal{F}$ and let $\mathcal{F}_{ij}, i = 1, \ldots, m, j = 1, \ldots, m_i$ be the non-geometrically constant constituants of $\mathcal{R}^\vee\mathcal{G}_i$. Then, since $\sum_{ij} m_i = 1$, we have $m = m_1 = 1$.

3. This follows from 2.

10 Legendre transform and Veronese embedding

11 Dimension

Lemma 11.0.1. *Let X be a smooth scheme over k and $\mathcal{F}$ be a sheaf on X.*

1. *The support of $\mathbf{P}(SS\mathcal{F})$ equals the complement of the largest open subset on which the restriction of $\mathcal{F}$ is locally constant.*
2. *Let $\mathcal{F}_i, i = 1, \ldots, n$ be the constituants of $\mathcal{F}$. Then, the support of $\mathbf{P}(SS\mathcal{F})$ equals the supports of $\bigcup_{i=1}^n \mathbf{P}(SS\mathcal{F}_i)$.*

Proof. 1. $SS\mathcal{F}$ is a subset of the 0-section if and only if $\mathcal{F}$ is locally constant.

2. This follows from 1 and Lemma 9.0.2.2. □

Theorem 11.0.2. 1. $\dim SS\mathcal{F} = \dim X$.

2. *Let $\mathcal{F}_i, i = 1, \ldots, n$ be the constituants of $\mathcal{F}$. Then, $SS\mathcal{F}$ equals the union $\bigcup_{i=1}^n SS\mathcal{F}_i$.*

Proof. 1. By Lemma 11.0.1.2, we may assume $\mathcal{F}$ is a simple perverse sheaf. If $R\mathcal{F}$ is generically 0, then the support of $\mathbf{P}(SSR\mathcal{F})$ equals the support of $R\mathcal{F}$ and is irreducible. Since $\mathbf{P}(SSR\mathcal{F}) = \mathbf{P}(SS\mathcal{F})$ is irreducible, this equals the $\mathbf{P}(T^*_Z\mathbf{P})$ and is of dimension n. If otherwise the complement of the support $D_{\mathcal{F}}$ of $\mathbf{P}(SSR\mathcal{F})$ is the largest open subset of $\mathbf{P}^\vee$ on which $R\mathcal{F}$ is locally constant. Since $R\mathcal{F}_0$ is a simple perverse sheaf, $D_{\mathcal{F}}$ is a divisor.

2. By Lemma 11.0.1.2, we have $SS\mathcal{F} = \bigcup_{i=1}^n SS\mathcal{F}_i$ except for the 0-section. Since $\mathcal{F} = 0$ generically if and only if $\mathcal{F}_i = 0$ generically for every $i = 1, \ldots, n$, we have $SS\mathcal{F} = \bigcup_{i=1}^n SS\mathcal{F}_i$ including the 0-section. □

References

[1] M. Artin, *Faisceaux constructibles, cohomologie d'une courbe algébrique*, SGA4 Exposé IX. Théorie des Topos et Cohomologie Étale des Schémas, Lecture Notes in Mathematics, vol. 305, pp. 1-42 (1973)

[2] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Math., 22 (4), 1797–1819 (2016).

[3] A. Braverman, D. Gaitsgory, *Geometric Eisenstein series*, Invent. Math., 150, 287–384 (2002).

[4] J.-L. Brylinski, *Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques*, Astérisque 140–141, 3–134 (1986).

[5] P. Deligne, *La formule de dualité globale, Théorie des topos et cohomologie étale des schémas*, SGA 4 Exposé XVIII. Springer Lecture Notes in Math. 305, 480–587 (1972)

[6] P. Deligne, *Théorèmes de finitude en cohomologie ℓ-adique*, Cohomologie étale SGA $4\frac{1}{2}$. Springer Lecture Notes in Math. 569, 233–251 (1977)

[7] L. Fu, Etale cohomology theory, Nankai tracts in Mathematics Vol. 13, World Scientific, 2011, revised ed. 2015.

[8] H. Hu, E. Yang, *Relative singular support and the semi-continuity of characteristic cycles for étale sheaves*, Selecta Math., 24, 2235–2273 (2018).

[9] L. Illusie, *Appendice à Théorème de finitude en cohomologie ℓ-adique*, Cohomologie étale SGA $4\frac{1}{2}$. Springer Lecture Notes in Mathematics, vol. 569, pp. 252–261 (1977)

[10] Q. Lu and W. Zheng, *Categorical traces and a relative Lefschetz–Verdier formula*, Forum of Mathematics, Sigma (2022) Vol. 10, 1–24.

[11] T. Saito, *The characteristic cycle and the singular support of a constructible sheaf*, Inventiones Math. 207(2) (2017), 597–695.

[12] —–, *Characteristic cycle of the external product of constructible sheaves,* Manuscripta Math., 154, 1–12 (2017)

[13] —–, *Cotangent bundle and microsupports in the mixed characteristic case,* Algebra and number theory 16:2 (2022) 335–368.