1. Singular support & characteristic cycle.
2. Proper direct image.
3. Product

... p. Deligne, Kashiwara–Schapira, Brylinski–Laumon, Beilinson...

k perfect field. Λ finite field of char $\ell \neq$ char k.
(or $\mathbb{Z}_\ell, \mathbb{Q}_\ell, \ldots$)

X smooth $/k$ $\mathcal{F}$ constructible complex of Λ-modules $/X$
$\dim X = n$

$SS\mathcal{F} = C = \bigcup C_a \subset T^*X$ cotangent b'dle
closed conical subset
(i.e. stable under $\mathbb{G}_m$)

C_a irred. cpt $\dim C_a = n$.

$CC\mathcal{F} = \sum m_a C_a$ $m_a \in \mathbb{Z}$ index formula, Milnor formula

Example. 1 $D \subset X$ div. w. SNC. $D = \bigcup D_i$ $X_I = \bigcap_{i \in I} D_i$

$\mathcal{F} = j_!\mathcal{G}$ $j: U = X - D \hookrightarrow X$

$\mathcal{G}$ loc. cst on U. tamely ramified along D

$SS\mathcal{F} = \bigcup_I T^*_{X_I}X$ conormal b'dles.

$CC\mathcal{F} = (-1)^n \operatorname{rk}\mathcal{G} \sum_I T^*_{X_I}X$. (Euler Yang)

$X = U, \mathcal{F} = \Lambda$ $CC\mathcal{F} = (-1)^n T^*_X X$ largest

2 $\dim X = 1$. $U \subset X$ dense open where $\mathcal{F}|_U$ loc. const $(= \bigoplus_q H^q(\mathcal{F}))$
$\neq 0$

$SS\mathcal{F} = T^*_X X \cup \sum_{x \in X - U} T^*_x X$
0-section fiber

$CC\mathcal{F} = -(\operatorname{rk}\mathcal{F}|_U \cdot T^*_X X + \sum_{x \in X-U} a_x\mathcal{F} \cdot T^*_x X)$
GOS

$a_x\mathcal{F} = \operatorname{rk}\mathcal{F}|_U - \operatorname{rk}\mathcal{F}_x + \operatorname{Sw}_x\mathcal{F}$
Artin conductor Swan conductor.

3 $X = \mathbb{A}^2 \overset{j}{\supset} U = \mathbb{G}_m \times \mathbb{A}^1 = \operatorname{Spec} k[x^{\pm 1}, y]$ $= X - D$ 1.2 Lagrangean. $p \neq 2$

$\mathcal{G}$ loc. cst rk 1 on U defined by $t^p - t = y/x^p$

$\mathcal{F} = j_!\mathcal{G}$

$SS\mathcal{F} = T^*_X X \cup \langle dy/D \rangle \subset T^*D \neq T^*_D X$

$CC\mathcal{F} = T^*_X X + p \cdot \langle dy/D \rangle$. not Lagrangean.

Definition ~~8~~. $C \subset T^*X$ closed conical subset

1. $f: X \to Y$. Y smooth $C' \subset T^*Y$ closed conical
f is (C, C')-transversal if
$$f^*C' \cap df^{-1}(C) \subset X \times_Y T^*Y \xrightarrow{df} T^*X \quad (\supset C)$$
is a subset of the 0-section. $(C' \subset) T^*Y$ Eg. $(T^*_X X, T^*Y) \Leftrightarrow$ f-lisse

2. $h: W \to X$ W smooth is C-transversal if
h is $(T^*_W W, C)$-transversal.

3. $f: X \to Y$ Y smooth is C-transversal if
f is (C, T^*Y)-transversal

4. $h: W \to X$. $f: W \to Y$ is C-transversal if
$(h, f): W \to X \times Y$ is $C \times T^*Y$-transversal.

5. $j: U \to X$ étale. $f: U \to Y$ Y smooth curve. $u \in U$ closed pt
is an isolated char pt. if $U - \{x\} \to X$, $U - \{x\} \to Y$ is C-transversal
but $U \to X$. $U \to Y$ is not.   Eg $C = T^*_X X$

Def $\mathcal{F}$: constructible complex of Λ-modules on X

1. $C \subset T^*X$ closed conical   We say $\mathcal{F}$ is micro supported
if $\forall h: W \to X$, $f: W \to Y$ C-transversality implies that
f is locally acyclic relatively to $h^*\mathcal{F}$

2. $SS\mathcal{F}$ is the smallest closed conical subset of T^*X on which ~~st~~
$\mathcal{F}$ is micro supported

Theorem (Beilinson) The smallest exists and $\dim C_a = n$. $(C = \cup C_a)$

Theorem. Suppose $\mathcal{F}$ is micro supp. on $C = \cup C_a$. $\dim C_a = n$ e.g. $C = SS\mathcal{F}$
Then there exists a unique $\mathbb{Z}$-linear combination $A = \sum m_a C_a$ s.t.
$\forall j: U \to X$ $f: U \to Y$ with atmost isolated char pt $u \in U$, we have
$$-\operatorname{dimtot} \phi_u(j^*\mathcal{F}, f) = (A, df)_{T^*U, u} \quad \text{(Milnor formula)}$$
$\operatorname{dimtot} = \dim + \mathrm{Sw}$. ϕ_u: stalk of cx of vanishing cycles
$df = f^*dt$ t: local coord. at $v = f(u)$, $($ $)_{T^*U, u}$ int# supp'd on
$\mathcal{F} = \Lambda$. Deligne-Milnor SGA7 Exp XVI. $A = CCF$
additive

2. Proper push forward.
$f: X \to Y$ proper morphism of smooth schemes / k.
$C \subset T^*X$ closed conical.
$f_\circ C \subset T^*Y$ defined by the algebraic correspondence $T^*X \leftarrow X \times_Y T^*Y \to T^*Y$ (proper)

$\mathcal{F}$ on X micro supp'd on $C \Rightarrow Rf_*\mathcal{F}$ micro supp'd on $f_\circ C$
(proper b.c.)

$\dim X = n = \dim C_a$. $\dim Y = m$
$f_* : Z_n(C) \to CH_m(f_\circ C)$. def'd by alg corr.
If $\dim f_\circ C = m \Rightarrow$ " $Z_m(f_\circ C)$

Conjecture We have $CC\, Rf_*\mathcal{F} = f_* CC\mathcal{F}$
In particular if $\dim f_\circ C = m$, we have an equality of cycles i.e. in $Z(f_\circ C)$

If $Y = \operatorname{Spec} k$. Conj means an index formula $\chi(X_{\bar k}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$
Proved if X is projective (& smooth)
Further if $\dim X = 1$. index formula $=$ Grothendieck-Ogg-Shafarevich.

Theorem. Conj holds if $\dim X = 2$, $\dim Y = 1$. & $\exists V \subset Y$ dense open s.t.
$f_V: X_V \to V$ is smooth & C-transversal for $C = SS\mathcal{F}$
projective

Sketch of Pf. $\dim Y = 1 + \exists V \Rightarrow \dim f_\circ C = 1$. eq. as cycles.
Both sides are linear combination of $T^*_Y Y$ & $T^*_y Y$. $y \in Y - V$.
coeff of $T^*_Y Y$ — G.O.S for generic fiber.
coeff of $T^*_y Y$. conductor formula
$-a_y\, Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, y}$.
If f has at most iso. char pt conseq. of Milnor formula
↑ no ass on $\dim X$

local at each pt $y \in Y - V$. Kill ramification at other points by
Epp's theorem. $\Rightarrow \dim X = 2 \Rightarrow$ iso. char. pts $\Rightarrow$ conductor formula at $y' \neq y$
$\Rightarrow$ index formula $\Rightarrow$ cond formula at y.

3. Product

Theorem $\mathcal{F}, \mathcal{G}$ constructible cx. on X, Y smooth/k

$\mathcal{F}\boxtimes\mathcal{G} = pr_1^*\mathcal{F}\otimes pr_2^*\mathcal{G}$ on $X\times Y$

1. $SS(\mathcal{F}\boxtimes\mathcal{G}) = SS\mathcal{F}\times SS\mathcal{G}$
   $\subset T^*X\times T^*Y = T^*X\times Y$
2. $CC(\mathcal{F}\boxtimes\mathcal{G}) = CC\mathcal{F}\boxtimes CC\mathcal{G}$ $\left(=\sum m_a m'_b\, C_a\times C'_b\right)$

Points in Proof 1. Proj formula for nearby cycles over general base scheme (Z. Weizhe) = W. Zheng

2 i) $\mathcal{F}$ or $\mathcal{G}$ loc. const

ii) $\dim X=\dim Y=1$. coeff. of $T^*_{(x,y)}X\times Y$ Laumon's thesis

global argument using the index formula

equiv. to $-a_0(\mathcal{F}*\mathcal{G}) = (-a_0\mathcal{F})*(-a_0\mathcal{G})$ Laumon.

$*$ .. additive convolution

$\phi_{(0,0)}(\mathcal{F}\boxtimes\mathcal{G}, a)$

iii) general case. reduction to ii).

Thom-Sebastiani Illusie

$$R\Phi_{f+g}(\mathcal{F}\boxtimes\mathcal{G}) \cong R\Phi_f(\mathcal{F}) * R\Phi_g(\mathcal{G})$$

Cor 1 $\mathcal{F}, \mathcal{G}$ constructible on X. $SS\mathcal{F}\cap SS\mathcal{G}\subset T^*_XX$

$\Rightarrow$ $\mathcal{G}\otimes R\mathcal{H}om(\mathcal{F},\Lambda)\to R\mathcal{H}om(\mathcal{F},\mathcal{G})$ is an isom

$\because$ $\delta: X\to X\times X$ is $SS(\mathcal{F}\boxtimes\mathcal{G})$-transversal.

Cor 2 $f: X\to Y$, $\mathcal{F}$ on X, $\mathcal{G}$ on Y, f (C, C')-trans.

$C=SS\mathcal{F}$, $C'=SS\mathcal{G}$ $\Rightarrow$

$\gamma^*(\mathcal{F}\boxtimes\mathcal{G})\otimes R\gamma^!\Lambda\to R\gamma^!(\mathcal{F}\boxtimes\mathcal{G})$ isom

and $SS(\mathcal{F}\otimes f^*\mathcal{G})\subset C+f^\circ C'$