Étale cohomology and micro-local analysis

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Contents

- etale cohomology: number theory, more precisely arithmetic geometry.
  Weil conjectures, an analogue of Riemann hypothesis.
- micro-local analysis: partial differential equations of complex variables.
  They have different origins but have close similarities.

1 Etale cohomology

Riemann zeta-function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p:\text{prime}} \frac{1}{1 - \dfrac{1}{p^s}}.$$

The second equality is a consequence of the prime factorization.

analytic continuation, functional equation, zeros and a pole at $s = 1$.

zeros; $s = -2, -4, \cdots$, and $\operatorname{Re} s = \dfrac{1}{2}$. Riemann hypothesis.

analogue in positive characteristic. $\zeta(s)$ is the zeta function of the ring $\mathbf{Z}$

p prime:

$$\zeta_{\mathbf{F}_p[t]}(s) = \sum_{f:\text{monic}} \frac{1}{p^{\deg f \cdot s}} = \prod_{f:\text{monic,irreducible}} \frac{1}{1 - \dfrac{1}{p^{\deg f \cdot s}}}$$
$$= \frac{1}{1 - p^{1-s}}.$$

poles $1 + \dfrac{2\pi n\sqrt{-1}}{\log p}$.

for rings A finitely generated over $\mathbf{F}_p$, or more generally for algebraic varieties X over $\mathbf{F}_p$, the zeta functions $\zeta_A(s), \zeta_X(s)$ are defined.

Weil conjectures. $\zeta_X(s)$ is a rational function of p^{-s} and ...

Weil's observation: Suppose X is projective $\subset \mathbf{P}^n$ and has no singularities. There should be a good cohomology theory satisfying the Lefschetz trace formula

$$\sum_{q=0}^{2d}(-1)^q \mathrm{Tr}(F^n\colon \mathrm{H}^q(X)) = \# X(\mathbf{F}_{p^n}).$$

F is the Frobenius operator induced by the endomorphism of X raising the coordinates to its p-th power.

RHS is the number of points of X with coordinates in $\mathbf{F}_{p^n}$, the field with p^n elements.

$d = \dim X$.

LTF implies the rationality

$$\zeta_X(s) = \frac{P_1(p^{-s})\cdots P_{2d-1}(p^{-s})}{P_0(p^{-s})\cdot P_2(p^{-s})\cdots P_{2d}(p^{-s})}.$$

$P_q(T) = \det(1 - FT\colon \mathrm{H}^q(X))$.

Weil's expectation was realized by Grothendieck's ℓ-adic étale cohomology $\mathrm{H}^q(X_{\overline{\mathbf{F}}_\ell}, \mathbf{Q}_\ell)$.

$\mathbf{Q}_\ell$ is the completion of $\mathbf{Q}$ with respect to the ℓ-adic topology. ℓ is a prime, $\neq p$.

Analogue of Riemann hypothesis: $P_q(T) \in \mathbf{Z}[T] \subset \mathbf{Q}_\ell[T]$ and

the real parts of the zeros of $P_q(p^{-s})$ is $\dfrac{q}{2}$. Deligne's theorem.

2 $\mathcal{D}$-modules

(algebraic) D-module on $\mathbf{C}^n$.

$D = \mathbf{C}[X_1, \ldots, X_n, D_1, \ldots, D_n]$ non-commutative ring.

$X_iX_j = X_jX_i$, $D_iD_j = D_jD_i$, $D_iX_j = X_jD_i$ if $i \neq j$, $D_iX_i = X_iD_i + 1$.

$P \in D$ differential operator. The solution of $P(f) = 0$ is interpreted as a morphism of D-modules: $\mathrm{Hom}_D(\mathcal{M}, \mathcal{O})$ for $\mathcal{M} = D/DP$. $\mathcal{O}$ holomorphic functions.

constant functions: $\mathcal{O} = D/(D_1, \ldots, D_n)$.

Example of (non-degenerate) $\mathcal{D}$-module $\mathcal{M}$; locally free $\mathcal{O}$-modules of finite rank equipped with integrable connections;

de Rham complex

$$DR\mathcal{M} = [\mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^2_X \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^d_X].$$

3 Analogies

- 6 functor formalism: 6 operations $f_*, f^*, f_!, f^!, \otimes, \mathcal{H}om$ vs 4 operations $+, -, \times, \div$.

Sheaf theory. vector spaces are sheaves on one point set. To study X variety, we need to work with sheaves. $D(X)$ derived category of étale sheaves on X.

$f\colon X \to Y$ morphism, $f^*\colon D(Y) \to D(X), f_*\colon D(X) \to D(Y), ...$

$H^q(X_{\overline{\mathbf{F}}_\ell}, \mathbf{Q}_\ell)$. special case of f_* where $Y = \operatorname{Spec} \mathbf{F}_p$ is 1 point.

For a family $f\colon X \to Y$ of varieties, $f_*\mathbf{Q}_\ell$ parametrizes a family of $H^q(X, \mathbf{Q}_\ell)$.

Similarly, X complex manifolds, $D(\mathcal{D}_X)$, ...

X affine and $\mathcal{M}$ $\mathcal{O}$-coherent,

de Rham cohomology $H^q(X, DR\mathcal{M}) =$

$$H^q([\Gamma(X, \mathcal{M}) \xrightarrow{\nabla} \Gamma(X, \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^1_X) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Gamma(X, \mathcal{M} \otimes_{\mathcal{O}_X} \Omega^d_X)])$$

is the special case of f_* where Y is 1 point.

For a family $f\colon X \to Y$ of varieties, we have Gauss–Manin connection parametrizing a family of $H^q(X, DR\mathcal{M})$.

- wild ramification and irregular singularity.

To study Riemann surfaces locally, we focus on a point and a disk surrounding it.

Algebraically, we consider the local ring at a point and its completion.

$\operatorname{Spec} \mathbf{F}_p((t))$ is an positive characteristic analogue of

the punctured disc $\Delta^* = \{z \in \mathbf{C} \mid 0 < |z| < 1\}$.

The absolute Galois group $G_{\mathbf{F}_p((t))}$ corresponds to the fundamental group $\pi_1(\Delta^*) = \mathbf{Z}$.

$G_{\mathbf{F}_p((t))}$ has a 3 step filtration. The graded quotients from above are:

the arithmetic part $G_{\mathbf{F}_p} = \widehat{\mathbf{Z}} = \langle F \rangle$,

the geometric (or tame) part $\widehat{\mathbf{Z}}'$ (away from p),

and the wild part P a huge pro-p-group.

Example of a sheaf with wild ramification:

$X = \mathbf{A}^1_{\mathbf{F}_p} = \operatorname{Spec} \mathbf{F}_p[t] \to Y = \mathbf{A}^1_{\mathbf{F}_p} = \operatorname{Spec} \mathbf{F}_p[x]$ sending t to $x = t^p - t$.

The Galois group $G = \operatorname{Aut}_Y X$ is $\mathbf{F}_p$. $a \in \mathbf{F}_p$ acts on $X = \mathbf{A}^1_{\mathbf{F}_p}$ by $t \mapsto t + a$.

a character of the additive group $G = \mathbf{F}_p \to \mathbf{Q}_\ell(\zeta_p)^\times$ defines a locally constant rank 1 sheaf $\mathcal{L}$ on $Y = \mathbf{A}^1_{\mathbf{F}_p}$.

The Euler–Poincaré characteristic $\chi(\mathbf{A}^1_{\mathbf{F}_p}, \mathcal{L}) = \sum_{q=0}^1 (-1)^q \dim H^q(\mathbf{A}^1_{\mathbf{F}_p}, \mathcal{L})$ satisfies

$\chi(\mathbf{A}^1_{\mathbf{F}_p}, \mathcal{L}) = \chi(\mathbf{A}^1_{\mathbf{F}_p}, \mathbf{Q}_\ell) - 1$.

a special case of the Grothendieck–Ogg–Shafarevich formula.

1 is the contribution of the wild ramification at the infinity.

The action of $P \subset G_{\mathbf{F}_p[[1/x]]}$ is non-trivial because the function x has a pole at ∞.

character of the additive group $\exp\colon \mathbf{C} \to \mathbf{C}^\times$ is a solution of the differential equation $Pf = 0$ for $P = D_1 - 1 \in \mathbf{C}[X_1, D_1]$.

corresponding D-module $\mathcal{M} = D/DP$ is a free $\mathcal{O} = \mathbf{C}[X_1]$-module of rank 1 with integrable connection $\nabla\colon \mathcal{M} \to \mathcal{M} \otimes \Omega^1_X$ sending f to $(f' - f)dX_1$.

The Euler–Poincaré characteristic

$$\chi(\mathbf{A}^1_{\mathbf{C}}, DR\mathcal{M}) = \dim \operatorname{Ker}(\nabla\colon \mathbf{C}[X_1] \to \mathbf{C}[X_1]dX_1) - \dim \operatorname{Coker}(\nabla\colon \mathbf{C}[X_1] \to \mathbf{C}[X_1]dX_1)$$

satisfies

$$\chi(\mathbf{A}^1_{\mathbf{C}}, DR\mathcal{M}) = 0 - 0 = \chi(\mathbf{A}^1_{\mathbf{C}}, DR\mathcal{O}) - 1.$$

1 is the contribution of the irregular singularity at ∞.

$d\dfrac{1}{X_1}$ has a pole of order 2.

order at most 1: regular singularity

order greater than 1: irregular singularity. order -1 = irregularity.

4 Micro-local analysis

- $\mathcal{M}$ coherent (=finitely generated) $\mathcal{D}$-module on a complex manifold X.
  Singular support of $\mathcal{M}$,
  $SS\mathcal{M} \subset T^*X$ is defined as a closed conical (=stable under $\mathbf{C}^\times$-action) subset in the cotangent bundle T^*X.
  We have $\dim SS\mathcal{M} \geqq \dim X$. Involutivity (Gabber).
  We say that $\mathcal{M}$ is holonomic if the equality holds.
  For an integrable connection $\mathcal{M} \neq 0$,
  $SS\mathcal{M}$ is the 0-section $T^*_X X$ and $\mathcal{M}$ is holonomic.
  Assume that $\mathcal{M}$ is holonomic.
  Characteristic cycle $CC\mathcal{M} = \sum_a n_a C_a$ of $\mathcal{M}$ is defined as
  a $\mathbf{Z}$-linear combination of irreducible components of $SS\mathcal{M} = \bigcup_a C_a$.
  Further if X is compact, we have the index formula

$$\chi(X, DR\mathcal{M}) = \deg(CC\mathcal{M}, T^*_X X).$$

RHS is the intersection number with the 0-section.

Example For the open immersion $j\colon \mathbf{C} \to X = \mathbf{P}^1_{\mathbf{C}}$ and $\mathcal{M} = D/D(D_1 - 1)$ on $\mathbf{C}$ as in Example. We have

$$CCj_*\mathcal{M} = -(T^*_X X(\text{ 0-section}) + 2T^*_\infty X(\text{ fiber at } \infty)).$$

$2 = 1(\text{rank}) + 1(\text{irregularity})$. In the index formula, RHS is $-(-2+2) = 0$.

- X variety over $\mathbf{F}_p$ without singular points.
  $\mathcal{F}$ étale sheaf on X.
  Beilinson defined $SS\mathcal{F}$ as a closed conical subset of the cotangent bundle T^*X.
  and proved the holonomicity $\dim C_a = \dim X$ for irreducible components of $SS\mathcal{F} = \bigcup_a C_a$.
  S. defined $CC\mathcal{F} = \sum_a n_a C_a$, $n_a \in \mathbf{Z}$.
  and for X projective proved the index formula

$$\chi(X_{\bar{\mathbf{F}}_p}, \mathcal{F}) = \deg(CC\mathcal{F}, T^*_X X).$$

Example For the open immersion $j\colon \mathbf{A}^1_{\mathbf{F}_p} \to X = \mathbf{P}^1_{\mathbf{F}_p}$ and $\mathcal{L}$ as in Example. We have

$$CCj_*\mathcal{L} = -(T^*_X X(\text{ 0-section}) + 2T^*_\infty X(\text{ fiber at } \infty)).$$

$2 = 1(\text{rank}) + 1(\text{wild ramification})$.

-mixed characteristic.
replace $\mathbf{F}_p$ by $\mathbf{Z}_{(p)} = \{m/n \in \mathbf{Q} \mid n \text{ prime to } p\}$.
What is the cotangent bundle.
Geometric case, Kähler differential:
$d(x+y) = dx + dy, \qquad d(xy) = xdy + ydx.$
replace this by Frobenius–Witt differential:
$w(x+y) = wx + wy - P(x,y)wp, \qquad w(xy) = x^p wy + y^p wx.$
$P(x,y) = \dfrac{1}{p}((x+y)^p - x^p - y^p).$