Abstracts

Singular supports in positive and mixed characteristics

Takeshi Saito

1. Positive characteristic

Let X be a smooth scheme over a field k of characteristic $p > 0$. Let Λ be a finite field of characteristic $\ell \neq p$ and by abuse of terminology we call a bounded constructible complex $\mathcal{F}$ of Λ-modules on $X_{\text{ét}}$ a sheaf on X.

A closed subset C of a vector bundle E on X is said to be conical if it is stable under the $\mathbb{G}_m$-action. A closed conical subset C is uniquely determined by the intersection $C \cap X$ with the 0-section, called the base of C, and the projectivization $\mathbb{P}(C) \subset \mathbb{P}(E)$.

Let $h\colon W \to X$ be a morphism of smooth schemes over k. For a closed conical subset $C \subset T^*X$, its pull-back $h^*C \subset T^*X \times_X W$ is defined to be the inverse image by $T^*X \times_X W \to T^*X$. The morphism h is called C-transversal if the intersection $h^*C \cap \operatorname{Ker}(T^*X \times_X W \to T^*W)$ is a subset of the 0-section of $T^*X \times_X W$.

For example, if C is the conormal bundle $T^*_Z X = \operatorname{Ker}(T^*X|_Z \to T^*Z)$ of a closed subscheme $Z \subset X$ smooth over k, then h is C-transversal if and only if h is transversal to $Z \to X$; namely, $V = Z \times_X W$ is smooth over k and $\operatorname{codim}_W V = \operatorname{codim}_X Z$.

We say that a separated morphism h is $\mathcal{F}$-transversal, if the canonical morphism $h^*\mathcal{F} \otimes Rh^!\Lambda \to Rh^!\mathcal{F}$ is an isomorphism. We say that $\mathcal{F}$ is micro supported on C if the following conditions is satisfied:

For every pair of separated morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes over k, if $(h, f)\colon W \to X \times Y$ is $C \times T^*Y$-transversal, then (h, f) is $\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$-transversal for every sheaf $\mathcal{G}$ on Y.

If the smallest closed conical subset $C \subset T^*X$ on which $\mathcal{F}$ is micro supported exists, we call $C = SS\mathcal{F}$ the singular support of $\mathcal{F}$. The existence is non-trivial because $\mathcal{F}$ being micro supported on C_1 and C_2 does not imply a priori $\mathcal{F}$ being micro supported on the intersection $C_1 \cap C_2$.

Theorem 1 (Beilinson [1]) 1. *$SS\mathcal{F}$ always exists.*

2. *Every irreducible component of $SS\mathcal{F}$ has the same dimension as X.*

If X is a curve, an irreducible component of dimension 1 of a closed conical subset of a line bundle T^*X over X is either the 0-section or the fiber of a closed point. The 0-section appears in $SS\mathcal{F}$ if and only if the sheaf $\mathcal{F}$ is generically non-zero. The fiber of a closed point appears if and only if the sheaf ramifies there. In higher dimension, $SS\mathcal{F}$ can be more complicated.

A key tool in the proof by Beilinson of Theorem 1 is the Radon transform. First, we reduce the proof to the case where X is a projective space $\mathbb{P}^n$. The dual projective space $\mathbb{P}^{n\vee}$ is the moduli of hyperplanes in $\mathbb{P}^n$. The universal family Q of hyperplanes is canonically identified with the projectivizations $\mathbb{P}(T^*\mathbb{P}^n) =$

$\mathbb{P}(T^*\mathbb{P}^{n\vee})$. Since the base of $SS\mathcal{F}$ equals the support of $\mathcal{F}$, the singular support is essentially determined by its projectivization $\mathbb{P}(SS\mathcal{F}) \subset Q$. Using this fact and analyzing the projections $Q \to \mathbb{P}^n, \mathbb{P}^{n\vee}$ as h and f in the definition of micro support, one can prove Theorem 1.

2. Mixed characteristic

Let X be a regular noetherian scheme over $\mathbb{Z}_{(p)}$. To consider singular supports in mixed characteristic case, we first need to solve the problem: Where $SS\mathcal{F}$ should live? In the geometric case, the vector bundle T^*X is defined by $\Omega^1_{X/k}$. In mixed characteristic, Ω^1_X may not be locally free. Even if it is, it will be too small, e.g. for $X = \operatorname{Spec} \mathbb{Z}_{(p)}$.

A solution is given by the Frobenius–Witt differentials. The sheaf Ω^1_X of Kähler differentials is defined by the universality for the usual derivations satisfying $d(x+y) = dx+dy$ and $d(xy) = xdy+ydx$. The sheaf $F\Omega^1_X$ of FW differentials is defined by replacing these relations by $d(x+y) = dx + dy + ((x+y)^p - x^p - y^p)/p \cdot dp$ and $d(xy) = x^p dy + y^p dx$. The fraction in the first equality means the substitution to the quotient as a polynomial.

We assume the following finiteness condition:

(F) The reduced part $X_{\mathbb{F}_p,\mathrm{red}}$ of the characteristic p fiber is of finite type over a field k of finite p-basis $[k : k^p] < \infty$.

Then, the $\mathcal{O}_X$-module $F\Omega^1_X$ is a locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-module of finite type. For $x \in X_{\mathbb{F}_p}$, we have a short exact sequence $0 \to F^*\mathfrak{m}_x/\mathfrak{m}_x^2 \to F\Omega^1_{X,x} \otimes k(x) \to F^*\Omega^1_{k(x)} \to 0$ where F^* denotes the Frobenius pull-back. For example, if X is of finite type over a complete discrete valuation ring of mixed characteristic with perfect residue field, the rank of the locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-module $F\Omega^1_X$ is $\dim X$.

In the following, we assume the finiteness condition (F) above and define the FW-cotangent bundle FT^*X on $X_{\mathbb{F}_p}$ to be the vector bundle corresponding to $F\Omega^1_X$. Although it is restricted to the characteristic p fiber, the vector bundle has the correct rank.

Let $h\colon W \to X$ be a separated morphism of finite type of regular noetherian schemes. For a closed conical subset $C \subset FT^*X$, we say that h is C-transversal if the intersection $h^*C \cap \operatorname{Ker}(FT^*X \times_X W \to FT^*W)$ is a subset of the 0-section of $FT^*X \times_X W$. We say that a sheaf $\mathcal{F}$ on X is micro supported on C if the following conditions are satisfied:

(i) The intersection of the support of $\mathcal{F}$ with $X_{\mathbb{F}_p}$ is a subset of the base of C.

(ii) For every separated morphism $h\colon W \to X$ of finite type of regular noetherian scheme if h is C-transversal, then h is $\mathcal{F}$-transversal on a neighborhood of $W_{\mathbb{F}_p}$.

To use the Radon transform, we fix a regular noetherian scheme S over $\mathbb{Z}_{(p)}$ satisfying (F) and introduce a relative version. For a closed conical subset $C \subset FT^*X$, we say that a pair (h, f) of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of regular schemes of finite type over S such that Y is smooth over S is C-acyclic if we have an inclusion

$$
\begin{aligned}
&(h^*C \times_W (FT^*Y \times_Y W)) \cap \mathrm{Ker}((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*W) \\
&\subset \mathrm{Ker}((FT^*X \times_X W) \times_W (FT^*Y \times_Y W) \to FT^*(X \times_S Y) \times_{X \times_S Y} W).
\end{aligned}
$$

We say that a sheaf $\mathcal{F}$ on X is S-micro supported on C if the following condition is satisfied:

For every C-acyclic pair (h, f) as above and for every sheaf $\mathcal{G}$ on Y micro supported on some closed conical subset $C' \subset FT^*Y$ such that $C' \cap \mathrm{Im}(FT^*S \times_S Y \to FT^*Y)$ is a subset of the 0-section of FT^*Y, the morphism $(h, f)\colon W \to X \times_S Y$ is $\mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$-transversal on a neighborhood of $W_{\mathbb{F}_p}$.

We define $SS\mathcal{F}$ and $SS_S\mathcal{F}$ to be the smallest closed conical subsets of FT^*X on which $\mathcal{F}$ is micro supported and is S-micro supported respectively. We say that a closed conical subset $C \subset FT^*X$ is S-stable if C is stable under the action of $FT^*S \times_S X$. We also define $SS_S^{\mathrm{sat}}\mathcal{F}$ to be the smallest S-stable closed conical subset of FT^*X on which $\mathcal{F}$ is S-micro supported. Although we don't know the existence, we expect to have inclusions $SS_S\mathcal{F} \subset SS\mathcal{F} \subset SS_S^{\mathrm{sat}}\mathcal{F}$.

By adopting Beilinson's argument using Radon transform, we obtain the following.

Theorem 2 *If X is smooth over S, $SS_S^{\mathrm{sat}}\mathcal{F}$ exists.*

References

[1] A. Beilinson, *Constructible sheaves are holonomic*, Selecta Math. **22** (4), (2016), 1797–1819.

[2] T. Saito, *Cotangent bundles and micro-supports in mixed characteristic case*, Algebra & Number Theory **16**-2 (2022), 335–368.

Reporter: Qi Ge