Singular support

X regular schm / $\mathbb{Z}_{(p)}$

Λ / $\mathbb{F}_\ell$ finite. $\ell \neq p$.

$\mathcal{F}$ $D^b_c(X_{et}, \Lambda)$ $\mathcal{H}^q(\mathcal{F})$ constructible

$= 0$ for almost all q

positive / mixed

$SS\mathcal{F}$ established / incomplete

positive char Beilinson with slight modification

k ch $= p$

X/k smooth T^*X cotangent bdle $\Omega^1_{X/k}$

rk $=$ dim X

C closed conical subset $\subset T^*X$

stable under $\mathbb{G}_m$-action

Control property of $\mathcal{F}$ by property of C.

$h: W \to X$ C-transversal

h^*C inv. image of C by $T^*X \times_X W \to T^*X$

$h^*C \cap \mathrm{Ker}(T^*X \times_X W \to T^*W)$

$\subset$ 0-section

e.g. $Z \subset X$ closed subschm smooth/k

$T^*_Z X = \mathrm{Ker}(T^*X|_Z \to T^*Z)$ conormal bdle

$C = T^*_Z X$ C-transversal

$\Leftarrow$ $V = Z \times_X W$ smooth $/k$

& $\mathrm{codim}_W V = \mathrm{codim}_X Z$

h smooth $\Rightarrow$ $\forall C$ h C-transversal

corresponding property $\mathcal{F}$-transversal

$$h^* \mathcal{F} \otimes h^! \Lambda \to h^! \mathcal{F}$$ omit to write R

is an isom

h smooth $\overset{P.D}{\Rightarrow}$ $\forall \mathcal{F}$ h $\mathcal{F}$-transversal.

Def. (Beilinson + ε.)

1. We say $\mathcal{F}$ is micro supported on C if

$\forall h : W \to X$, $\forall f : W \to Y$

$(h, f) : W \to X \times Y$ $C \times T^*_Y Y$-transversal

$\Rightarrow$ $\forall \mathcal{G}$ on Y

$\mathcal{F} \boxtimes \mathcal{G}$-transversal.

2. If smallest $\underline{C \text{ exists}}$, we call it $SS\mathcal{F}$.

non trivial $C, C' \Rightarrow C \cap C'$

Thm (Beilinson)

1. $SS\mathcal{F}$ always exists.
2. $SS\mathcal{F} = \bigcup C_a$ $\forall C_a$ $\dim C_a = \dim X$

$$CC\mathcal{F} = \sum m_a C_a.$$

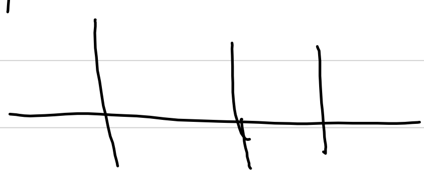
X proj smooth $\quad \chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$

Example of $SS\mathcal{F}$

$SS\mathcal{F} = \emptyset \iff \mathcal{F} = 0$

$SS\mathcal{F} = T^*_X X \iff \mathcal{F}$ l.c. & $\neq 0$

$\dim X = 1$



($\dim X > 1$. not necessarily Lagrangean
$\uparrow$
$T^*_Z X$)

Key tool of the pf.

After reduction to $X = \mathbb{P}^n$

use Radon transform

$$\mathbb{P} \xleftarrow{h} Q \xrightarrow{f} \mathbb{P}^\vee \quad \text{moduli of hyperplanes}$$

univ. family of hyperplanes

$\|$

$\mathbb{P}(T^*\mathbb{P})$

$C \subset T^*X \quad \leftrightarrow \quad B = C \cap T^*_X X \subset X \quad$ base

closed conical $\quad \mathbb{P}(C) \subset \mathbb{P}(T^*X)$ projectivization

base of $SS\mathcal{F}$ = supp of $\mathcal{F}$

much more complicated

mixed char. Where should SST live?

Ω^1_X may not be locally free

$\Omega_{\mathbb{Z}_{(p)}} = 0$ too small.

$F\Omega^1_X$ Frobenius-Witt diff.

modify Kähler diff

$d(x+y) = dx + dy$, $d(xy) = x\,dy + y\,dx$.

$X/\mathbb{Z}_{(p)}$ regular noetherian.

$X_{\mathbb{F}_p}$ red f.t / k s.t $[k:k^p] < \infty$

$F\Omega^1_X$ locally free $\mathcal{O}_{X_{\mathbb{F}_p}}$-mod of fin. rk

$x \in X_{\mathbb{F}_p}$ $0 \to F^* m_x/m_x^2 \to F\Omega^1_{X,x} \otimes k(x) \to F^*\Omega^1_{k(x)} \to 0$

Frobenius pull-back

If X/k smooth & perfect

$\Rightarrow F\Omega^1_X = F^*\Omega^1_{X/k}$ | $X/\mathbb{Z}_p$ f.t. rk $= \dim X$.

(relation with $H_1(L\Omega)$, δ-ring, norm of local fd.)

FT^*X. v.b on $X_{\mathbb{F}_p}$.

How to define SS $\mathcal{F}$

format micro supp C controls $\mathcal{F}$
(good property for C
$\Rightarrow$ corresponding property for $\mathcal{F}$)

SS smallest C on which $\mathcal{F}$ is m.s.
Existence?

$f: X \to \mathbb{A}^1_S$

imitate Beilinson's def. $X \xleftarrow{h} W \xrightarrow{f} Y$
problem not enough f. f def
modify in 2 ways

1. absolute forget f.

Def $\mathcal{F}$ is m.s on C if $\mathrm{supp}\,\mathcal{F} \subset \mathrm{base}\, C$ &
$\forall h: W \to X$ sep f.t of reg sch
C-trans $h^\circ C \cap \mathrm{Ker}(FT^*X \times_X W \to FT^*W)$
$\subset 0$-section
$\Rightarrow$ $\mathcal{F}$-trans on a nbd of $W_{\mathbb{F}_p}$

If X/k sm. k perfect $\Rightarrow$ Equiv to Beilinson's def

Need f for Radon transform.

2. relative fix $S/\mathbb{Z}_{(p)}$ reg noeth. + finiteness cond

Def. $\mathcal{F}$ is S-mcs on C if

$\forall\, h: W \to X$, $f: X \to Y$, Y smooth$/S$

(h, f) $\underline{C\text{-acyclic over } S}$ skip

$\Rightarrow (h, f)$ $\mathcal{F}$-acyclic over S and!

$\forall\, \mathcal{G}$ on Y s.t. $\exists\, C' \subset FT^*Y$ on which $\mathcal{G}$ mcs on C'

$C' \cap \mathrm{Im}(FT^*S \to FT^*Y)$

$\subset 0$-sect

$(h, f): W \to X \times_S Y$ $\mathcal{F} \boxtimes \mathcal{G}$-transversal

smallest $SS_S\mathcal{F}$

C $\underline{S\text{-saturated}}$ stable under $FT^*S \times_S X$-action

smallest $SS_S^{sat}\mathcal{F}$

Expect. $SS_S\mathcal{F} \subset SS\mathcal{F} \subset SS_S^{sat}\mathcal{F}$

$\underline{\text{Thm}}$ If X/S smooth, $SS_S^{sat}\mathcal{F}$ exists

$$0 \to FT^*S \times_S X \to FT^*X \to F^*(T^*X/S\,|\,X_{\mathbb{F}_p}) \to 0$$

$\cup$ $\quad \overset{1:1}{\longleftrightarrow}$ $\quad \cup$ Beilinson.

S-saturated