not familiar

microlocal analysis in algebraic geometry

ℓ-adic, mod ℓ

Characteristic cycle

X smooth / k perfect. $\dim X = n$.

Λ finite field char ℓ inv. in k

$\mathcal{F}$ constructible complex of Λ-mod on X.

$\operatorname{supp} \mathcal{F} \subset X$ closed subset

Beilinson $SS\mathcal{F} \subset T^*X$ singular support $\dim SS\mathcal{F} = n$

closed conical subset of the cotangent b'dle . $\dim T^*X = 2n$

stable under $\mathbb{G}_m$-action

$SS\mathcal{F} = \bigcup C_a$. C_a irred dim n. $SS\mathcal{F} \cap T^*_X X = \operatorname{supp} \mathcal{F}$

$CC\mathcal{F} = \sum m_a C_a$. $m_a \in \mathbb{Z}$. characteristic cycle.

$\mathcal{F}$ perverse $\Rightarrow$ $m_a > 0$ $\forall a$.

1. Classical example.
2. Characteristic class.
3. Direct image.

smallest

Example. $\dim X = 1$. $D \subset X$ $\mathcal{F}|_{X-D}$ locally const $\neq 0$

$SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in D} T^*_x X$

0-section fiber

$$CC\mathcal{F} = -\left(\operatorname{rk} \mathcal{F}|_{X-D} \cdot T^*_X X + \sum_{x \in D} a_x \mathcal{F} \cdot T^*_x X\right)$$

$a_x \mathcal{F}$ Artin conductor

$= \operatorname{rk} \mathcal{F} - \operatorname{rk} \mathcal{F}_{\bar{x}} + \operatorname{Sw}_x \mathcal{F}$ Swan conductor.

not Lagrangean in higher dim.

2. Direct image

$f: X \to Y$ proper morphism of smooth curves $/k$

$\mathcal{F}$ on X $\dim X = n$. $\dim Y = m$

Relation between

$SS\mathcal{F}, CC\mathcal{F}$ on T^*X and $SSRf_*\mathcal{F}, CCRf_*\mathcal{F}$ on T^*Y

$$\begin{array}{ccccc} T^*X & \xleftarrow{a} & X \times_Y T^*Y & \xrightarrow{b} & T^*Y \\ \cup & & \cup & & \cup \\ C & & a^{-1}(C) & & b(a^{-1}(C)) = f_\circ(C) \end{array}$$ b proper

Beilinson $SSRf_*\mathcal{F} \subset f_\circ SS\mathcal{F}$. closed conical

A $a^!A$ $f_!A = b_!a^!A \in CH_m(f_\circ C)$

algebraic cycle / rational equiv.

$CH_m = Z_m$ if $\dim f_\circ C \leqq m$.

Conjecture 1. $CCRf_*\mathcal{F} = f_!CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$.
Satisfied if $CH_m = 0$

Theorem 1. Assume X & Y proj. $f: X \to Y$ proj & $\dim f_\circ SS\mathcal{F} \leqq m$. Then

$$CCRf_*\mathcal{F} = f_!CC\mathcal{F} \text{ in } Z_m(f_\circ SS\mathcal{F})$$

Example 1 $Y = \operatorname{Spec} k$. $\dim f_\circ SS\mathcal{F} = 0$ always.

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$$ index formula

Further if $\dim X = 1$, Grothendieck-Ogg-Shafarevich.

2. $\dim Y = 1$.

$$-a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, X_y}$$ for each y.

If $\mathcal{F} = \Lambda$, Bloch's conductor formula.

Pf. G-O-S $\Rightarrow$ index formula $\Rightarrow$ conductor formula $\Rightarrow$ Thm 1.

i.f $\Rightarrow$ c.f $\sum_{y'} = \sum_{y'}$.

Fix y. Find $Y' \to Y$ étale at y. Killing other terms

analog of stable reduction theorem

3 Characteristic class

X possibly singular $i: X \to M$ closed immersion

M smooth $N = \dim M$

$CC\, i_*\mathcal{F} = \sum m_a C_a \quad C_a \subset X \times_M T^*M$

capital

lower case proj. completion $\cap$

$cc_X \mathcal{F} = \sum m_a \overline{C_a} \quad \overline{C_a} \subset \mathbb{P}(X \times_M T^*M \oplus \mathbb{A}^1_X)$

characteristic $\in CH_N(\mathbb{P}(\text{——}))$

class $= \bigoplus CH_q(X) = CH_\bullet(X)$

indep of M, i.

$K(X.\Lambda)$ Grothendieck group of cat of const. complexes.

$$cc_X : K(X.\Lambda) \to CH_\bullet(X)$$

char $k = 0$ Sabbah's construction of the MacPherson Chern class

$cc_X \mathcal{F} = (-1)^? c_M \mathcal{F}$.

Grothendieck's question at SGA5. (R et S)

$f: X \to Y$ proper

$$\begin{tikzcd} K(X.\Lambda) \arrow[r, "cc_X"] \arrow[d, "f_*"'] \arrow[dr, phantom, "\circlearrowleft"] & CH.(X) \arrow[d, "f_!"] \\ K(Y.\Lambda) \arrow[r, "cc_Y"] & CH.(Y) \end{tikzcd}$$

~~Elementary~~ Counterexample need to replace $CH_\bullet$ by $CH_{\mathbb{Q}}$

Conj 1 $\Rightarrow$ C.D. for $CH_{\mathbb{Q}}$

Umezaki - Yang - Zhao

k finite $X.Y$ proj smooth $\Rightarrow$ CD for $CH_{\mathbb{Q}}$.