Singular Supports in Equal and Mixed Characteristics

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Introduction

In the geometric case, the theory was established by Beilinson in [1]. In the mixed characteristic case, the theory is far from complete [14]. First, we review the theory of Beilinson; the definition and the existence of singular support. Later, we discuss its variations to adjust it in mixed characteristic situation.

Let k be a field and X a smooth scheme over k. The cotangent bundle T^*X is the covariant vector bundle associated to $\Omega^1_{X/k}$.

Let Λ be a finite field of characteristic different from that of k. An object of $D^b_{(c)}(X_{\text{ét}}, \Lambda)$ will be called a sheaf on X.

A closed subset $C \subset E$ of a vector bundle is called conical if it is stable under the $\mathbf{G}_m$-action.

The singular support $SS\mathcal{F}$ of $\mathcal{F}$ is defined as a closed subset $C \subset T^*X$. The relation between $\mathcal{F}$ and C is indirect, due to the lack of micro local sheaf theory. It goes along the following general format:

First step: We say that $\mathcal{F}$ is micro supported on C if a certain good functorial property for C implies the corresponding property for $\mathcal{F}$.

Definition 0.1. If the smallest C on which $\mathcal{F}$ is micro supported exists, we call such C the singular support of $\mathcal{F}$.

The existence is non-trivial because it is not clear from the definition if the condition that $\mathcal{F}$ is micro supported on C and on C' implies that $\mathcal{F}$ is micro supported on the intersection $C \cap C'$.

The existence is proved in [1] by the reduction to the case where $X = \mathbf{P}^n$. A closed conical subset C is determined by its base $C \cap X$ and the projectivization $\mathbf{P}(C) \subset \mathbf{P}(T^*X)$. The base of the singular support equals the support of the sheaf. If $X = \mathbf{P}$, the projective space bundle $\mathbf{P}(T^*\mathbf{P})$ is canonically identified with the universal family Q of hyperplanes over the moduli $\mathbf{P}^\vee$ of hyperplanes. Using the diagram

$$\begin{array}{ccc} \mathbf{P} & \xleftarrow{\;q\;} & Q \\ & & \downarrow{\scriptstyle q^\vee} \\ & & \mathbf{P}^\vee \end{array}$$

and the (naive) Radon transform defined by $R\mathcal{F} = q^\vee_* q^* \mathcal{F}$, we will capture $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ geometrically.

1 Original definition

1.1 Local acyclicity

Definition 1.1 ([5])**.** Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a sheaf on X. We say that f is $\mathcal{F}$-acyclic (or equivalently, locally acyclic relatively to $\mathcal{F}$) if the following condition is satisfied:

Let $t \to s$ be a specialization of geometric points of Y and let

$$\begin{array}{ccccc} X_s & \xrightarrow{i} & X \times_Y Y_{(s)} & \xleftarrow{j} & X_t \\ \downarrow & & \downarrow & & \downarrow \\ s & \longrightarrow & Y_{(s)} & \longleftarrow & t \end{array}$$

be the cartesian diagram. Then the morphism

$$\mathcal{F}_{X_s} \to i^* j_* \mathcal{F}_{X_t} \tag{1.1}$$

is an isomorphism.

We say that f is universally $\mathcal{F}$-acyclic if for every $g\colon Y' \to Y$, the base change $f'\colon X' = X \times_Y Y' \to Y'$ is $g'^*\mathcal{F}$-acyclic.

The condition that the morphism (1.1) is an isomorphism means that for every geometric point x of X_s, the morphism $\mathcal{F}_x \to R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{F}_{X_{(x)} \times_{S_{(s)}} t})$ is an isomorphism. The fiber $X_{(x)} \times_{S_{(s)}} t$ of the morphism $X_{(x)} \to S_{(s)}$ is called the Milnor fiber.

To check the universal local acyclicity (abbreviated ula), it suffices to consider only smooth morphisms $Y' \to Y$.

Examples 1.2. 1. $f\colon X \to Y$ smooth, $\mathcal{F}$ locally constant. Local acyclicity of smooth morphisms.

2. $f\colon X \to \operatorname{Spec} k$, $\mathcal{F}$ arbitrary. f is universally $\mathcal{F}$-acyclic. Generic universal local acyclicity ([5]).

3. If $f\colon X \to Y$ is $\mathcal{F}$-acyclic and if $g\colon Y \to Z$ smooth, then $gf\colon X \to Z$ is $\mathcal{F}$-acyclic ([8]). This is a generalization of 1.

4. $f = 1_X$ is $\mathcal{F}$-acyclic if and only if $\mathcal{F}$ is locally constant.

5. If $f\colon X \to Y$ is proper and if $gf\colon X \to Z$ is $\mathcal{F}$-acyclic, then g is $f_*\mathcal{F}$-acyclic. This is a consequence of the proper base change theorem.

1.2 Definition

Definition 1.3. Let X be a smooth scheme over k, $\mathcal{F}$ be a sheaf on X and $C \subset T^*X$ be a closed conical subset. We say that $\mathcal{F}$ is micro supported on C if the following condition is satisfied: Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of smooth schemes over k. If h is <u>C-transversal</u> and if f is <u>$h^\circ C$-acyclic</u>, then f is $h^*\mathcal{F}$-acyclic.

In the original terminology in [1], the C-acyclicity was also called the C-transversality.

The condition that f is $h^*\mathcal{F}$-acyclic can be equivalently replaced by the condition that f is universally $h^*\mathcal{F}$-acyclic. As h, it suffices to consider étale morphisms. As f, it suffices to consider morphisms to $\mathbf{A}^1$.

For a closed conical subset C of a vector bundle E over a scheme X, we define the base B and the support S by

$$B = \{x \in X \mid C_x \neq \varnothing\} \supset S = \{x \in X \mid C_x \not\subset \{0\}\}$$

where C_x denotes the fiber of C at x. Both B and S are closed subsets of X since $B = C \cap X$ is the intersection with the 0-section and S is the image of the projectivization $\mathbf{P}(C) \subset \mathbf{P}(E)$.

Definition 1.4. Let X be a smooth scheme over k and $C \subset T^*X$ be a closed conical subset. Let $h\colon W \to X$ be a morphism of smooth schemes over k. We say that h is C-transversal if the support of the intersection

$$h^*C \cap \operatorname{Ker}(T^*X \times_X W \to T^*W) \tag{1.2}$$

is empty.

h^*C is defined to be the inverse image of C by $T^*X \times_X W \to T^*X$.

Examples 1.5. 1. Let $Z \subset X$ be a closed subscheme smooth over k and let $C = T^*_Z X$ be the conormal bundle. Then h is C-transversal if and only if h is transversal to $Z \to X$. Namely, $V = W \times_X Z \subset W$ is smooth over k and the codimensions $\operatorname{codim}_W V$ and $\operatorname{codim}_X Z$ are the same.

2. If h is smooth, h is C-transversal for every C.
3. If C is a subset of the 0-section $T^*_X X$, every h is C-transversal.
4. If h is C-transversal and if $C' \subset C$, then h is C'-transversal.
5. If h is an immersion, $\operatorname{Ker}(T^*X \times_X W \to T^*W)$ is the conormal sheaf $T^*_W X$.

Lemma 1.6. *Let $E \to F$ be a linear morphism of vector bundles on a scheme X and let $C \subset E$ be a closed conical subset. If the intersection $C \cap \operatorname{Ker}(E \to F)$ is a closed subset of the 0-section, then $E \to F$ is finite on C and consequently, the image $C' \subset F$ is a closed conical subset.*

Proof. It is sufficient to prove the following statement: Let $A \to B$ be a morphism of graded rings and $J \subset B$ be a graded ideal. Let $S \subset B$ a set of homogeneous elements. Then, the following conditions are equivalent:

(1) B/J is generated by S as an A-module.

(2) $B/(J + A_{\geqq 1}B)$ is generated by S as an $A/A_{\geqq 1}$-module.

By induction on n, condition (2) is equivalent to

(3) $B/(J + A_{\geqq n}B)$ is generated by S over $A/A_{\geqq n}$ for every n.

It is reduced to the equivalence of the following conditions applied to the graded A-module $M = B/(J + AS)$:

(1′) $M = 0$.

(3′) $M/A_{\geqq n}M = 0$ for every n. □

By Lemma 1.6, if h is C-transversal, the image $h^\circ C \subset T^*W$ of h^*C by $T^*X \times_X W \to T^*W$ is a closed conical subset.

Definition 1.7. Let X be a smooth scheme over k and $C \subset T^*X$ be a closed conical subset. Let $f\colon X \to Y$ be a morphism of smooth schemes over k. We say that f is C-acyclic if the support of the inverse image of C by $T^*Y \times_Y X \to T^*X$ is empty.

C-acyclicity is also an open condition.

Examples 1.8. 1. If C is a subset of the 0-section and if f is smooth, then f is C-acyclic.

2. The morphism $X \to \operatorname{Spec} k$ is C-acyclic for every $C \subset T^*X$.
3. If $f\colon X \to Y$ is C-acyclic and if $g\colon Y \to Z$ is smooth, then $gf\colon X \to Z$ is C-acyclic.
4. $f = 1_X$ is C-acyclic if and only if C is a subset of the 0-section.
5. If f is C-acyclic, then f is smooth on a nbd of the base of C.

The following conditions are equivalent:
(1) $h\colon W \to X$ is C-transversal and $f\colon W \to Y$ is $h^\circ C$-acyclic.
(2) $(h,f)\colon W \to X \times Y$ is $C \times T^*Y$-transversal.
We say that (h,f) is C-acyclic if (2) (and hence (1)) is satisfied.

Examples 1.9. 1. Every sheaf on X is micro supported on T^*X. (h,f) is T^*X-acyclic if and only if $(h,f)\colon W \to X \times Y$ is smooth: We may assume $W = X \times Y \times \mathbf{A}^n$ since the local acyclicity is an étale local condition. Consider the cartesian diagram

$$\begin{array}{ccccc} X & \longleftarrow & X \times Y & \longleftarrow & W \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec} k & \longleftarrow & Y & \longleftarrow & Y \times \mathbf{A}^n. \end{array}$$

By the generic local acyclicity (Examples 1.2.1), the right vertical arrow is $h^*\mathcal{F}$-acyclic. Since $Y \times \mathbf{A}^n \to Y$ is smooth, the composition $W \to Y$ is $h^*\mathcal{F}$-acyclic by Examples 1.2.3.

2. If $\mathcal{F}$ is ms on C, then $\operatorname{supp} \mathcal{F}$ is a subset of the base B of C. In other words, $\mathcal{F}|_U = 0$ on the complement $U = X - B$: Let $h\colon U \to X$ be the open immersion and $f\colon U \to Y$ be the 0-map. Then (h,f) is C-acyclic and hence f is $\mathcal{F}|_U$-acyclic. This means $\mathcal{F}|_U = 0$.

3. Assume that $\mathcal{F}$ is ms on C. Let $U \subset X$ be an open subset and assume that $\mathcal{F}|_U$ is ms on C'. Then, $\mathcal{F}$ is ms on the union $C_1 = C|_{X - U} \cup \overline{C'}$. In particular, if $\mathcal{F}|_U = 0$, then $\mathcal{F}$ is ms on $C|_{X - U}$: Assume (h,f) is C_1-acyclic. Then, by the open condition property, on a neighborhood V of the inverse image of the complement $X - U$, (h,f) is C-acyclic. On the inverse image of U, (h,f) is C'-acyclic. Hence f is $h^*\mathcal{F}$-acyclic on $h^{-1}(U) \cup V = W$.

4. $\mathcal{F} = 0$ if and only if $\mathcal{F}$ is ms on $\varnothing$.

5. $\mathcal{F}$ is locally constant if and only if $\mathcal{F}$ is ms on the 0-section $T^*_X X$: If (h,f) is $T^*_X X$-acyclic, then f is smooth by Examples 1.8.5. Hence if $\mathcal{F}$ is locally constant, f is $h^*\mathcal{F}$-acyclic by Examples 1.2.2.

Conversely, assume that $\mathcal{F}$ is ms on $T^*_X X$. Since $(1,1)$ is $T^*_X X$-acyclic, 1_X is $\mathcal{F}$-acyclic. This means that $\mathcal{F}$ is lcc by Examples 1.2.4.

2 Proof of existence

2.1 Reduction to $\mathbf{P}^n$

The existence is reduced to the case where $X = \mathbf{P}^n$ by the following steps.

1. reduction to affine schemes: Since the assertion is local, we may assume X is affine.

2. reduction to $\mathbf{A}^n$: Since the existence of $SSi_*\mathcal{F}$ implies that of $SS\mathcal{F}$ for a closed immersion i, we may assume $X = \mathbf{A}^n$.

3. reduction to $\mathbf{P}^n$: Since the assertion is local, we may assume $X = \mathbf{P}^n$.

Local.

Proposition 2.1. 1. *Let $U \subset X$ be an open subset. If $\mathcal{F}$ is micro supported on C, then $\mathcal{F}|_U$ is micro supported on $C|_U$. If $SS\mathcal{F} = C$, then $SS\mathcal{F}|_U = C|_U$.*

2. *Let (U_i) be an open covering of X. Let C be a closed conical subset. If $\mathcal{F}|_{U_i}$ is micro supported on $C|_{U_i}$ for every i, then $\mathcal{F}$ is micro supported on C. If $SS\mathcal{F}|_{U_i} = C_i$, then $SS\mathcal{F} = \bigcup C_i$.*

Proof. 1. Assume that $(h\colon W \to U, f\colon W \to Y)$ is $C|_U$-acyclic. Then $(h\colon W \to X, f\colon W \to Y)$ is C-acyclic. Hence f is $h^*\mathcal{F}$-acyclic.

Assume that $\mathcal{F}|_U$ is ms on C'. Then $\mathcal{F}$ is ms on $C|_{X - U} \cup \overline{C'}$ by Examples 1.9.3. Hence we have $C \subset C|_{X - U} \cup \overline{C'}$ and $C|_U \subset C'$.

2. Assume that (h, f) is C-acyclic. Then $(h_i\colon h^{-1}(U_i) \to U_i, f_i\colon h^{-1}(U_i) \to Y)$ is C_i-acyclic for every i. Hence f_i is $h_i^*\mathcal{F}$-acyclic for every i and f is $h^*\mathcal{F}$-acyclic.

By 1., we have $C_i|_{U_i \cap U_j} = C_j|_{U_i \cap U_j}$ for every i, j. Hence $C = \bigcup C_i$ is a closed conical subset and $C|_{U_i} = C_i$. Assume that $\mathcal{F}$ is ms on C'. Then $\mathcal{F}|_{U_i}$ is ms on $C'|_{U_i}$ and we have $C_i \subset C'|_{U_i}$. Thus we have $C \subset C'$. □

Closed immersion. Let $i\colon X \to P$ be a closed immersion of smooth schemes. For a closed conical subset $C \subset T^*X$, define $i_\circ C \subset T^*P \times_P X \subset T^*P$ to be the inverse image of C by the surjection $T^*P \times_P X \to T^*X$.

Examples 2.2. Let $i\colon Z \to X$ be a closed immersion of smooth schemes over k. Then we have $T^*_Z X = i_\circ T^*_Z Z$.

Let $C \subset T^*Z$ be a closed conical subset. Assume that $h\colon W \to X$ is $i_\circ C$-transversal. Then h is transversal to $Z \to X$ on a neighborhood V' of the inverse image of the base of C. If h is transversal to $Z \to X$ and if $h'\colon V' \to Z$ denotes the restriction of the base change, then h' is C-transversal.

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recall

Definition of ms. If (h, f) is C-acyclic (i.e. (h, f) is $C \times T^*Y$-transversal), f is $h^*\mathcal{F}$-acyclic.

Proof of the existence of SS.

Reduction to $\mathbf{P}^n$. Local + closed immersion.

$\mathbf{P}^n$. Radon transform.

Proposition 2.3. *Let $i\colon X \to P$ be a closed immersion of smooth schemes over k.*

1. *If $\mathcal{F}$ is micro supported on C, then $i_*\mathcal{F}$ is micro supported on $i_\circ C$.*
2. *Let $C_P \subset T^*P$ be a closed conical subset and let $C \subset T^*X$ be the closure of the image of $C_P|_X$ by the surjection $T^*P \times_P X \to T^*X$. If $i_*\mathcal{F}$ is micro supported on C_P, then $\mathcal{F}$ is micro supported on C. If $SSi_*\mathcal{F} = C_P$, then $SS\mathcal{F} = C$.*

Proof. 1. Assume that (h, f) is $i_\circ C$-acyclic. We consider the commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{h'} & V & & \\ {\scriptstyle i}\downarrow & & \downarrow & \overset{f'}{\searrow} & \\ P & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array}$$

with cartesian square. By replacing W by a neighborhood of the inverse image of the base of $i_\circ C$, we may assume that h is transversal to $X \to P$ by Example 2.2. Then, V is smooth over k and further (h', f') is C-acyclic by Example 2.2. Hence f' is $h'^*\mathcal{F}$-acyclic and f is $h^*i_*\mathcal{F}$-acyclic by Examples 1.2.5.

2. Assume that (h, f) is C-acyclic. Replacing $W \to X \to P$ by $W \to X \times W \to P \times W$ and $\mathcal{F}$ by the pull-back, we may assume that h is an immersion. Then, locally, by lifting the defining equations, we find a cartesian diagram

$$\begin{array}{ccc} X \times Y & \xleftarrow{(h,f)} & W \\ \downarrow & & \downarrow \\ P \times Y & \xleftarrow{(\widetilde{h},\widetilde{f})} & V. \end{array}$$

Since the commutative diagram

$$\begin{array}{ccc} T^*P \times T^*Y & \longrightarrow & T^*V \\ \downarrow & & \downarrow \\ T^*X \times T^*Y & \longrightarrow & T^*W \end{array}$$

induces an isomorphism on the kernels, the assumption that (h, f) is C-acyclic implies that $(\widetilde{h}, \widetilde{f})$ is C_P-acyclic on a neighborhood of W. Hence $\widetilde{f}$ is $\widetilde{h}^*i_*\mathcal{F}$-acyclic, $f = \widetilde{f} \circ i$ is $h^*\mathcal{F}$-acyclic by Examples 1.9.5.

Assume that $\mathcal{F}$ is ms on C'. Then, by 1, $i_*\mathcal{F}$ is ms on $i_\circ C'$. Hence we have $C_P \subset i_\circ C'$ and $C \subset C'$. □

2.2 Radon transform

Let V be a k-vector space of dimension $n+1$ and $\mathbf{P} = \mathbf{P}(V)$ be the projective space of dimension n parametrizing lines in V. The dual projective space $\mathbf{P}^\vee = \mathbf{P}(V^\vee)$ is the moduli space of hyperplanes in $\mathbf{P}$. Let $Q \subset \mathbf{P} \times \mathbf{P}^\vee$ be the universal family of hyperplanes and let $q\colon Q \to \mathbf{P}$ and $q^\vee\colon Q \to \mathbf{P}^\vee$ be the projections.

We define the naive Radon transform $R\mathcal{F}$ to be $Rq^\vee_* q^*\mathcal{F}$ and the naive inverse Radon transform $R^\vee\mathcal{G}$ to be $Rq_*q^{\vee *}\mathcal{G}$. They are inverse to each other up to geometrically constant sheaves. We compute $R^\vee R\mathcal{F}$.

Proposition 2.4. 1. *We have a canonical isomorphism*

$$R^\vee R\mathcal{F} \to R\mathrm{pr}_{2*}\big(\mathrm{pr}_1^*\mathcal{F} \otimes R(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q}\big). \tag{2.1}$$

2. *The isomorphism* (2.1) *induces a distinguished triangle*

$$\to \bigoplus_{s=0}^{n-2} p^*p_*\mathcal{F}(-s)[-2s] \to R^\vee R\mathcal{F} \to \mathcal{F}(-(n-1))[-2(n-1)] \to \tag{2.2}$$

where $p\colon \mathbf{P} \to \operatorname{Spec} k$ denote the canonical morphism.

Proof. 1. By the definition, we have $R^\vee R\mathcal{F} = Rq_*q^{\vee *}Rq^\vee_*q^*\mathcal{F}$. By the proper base change theorem to the cartesian diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{\ q\ } & Q & \xleftarrow{\ \mathrm{pr}_1\ } & Q\times_{\mathbf{P}^\vee} Q \\ && {\scriptstyle q^\vee}\big\downarrow && \big\downarrow{\scriptstyle \mathrm{pr}_2} \\ && \mathbf{P}^\vee & \xleftarrow{\ q^\vee\ } & Q \\ &&&& \big\downarrow{\scriptstyle q} \\ &&&& \mathbf{P}, \end{array}$$

we have a canonical isomorphism $Rq_*q^{\vee *}Rq^\vee_*q^*\mathcal{F} \to R(q\circ\mathrm{pr}_2)_*(q\circ\mathrm{pr}_1)^*\mathcal{F}$. In the notation of the diagram

$$\begin{array}{ccccc} \mathbf{P} & \xleftarrow{\ \mathrm{pr}_1\ } & \mathbf{P}\times\mathbf{P} & \xleftarrow{\ q\times q\ } & Q\times_{\mathbf{P}^\vee} Q \\ && {\scriptstyle \mathrm{pr}_2}\big\downarrow && \\ && \mathbf{P}, && \end{array} \tag{2.3}$$

the latter is identified with $R(\mathrm{pr}_2\circ(q\times q))_*(\mathrm{pr}_1\circ(q\times q))^*\mathcal{F}$. This is identified with $R\mathrm{pr}_{2*}(\mathrm{pr}_{1*}\mathcal{F}\otimes R(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q})$ by the projection formula.

2. By the isomorphisms (2.1) and (2.4) below, we have a distinguished triangle

$$\to \tau_{\leqq 2(n-2)}R\Gamma(\mathbf{P}^\vee_{\bar k},\Lambda)\otimes\Lambda_{\mathbf{P}\times\mathbf{P}} \to R(q\times q)_*\Lambda_{Q\times_{\mathbf{P}^\vee}Q} \to \Lambda_{\mathbf{P}}(n-1)[2(n-1)] \to .$$

Applying $R\mathrm{pr}_{2*}(\mathrm{pr}_{1*}\mathcal{F}\otimes -)$ to this, we obtain (2.2).

Lemma 2.5. *We consider the commutative diagram*

$$\begin{array}{ccccc} Q \times_{\mathbf{P}^\vee} Q & \xrightarrow{i} & \mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P} & & \\ & \searrow^{q \times q} & \downarrow{\scriptstyle \mathrm{pr}_{13}} & & \\ & & \mathbf{P} \times \mathbf{P} & \xleftarrow{\delta_{\mathbf{P}}} & \mathbf{P} \end{array}$$

where $\delta_{\mathbf{P}} \colon \mathbf{P} \to \mathbf{P} \times \mathbf{P}$ *is the diagonal immersion. Then the closed immersion* i *induces isomorphisms*

$$R^s(q \times q)_* \Lambda_{Q \times_{\mathbf{P}^\vee} Q} \to \begin{cases} \Lambda(-i)[-2i] & \text{if } s = 2i, 0 \leqslant i \leqslant n-2, \\ \delta_{\mathbf{P}*}\Lambda(-(n-1))[-2(n-1)] & \text{if } s = 2(n-1) \end{cases} \tag{2.4}$$

and $R^s(q \times q)_* \Lambda_{Q \times_{\mathbf{P}^\vee} Q} = 0$ *otherwise.*

Proof. The immersion i induces morphisms

$$R^s \mathrm{pr}_{13*} \Lambda_{\mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P}} \to R^s(q \times q)_* \Lambda_{Q \times_{\mathbf{P}^\vee} Q} \tag{2.5}$$

and we have isomorphisms $R^s \mathrm{pr}_{13*} \Lambda_{\mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P}} \to \Lambda(-i)[-2i]$ for $s = 2i, 0 \leqslant i \leqslant n$ and $R^s \mathrm{pr}_{13*} \Lambda_{\mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P}} = 0$ otherwise. The restriction of the closed immersion $i \colon Q \times_{\mathbf{P}^\vee} Q \to \mathbf{P} \times \mathbf{P}^\vee \times \mathbf{P}$ on the diagonal $\mathbf{P} \subset \mathbf{P} \times \mathbf{P}$ is the sub $\mathbf{P}^{n-1}$-bundle $Q \subset \mathbf{P} \times \mathbf{P}^\vee$. On the complement $\mathbf{P} \times \mathbf{P} - \mathbf{P}$, it is a sub $\mathbf{P}^{n-2}$-bundle. Hence (2.5) is an isomorphism for $s \neq 2(n-1)$ and induces an isomorphism $\delta_{\mathbf{P}*} R^{2(n-1)} q_* \Lambda_Q \to R^{2(n-1)}(q \times q)_* \Lambda_{Q \times_{\mathbf{P}^\vee} Q}$. □

Corollary 2.6. *We have the implications* (1)⇒(2).

(1) $\mathcal{F}$ *is micro supported on* C.

(2) $R^\vee R\mathcal{F}$ *is micro supported on* $C^+ = C \cup T^*_{\mathbf{P}}\mathbf{P}$.

We will prove the existence of singular support by proving a refinement involving the micro support of the Radon transform $R\mathcal{F}$.

2.3 Proof of existence

Let $g \colon X \to X'$ be a proper morphism of smooth schemes over k and $C \subset T^*X$ be a closed conical subset. Then, $g_\circ C \subset T^*X'$ is defined as a closed conical subset. The construction $i_\circ C$ for a closed immersion is a special case.

Lemma 2.7. *Let* $g \colon X \to X'$ *be a proper morphism of smooth schemes over* k *and* $C \subset T^*X$ *be a closed conical subset. Assume that* (h', f') *is* $g_\circ C$*-acyclic. Then* h' *is transversal to* $X \to X'$ *on a neighborhood of the inverse image of the base of* C. *Namely,* $W = X \times_{X'} W'$ *is smooth over* k *on a neighborhood of the inverse image* W_1 *of the base of* C *and* $\dim W - \dim W' = \dim X - \dim X'$. *Further* (h_1, f_1) *is* C*-acyclic.*

Proposition 2.8. *Let* X *be a smooth scheme over* k *and* $\mathcal{F}$ *be a sheaf on* X. *Assume that* $\mathcal{F}$ *is ms on* $C \subset T^*X$.

1. *If* $g \colon X' \to X$ *is* C*-transversal, then* $g^*\mathcal{F}$ *is ms on* $g^\circ C$.
2. *If* $g \colon X \to X'$ *is proper, then* $g_*\mathcal{F}$ *is ms on* $g_\circ C$.

Proof. 1. Assume that a pair $h\colon W \to X'$ and $f\colon W \to Y$ is $g^\circ C$-acyclic. Then $gh\colon W \to X$ and $f\colon W \to Y$ is C-acyclic and f is $h^*g^*\mathcal{F}$-acyclic.

2. Proof is similar to that of Proposition 2.3.1. Assume that a pair $h'\colon W' \to X'$ and $f'\colon W' \to Y$ is $g_\circ C$-acyclic. Then on an inverse image of the base of C, $W = X \times_{X'} W'$ is smooth over k and the pair of $h\colon W \to X$ and $f\colon W \to Y$ is C-acyclic by Lemma 2.7. Hence $f\colon W \to Y$ is $h^*\mathcal{F}$-acyclic and and f' is $g'_*h^*\mathcal{F} = h'^*g^*\mathcal{F}$-acyclic. □

We call

$$C^\vee = q^\vee_\circ q^\circ C \subset T^*\mathbf{P}^\vee$$

the Legendre transform of C.

Corollary 2.9. *We have the implications* (1)⇒(2)⇒(3).

(1) *$\mathcal{F}$ is micro supported on C.*

(2) *The Radon transform $R\mathcal{F}$ is micro supported on $C^\vee$.*

(3) *$\mathcal{F}$ is micro supported on $C^{\vee\vee+}$.*

Proof. (1)⇒(2): By Proposition 2.8, The Radon transform $R\mathcal{F} = p^\vee_* p^*\mathcal{F}$ is micro supported on $C^\vee = p^\vee_\circ p^\circ C$.

(2)⇒(3): By (1)⇒(2) and symmetry, $R^\vee R\mathcal{F}$ is micro supported on $C^{\vee\vee}$. □

We will prove a refinement of (1)⇒(2) involving the local acyclicity.

By the exact sequence $0 \to \Omega^1_{\mathbf{P}/k}(1) \to \mathcal{O}_\mathbf{P} \otimes V^\vee \to \mathcal{O}_\mathbf{P}(1) \to 0$ of locally free $\mathcal{O}_\mathbf{P}$-modules, we identify $Q = \mathbf{P}(T^*\mathbf{P})$. By symmetry, we also identify $Q = \mathbf{P}(T^*\mathbf{P}^\vee)$. The kernel $L_Q = \mathrm{Ker}\big((T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*Q\big)$ equals the conormal bundle

$$L_Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee) \subset (T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q$$

and its image $L \subset T^*\mathbf{P} \times_\mathbf{P} Q$ is the universal sub line bundle. The kernel $L_Q = T^*_Q(\mathbf{P} \times \mathbf{P}^\vee)$ is also identified with the intersection $q^*T^*\mathbf{P} \cap q^{\vee *}T^*\mathbf{P}^\vee \subset T^*Q$. We consider projectivizations $\mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P})$ and $\mathbf{P}(C^\vee) \subset \mathbf{P}(T^*\mathbf{P}^\vee)$ as closed subsets of Q. The projectivization $\mathbf{P}(C) \subset Q$ is the support of the intersection $(C \times_\mathbf{P} Q) \cap L \subset T^*\mathbf{P} \times_\mathbf{P} Q$. If we regard $T^*\mathbf{P} \times_\mathbf{P} Q$ as a sub vector bundle of T^*Q, it is also the support of the intersection $(C \times_\mathbf{P} Q) \cap L_Q \subset T^*Q$.

Proposition 2.10. *Let $C \subset T^*\mathbf{P}$ be a closed conical subset. Let $E = \mathbf{P}(C) \subset Q = \mathbf{P}(T^*\mathbf{P})$ be the projectivization.*

1. *The projectivization $E = \mathbf{P}(C) \subset Q$ is the complement of the largest open subset where $(q, q^\vee)$ is C-acyclic.*

2. *The Legendre transform $C^\vee$ equals the image of the union of $L_Q|_E \subset L_Q \subset T^*\mathbf{P}^\vee \times_{\mathbf{P}^\vee} Q$ and its base. We have $\mathbf{P}(C) = \mathbf{P}(C^\vee)$.*

3. *We have $C^{\vee\vee} \subset C^+$.*

Proof. 1. The largest open subset is the complement of the support of the intersection $(q^*C \times T^*\mathbf{P}^\vee)|_Q \cap \mathrm{Ker}((T^*\mathbf{P} \times T^*\mathbf{P}^\vee)|_Q \to T^*Q)$. This equals $(C \times_\mathbf{P} Q) \cap L_Q \subset T^*Q$ and its support is $E = \mathbf{P}(C) \subset \mathbf{P}(T^*\mathbf{P})$.

2. The Legendre transform $C^\vee$ is defined as follows: Take the intersection $q^*C \cap q^{\vee *}T^*\mathbf{P}^\vee \subset q^*T^*\mathbf{P} \cap q^{\vee *}T^*\mathbf{P}^\vee = L_Q \subset T^*Q$. Then, $C^\vee$ is the image by the projection $q^{\vee *}T^*\mathbf{P}^\vee \to T^*\mathbf{P}^\vee$. The intersection $q^*C \cap q^{\vee *}T^*\mathbf{P}^\vee$ equals $(C \times_\mathbf{P} Q) \cap L_Q \subset T^*Q$ and its support is E. Hence it is $L|_E$ up to the base.

By symmetry, we also identify L_Q as the universal sub line bundle $L^\vee$ of $T^*\mathbf{P}^\vee$ on $Q = \mathbf{P}(T^*\mathbf{P}^\vee)$. Then, we have $q^*C \cap L_Q = q^{\vee *}C^\vee \cap L_Q$ and $\mathbf{P}(C) = \mathbf{P}(C^\vee) = E$.

3. By 2 and symmetry, we have $\mathbf{P}(C) = \mathbf{P}(C^\vee) = \mathbf{P}(C^{\vee\vee})$. Hence we have $C^{\vee\vee} \subset C^+$. □

Proposition 2.11. *We have the implications* (1)⇒(2)⇒(3)⇒(4).

(1) $\mathcal{F}$ *is micro supported on* C.

(2) $q^\vee$ *is universally* $q^*\mathcal{F}$*-acyclic outside* $E = \mathbf{P}(C)$.

(3) $R\mathcal{F}$ *is micro supported on* $C^{\vee +} = C^\vee \cup T^*_{\mathbf{P}^\vee}\mathbf{P}^\vee$.

(4) $\mathcal{F}$ *is micro supported on* C^+.

Proof. (1)⇒(2): By Proposition 2.10.1, $(q, q^\vee)$ is C-acyclic outside E. Hence $(q, q^\vee)$ is C-acyclic outside E.

(2)⇒(3): Assume that (h, f) is $C^{\vee +}$-acyclic and consider the commutative diagram

$$\begin{array}{ccccccc} \mathbf{P} & \xleftarrow{q} & Q & \xleftarrow{h'} & Q_W & & \\ & & \downarrow{\scriptstyle q^\vee} & & \downarrow{\scriptstyle q^\vee_W} & \searrow^{f'} & \\ & & \mathbf{P}^\vee & \xleftarrow{h} & W & \xrightarrow{f} & Y \end{array}$$

with cartesian square. Since $q^\vee_W$ is proper, it suffices to show that f' is $(qh')^*\mathcal{F}$-acyclic.

By (2), outside E, $q^\vee$ is universally $q^*\mathcal{F}$-acyclic. Hence outside the inverse image E_W, the base change $q^\vee_W$ is $(qh')^*\mathcal{F}$-acyclic. Since (h, f) is $T^*_{\mathbf{P}^\vee}\mathbf{P}^\vee$-acyclic, f is smooth. Hence outside E_W, the composition $f' = fq^\vee_W$ is $(ph')^*\mathcal{F}$-acyclic by Examples 1.9.3.

Since (h, f) is $C^\vee$-acyclic, (h', f') is $q^\circ C$-acyclic by Lemma 2.7. Hence (qh', f') is C-acyclic. The conormal bundle of the immersion $Q_W \to \mathbf{P} \times W$ is the pull-back L_W of L. The support of $(C \times T^*W) \cap L_W$ is E_W and hence $(C \times T^*W) \cap L_W$ contains $(T^*\mathbf{P} \times T^*W) \cap L_W|_E$ as a subset. Since $f\colon W \to Y$ is smooth, this implies that (qh', f') is $T^*\mathbf{P}$-acyclic on a neighborhood U of E_W. Since $\mathcal{F}$ is micro supported on $T^*\mathbf{P}$ by Example 1.9.1, f' is $(qh)^*\mathcal{F}$-acyclic on U.

(3)⇒(4): By (3) and (1)⇒(3), $R^\vee R\mathcal{F}$ is micro supported on $(C^{\vee +})^{\vee +} = C^+$. By the distinguished triangle in Proposition 2.4.2, $\mathcal{F}$ is also micro supported on C^+. □

We prove the existence of singular support for $X = \mathbf{P}$.

Corollary 2.12. *Let* $\mathcal{F}$ *be a sheaf on* $\mathbf{P}$. *Let* $E \subset Q = \mathbf{P}(T^*\mathbf{P})$ *be the complement of the largest open subset on which* $q^\vee$ *is universally* $q^*\mathcal{F}$*-acyclic. Then, the closed conical subset* $C \subset T^*\mathbf{P}$ *such that the base is the support* B *of* $\mathcal{F}$ *and* $\mathbf{P}(C) = E$ *is the singular support of* $\mathcal{F}$.

Proof. By Proposition 2.11 (2)⇒(4), $\mathcal{F}$ is micro supported on $(C^{\vee +})^{\vee +} = C^+$. Hence $\mathcal{F}$ is micro supported on $C = C^+|_B$. If $\mathcal{F}$ is micro supported on C', then we have $E \subset E'$ by Proposition 2.11 (1)⇒(2). Since the base of C' contains $B = \operatorname{supp}\mathcal{F}$, we have $C \subset C'$. □

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3 Variations

We want to transplant Beilinson's theory to mixed characteristic case. There are two obstacles:

- There is no cotangent bundle of the correct rank.
- There are not sufficiently many morphisms f.

A solution to the first problem is given by the Frobenius–Witt cotangent bundle. Although it is supported on the characteristic p fiber, it has the correct rank. A solution to the second problem is given by the transversality.

In the original definition, the C-acyclicity corresponds to the $\mathcal{F}$-acyclicity. We introduce the property corresponding to the C-transversality. As we don't have enough morphisms f in the mixed characteristic case, we define micro support using only the transversality. However, to adapt the proof of existence, we need to use the second morphism f by fixing the base scheme. Thus, we introduce a relative variant of micro support using pairs of morphisms and an equivalent condition of the original definition.

3.1 $\mathcal{F}$-transversality

Let $h\colon W \to X$ be a separated morphism of finite type and $\mathcal{F}$ be a sheaf on X. We define a canonical morphism

$$c_{h,\mathcal{F}}\colon h^*\mathcal{F} \otimes h^!\Lambda \to h^!\mathcal{F}$$

comparing the two pull-backs $h^*\mathcal{F}$ and $h^!\mathcal{F}$. It is defined as the adjoint of $h_!(h^*\mathcal{F}\otimes h^!\Lambda) \to \mathcal{F}$. By the projection formula, this is defined as $1 \otimes \mathrm{adj}\colon \mathcal{F} \otimes h_!h^!\Lambda \to \mathcal{F}$.

Definition 3.1. Let $h\colon W \to X$ be a separated morphism of smooth schemes over k and $\mathcal{F}$ be a sheaf. We say that h is $\mathcal{F}$-transversal if $c_{h,\mathcal{F}}\colon h^*\mathcal{F} \otimes h^!\Lambda \to h^!\mathcal{F}$ is an isomorphism.

Without assuming h being separated, the $\mathcal{F}$-transversality can be defined locally on W.

Examples 3.2. 1. Let $Z \subset X$ be a closed subscheme smooth over k. If h is transversal to $Z \to X$, then h is Λ_Z-transversal.

2. If h is smooth, h is $\mathcal{F}$-transversal for every $\mathcal{F}$. Poincaré duality.

3. If $\mathcal{F}$ is locally constant, every h is $\mathcal{F}$-transversal.

Relation between the transversality and the acyclicity.

Let

$$\begin{array}{ccc} W & \xrightarrow{h} & X \\ \downarrow & & \downarrow{\scriptstyle f} \\ y & \longrightarrow & Y \end{array}$$

be a cartesian diagram of smooth schemes over k. Then h is C-transversal if and only if f is C-acyclic on a neighborhood of W since $T^*Y \times_Y W$ is canonically identified with $\mathrm{Ker}(T^*X \times_X W \to T^*W)$.

In a more general situation, we have the following

Lemma 3.3. *Let*

$$\begin{array}{ccc} X & \xleftarrow{h} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longleftarrow & V \end{array}$$

be a cartesian diagram of smooth schemes over k. Assume that f is smooth.

1. *If f is C-acyclic, then (h, g) is C-acyclic.*
2. *If f is universally $\mathcal{F}$-acyclic, then h is $\mathcal{F}$-transversal and g is $h^*\mathcal{F}$-acyclic.*

Proof. 1. Since the kernel $\mathrm{Ker}(T^*X \times_X W \oplus T^*V \times_V W \to T^*W)$ is canonically identified with $T^*Y \times_Y W$, the support of the intersection with $h^*C \times g^*T^*V$ is empty.

2. It suffices to show the $\mathcal{F}$-transversality. By the factorization $V \to X \times V \to X$ and by the transitivity, we may assume that $V \to X$ is an immersion and further a closed immersion. Let $j\colon U = X - W \to X$ be the open immersion and consider the commutative diagram

$$\begin{array}{ccccccc} \mathcal{F} \otimes h_! h^! \Lambda & \longrightarrow & \mathcal{F} \otimes \Lambda & \longrightarrow & \mathcal{F} \otimes j_* j^* \Lambda & \longrightarrow \\ \downarrow & & \| & & \downarrow & \\ h_! h^! \mathcal{F} & \longrightarrow & \mathcal{F} & \longrightarrow & j_* j^* \mathcal{F} & \longrightarrow \end{array}$$

of distinguished triangles. By the $\mathcal{F}$-acyclicity and Corollary 3.5 below, the right vertical arrow is an isomorphism. Hence the left vertical arrow is an isomorphism which means that h is $\mathcal{F}$-transversal. □

We will later state an inverse proved using alterations.

Proposition 3.4 (Fu Lei Theorem 7.6.9)**.** *Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a sheaf on X. Then the following conditions are equivalent:*

(1) *f is $\mathcal{F}$-acyclic.*

(2) *For every cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow{h} & U \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow{j} & V \end{array}$$

such that the horizontal arrows are immersions and for every constructible sheaf $\mathcal{G}$ on V, the morphism

$$\mathcal{F} \otimes f^* j_* \mathcal{G} \to h_*(h^*\mathcal{F} \otimes g^*\mathcal{G})$$

is an isomorphism.

Corollary 3.5. *Let*

$$\begin{array}{ccc} X & \xleftarrow{h} & W \\ {\scriptstyle f}\downarrow & & \downarrow \\ Y & \xleftarrow{i} & V \end{array}$$

be a cartesian diagram of schemes such that the vertical arrows are smooth and that the horizontal arrows are immersions. Let $\mathcal{F}$ be a sheaf on X and $\mathcal{G}$ be a sheaf on Y. Assume that $f\colon X \to Y$ is $\mathcal{F}$-acyclic and $i\colon V \to Y$ is $\mathcal{G}$-transversal. Then $h\colon W \to X$ is $\mathcal{F} \otimes f^*\mathcal{G}$-transversal.

In particular, for any immersion $i\colon V \to Y$, the morphism $h\colon W \to X$ is $\mathcal{F}$-transversal.

3.2 Equivalent definitions

Proposition 3.6. *Let X be a smooth scheme over k. Let $\mathcal{F}$ be a sheaf on X and C be a closed conical subset of T^*X. Then the following conditions are equivalent:*

(1) $\mathcal{F}$ is micro supported on C.

(2) Let (h, f) be a pair of morphisms $h\colon W \to X$ and $f\colon X \to Y$ of smooth schemes over k such that h is C-transversal and f is $h^\circ C$-acyclic. Then, h is $\mathcal{F}$-transversal and f is $h^*\mathcal{F}$-acyclic.

(3) Let (h, f) be a pair of morphisms $h\colon W \to X$ and $f\colon X \to Y$ of smooth schemes over k such that $(h, f)\colon W \to X \times Y$ is $C \times T^*Y$-transversal. Then, for any sheaf $\mathcal{G}$ on Y, the morphism $(h, f)\colon W \to X \times Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.

Proof. (2)⇒(1) is clear.

(1)⇒(2): It suffices to show that the C-transversality of h implies the $\mathcal{F}$-transversality of h. By the transitivity, by considering the factorization $W \to X \times W \to X$, we may assume that h is an immersion. Since the question is local, we may assume that there exists a cartesian diagram

$$\begin{array}{ccc} W & \xrightarrow{h} & X \\ \big\downarrow & & \big\downarrow f \\ y & \longrightarrow & Y. \end{array}$$

Since h is C-transversal, f is C-acyclic on a neighborhood of W. Hence f is $\mathcal{F}$-acyclic on a neighborhood of W. By Lemma 3.3.2, h is $\mathcal{F}$-transversal.

(2)⇔(3): We consider the decomposition $(h \times 1_Y) \circ (1_W, f)\colon W \to W \times Y \to X \times Y$ of (h, f) and compare

(h) the $\mathcal{F}$-transversality of h and the $\mathcal{F} \boxtimes \mathcal{G}$-transversality of $h \times 1_Y$ for every $\mathcal{G}$ on Y.

(f) the $h^*\mathcal{F}$-acyclicity of f and the $h^*\mathcal{F} \boxtimes \mathcal{G}$-transversality of $(1_W, f)$ for every $\mathcal{G}$ on Y.

(h) Let $p\colon W \to P = \operatorname{Spec} k$ be the canonical morphism. Then the C-transversality of h is equivalent to $C \times T^*P$-transversality of (h, p). By Lemma 3.7, (2) implies that h is $\mathcal{F}$-transversal and (3) implies that h is $\mathcal{F} \boxtimes \mathcal{G}$-transversal for every $\mathcal{G}$ on P. Thus, in both case h is $\mathcal{F}$-transversal. By Lemma 3.8, the $\mathcal{F}$-transversality of h implies the $\mathcal{F} \boxtimes \mathcal{G}$-transversality of $h \times 1_Y$ for every $\mathcal{G}$ on Y.

Hence by the transitivity, it suffices to show the equivalence in (f). This follows from Proposition 3.9. □

Lemma 3.7. *Let (h, f) be a pair of morphisms $h\colon W \to X$ and $f\colon W \to Y$ of smooth schemes and C be a closed conical subset. Then the following conditions are equivalent:*

(1) h is C-transversal and f is $h^\circ C$-acyclic.

(2) $(h, f)\colon W \to X \times Y$ is $C \times T^*Y$-transversal.

Proof. If (2) is satisfied, then $(h, f)\colon W \to X \times Y$ is $C \times T^*_Y Y$-transversal. Hence h is C-transversal. The condition (2) means that if $\xi \in C \subset T^*X$ and $\eta \in T^*Y$ have the same image in T^*W, then $\xi = 0$ and $\eta = 0$. Under the C-transversality of h, this is equivalent to the $h^\circ C$-acyclicity of f. □

Lemma 3.8. *Let $h\colon W \to X$ be a morphism and assume that h is $\mathcal{F}$-transversal. Then, for any $\mathcal{G}$ on Y, the morphism $h \times 1\colon W \times Y \to X \times Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.*

Proof. We consider the cartesian diagram

$$\begin{array}{ccc} X\times Y & \xleftarrow{h\times 1} & W\times Y \\ \downarrow & & \downarrow \\ X & \xleftarrow{h} & W. \end{array}$$

Since $Y \to \operatorname{Spec} k$ is universally $\mathcal{G}$-acyclic, the projection $X \times Y \to X$ is $\mathrm{pr}_2^*\mathcal{G}$-acyclic. Hence by Corollary 3.5, $h \times 1$ is $\mathcal{F} \boxtimes \mathcal{G} = \mathrm{pr}_1^*\mathcal{F} \otimes \mathrm{pr}_2^*\mathcal{G}$-transversal. □

Proposition 3.9. (cf. [2, Appendix Theorem B.2]) *Let $f\colon X \to Y$ be a morphism of schemes over an excellent regular noetherian scheme S and let $\mathcal{F}$ be a sheaf on X. If $X \to S$ is universally $\mathcal{F}$-acyclic and if Y is smooth over S, then the following conditions are equivalent:*

(1) *f is $\mathcal{F}$-acyclic.*

(2) *For any constructible sheaf $\mathcal{G}$ on Y, the morphism $\gamma = (1_X, f)\colon X \to X \times_S Y$ is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.*

The assumption that $X \to S$ is universally $\mathcal{F}$-acyclic is satisfied if $S = \operatorname{Spec} k$ for a field k by Example 1.2.2.

Proposition 3.10. *Let X be a smooth scheme over k. Let $\mathcal{F}$ be a sheaf on X and C be a closed conical subset of T^*X. We consider the following conditions:*

(1) *$\mathcal{F}$ is micro supported on C.*

(2) *The support of $\mathcal{F}$ is a subset of the base of C. Let $h\colon W \to X$ be a morphism of smooth schemes over k such that h is C-transversal. Then, h is $\mathcal{F}$-transversal.*

We have (1)⇒(2). If k is perfect, they are equivalent to each other.

Proof. The implication (1)⇒(2) is proved in Proposition 3.6 (1)⇒(2).

We show (2)⇒(1) assuming that k is perfect. Assume that (h, f) is C-acyclic. We show that f is $h^*\mathcal{F}$-acyclic. By the transitivity, we may assume that $h = 1_X$. By shrinking W, we may assume that f is smooth. In the diagram Lemma 3.11 below, f is smooth and C-acyclic. Further, since k is assumed *perfect*, k' is separable and X' and W are smooth over k and the morphisms p' and $p'h$ are C-transversal. Hence by (2), p' and $p'h$ are $\mathcal{F}$-transversal. By the transitivity, this implies that h is $p'^*\mathcal{F}$-transversal. Thus by Lemma 3.11, f is $\mathcal{F}$-acyclic. □

Lemma 3.11. *Let $f\colon X \to Y$ be a smooth morphism of schemes of finite type over a field k and let $\mathcal{F}$ be a sheaf on X . Assume that for every cartesian diagram*

$$\begin{array}{ccccc} X & \xleftarrow{p'} & X' & \xleftarrow{h} & W \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \longleftarrow & Y' & \xleftarrow{i} & Z \end{array}$$

satisfying the following condition, the immersion h is $p'^*\mathcal{F}$-transversal: Y' and Z are smooth schemes over a finite extension of k, $p\colon Y' \to Y$ is proper and generically finite and $i\colon Z \to Y'$ is a closed immersion.

Then, f is $\mathcal{F}$-acyclic.

4 Mixed characteristics

First, we need to find a vector bundle where the singular support is defined. The $\mathcal{O}_X$-module Ω^1_X may not be locally free. Even if it is so, it can be too small, eg for $X = \operatorname{Spec} \mathbf{Z}_{(p)}$. We expect to have a locally free sheaf such that the fiber at the closed point x with perfect residue field is the Zariski cotangent space $\mathfrak{m}_x/\mathfrak{m}_x^2$. We will define a sheaf $F\Omega_X$ that is locally free on $X_{\mathbf{F}_p}$ such that at every points $x \in X_{\mathbf{F}_p}$ we have an exact sequence

$$0 \to F^*(\mathfrak{m}_x/\mathfrak{m}_x^2) \to F\Omega_{X,x} \otimes k(x) \to F^*(\Omega^1_{k(x)}) \to 0.$$

4.1 Frobenius–Witt cotangent bundle

Definition 4.1. Let p be a prime number.

1. Define a polynomial $P \in \mathbf{Z}[X, Y]$ by

$$P = \sum_{p=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} \cdot X^i Y^{p-i}.$$

2. Let A be a ring and M be an A-module. We say that a mapping $w\colon A \to M$ is an Frobenius–Witt derivation or FW-derivation for short if the following condition is satisfied: For any $a, b \in A$, we have

$$w(a+b) = w(a) + w(b) - P(a,b) \cdot w(p), \qquad w(ab) = b^p \cdot w(a) + a^p \cdot w(b).$$

3. The module $F\Omega_A$ of Frobenius–Witt differentials is an A-module representing the functor sending an A-module M to the set $\{$FW-derivations $w\colon A \to M\}$.

The polynomial P appears in the addition of Witt vectors W_2 of length 2.

If $w\colon A \to M$ is an FW-derivation, for an integer n, we have $w(n) = \dfrac{n - n^p}{p} \cdot w(p)$. If A is a ring over $\mathbf{Z}_{(p)}$, we have $p \cdot w(a) = 0$ for any $a \in A$. By induction on n, we have $w(na) = n \cdot w(a) + a^p \cdot w(n)$. Comparing this with the defining equality, we have $(n^p - n) \cdot w(a) = 0$. FW-derivations naturally arise from the cotangent complex and also in a construction due to Gabber–Ramero. It is a linearization of δ-ring.

In the following, we assume that X is of finite type over $S = \operatorname{Spec} \mathcal{O}_K$ for a discrete valuation ring of characteristic $(0, p)$ with perfect residue field k. If X is regular, $F\Omega_X$ is a locally free $\mathcal{O}_{X_{\mathbf{F}_p}}$-module of rank $\dim X$. Define the FW-cotangent bundle to be the associated vector bundle. If X is smooth over S, we have an exact sequence

$$0 \to FT^*S \times_S X \to FT^*X \to F^*T^*X/S|_{X_{\mathbf{F}_p}} \to 0 \tag{4.1}$$

of vector bundles on $X_{\mathbf{F}_p}$.

4.2 Singular supports

Definition 4.2. Let X be a regular scheme of finite type over S. Let $\mathcal{F}$ be a sheaf on X and $C \subset FT^*X$ be a closed conical subset. We say that $\mathcal{F}$ is micro supported on C if the following conditions are satisfied:

The intersection of the support of $\mathcal{F}$ with $X_{\mathbf{F}_p}$ is a subset of the base of C.

Let $h\colon W \to X$ be a (separated) morphism of regular schemes of finite type over S. If h is C-transversal, then h is $\mathcal{F}$-transversal on a neighborhood of $W_{\mathbf{F}_p}$.

C-transversality: Support of $h^*C \cap \mathrm{Ker}(FT^*X \times_X W \to FT^*W)$ is empty. A property on a neighborhood of $W_{\mathbf{F}_p}$.

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Examples 4.3. 1. Every sheaf is ms on FT^*X. If $h\colon W \to X$ is FT^*X-transversal, then h is smooth on a neighborhood of $W_{\mathbf{F}_p}$.

2. locally constant $\mathcal{F}$ is ms on the 0-section $F^*T^*_X X$. Converse also holds but is less trivial.

Open problems.

- Existence of SS. We can't adapt the method of Beilinson to prove the existence because we don't have f. So, prove the existence, we introduce a relative variant. We will study the relation between the absolute one and the relative one.

- dimension of SS. In the geometric case, Beilinson proved that every irreducible component of S has the same dimension as X. The mixed characteristic is an analogue of the situation where a morphism $X \to C$ to a smooth curve is given and we have only the restriction to the fiber of a closed point of c. So, we expect that every irreducible component of S has the same dimension as X or 1 less. I don't know how to adapt Beilinson's method in the mixed characteristic case.

- description at the generic point of an irreducible component of the closed fiber. In the geometric case, we have a description in terms of the characteristic form. Assume that $\mathcal{F} = j_!\mathcal{G}$ for a locally constant sheaf $\mathcal{G}$ and the open immersion $j\colon U = X - D \to X$ of the complement of a divisor. Let K be the local field at the generic point ξ of an irreducible component of D and V be the representation of the absolute Galois group G_K defined by $\mathcal{F}$. Then, V has the slope decomposition $V = \bigoplus_{r\geqq 1} V^{(r)}$ by the filtration by ramification groups and the representation $V^{(r)}$ of the abelian graded quotient $\mathrm{Gr}^r G_K = G^r_K/G^{r+}_K$ has the decomposition by characters $V^{(r)} = \bigoplus \chi_i^{n_i}$. If $r > 1$, we have a canonical injection $\mathrm{char}\colon \mathrm{Hom}(\mathrm{Gr}^r G_K, \mathbf{F}_p) \to \mathrm{Hom}(\mathfrak{m}^r_{\overline{K}}/\mathfrak{m}^{r+}_{\overline{K}}, \Omega^1_{X,\xi} \otimes k(\bar{\xi}))$ called the characteristic form. The fiber of the singular support at ξ is given by the lines defined as the image of $\mathrm{char}\,\chi_i$ together with the conormal bundle if $V^{(1)} \neq 0$. In the mixed characteristic case, we expect to replace $\Omega^1_{X,\xi} \otimes k(\bar{\xi})$ by $F\Omega^1_{X,\xi} \otimes k(\bar{\xi})$.

- CC. It is not convenient to have two types of components. $\dim = n$ and $\dim = n-1$. To avoid this problem, assume that on the generic fiber the rank function is the constant function 0. The ramification in the generic fiber is tame as characteristic is 0. Then, we expect to define CC as a $\mathbf{Z}$-linear combination of components with dimension n. We further expect that they satisfy the following properties.

– Compatibility with pull-back for properly transversal morphisms.

– Compatibility with push-forward for proper morphisms.

– If $X = S$, $CC\mathcal{F} = (\mathrm{rank}\,\mathcal{F}_x - \mathrm{Sw}\,\mathcal{F})[T^*_s S]$.

In the case where $\dim X = 2$ and $\mathrm{rank}\,\mathcal{F} = 1$, Ooe defined a candidate of CC and proved a conductor formula [10].

Definition 4.4. Let X be a regular scheme of finite type over S. Let $\mathcal{F}$ be a sheaf on X and $C \subset FT^*X$ be a closed conical subset. We say that $\mathcal{F}$ is S-micro supported on C if the following conditions are satisfied:

Let $h\colon W \to X$ and $f\colon W \to Y$ be (separated) morphisms of regular schemes of finite type over S and assume Y is smooth over S. If (h, f) is C-acyclic over S, then (h, f) is $\mathcal{F}$-acyclic over S on a neighborhood of $W_{\mathbf{F}_p}$.

Heuristic observation: Pretend that S had a base field k, and let $f_0\colon W \to Y_0$ be a morphism to smooth scheme over k. Then, (h, f_0) would be $\mathcal{F}$-acyclic if $(h, f_0)\colon W \to X \times_k$

Y_0 is $\mathcal{F}\boxtimes\mathcal{G}_0$-transversal for every sheaf $\mathcal{G}_0$ on Y_0. The base change $Y = Y_0\times_k S$ is smooth over S and for the pull-back $\mathcal{G}$ to Y of $\mathcal{G}_0$, the $\mathcal{F}\boxtimes\mathcal{G}_0$-transversality of $(h, f_0)\colon W \to X\times_k Y_0$ means the $\mathcal{F}\boxtimes\mathcal{G}$-transversality of $(h, f)\colon W \to X\times_S Y$. Thus, to formulate the relative $\mathcal{F}$-acyclicity, it suffices to determine which sheaves on Y are obtained as the pull-back from Y_0. If $\mathcal{G}$ is a such sheaf, it must be micro supported on the pull-back $C' \subset FT^*Y$ of a closed conical subset of FT^*Y_0. If C' is obtained in this way, then $Y \to S$ should be C'-acyclic; the support of its inverse image by $FT^*S\times_S Y \to FT^*Y$ should be empty.

Definition 4.5. 1. Let Y be a smooth scheme over S and $\mathcal{G}$ be a sheaf on Y. We say that $\mathcal{G}$ is S-acyclic if there exists a closed conical subset $C' \subset FT^*Y$ satisfying the following conditions:

$\mathcal{G}$ is micro supported on C' and $Y \to S$ is C'-acyclic.

2. Let $h\colon W \to X$ and $f\colon W \to Y$ be (separated) morphisms of regular schemes of finite type over S and assume Y is smooth over S. Let $\mathcal{F}$ be a sheaf on X. We say that (h, f) is $\mathcal{F}$-acyclic over S if for every sheaf $\mathcal{G}$ on Y which is S-acyclic, the morphism $(h, f)\colon W \to X\times_S Y$ is $\mathcal{F}\boxtimes\mathcal{G}$-transversal.

Definition 4.6. Let X be a regular scheme of finite type over S and let $C \subset FT^*X$ be a closed conical subset.

1. Let $h\colon W \to X$ and $f\colon W \to Y$ be morphisms of regular schemes of finite type over S and assume Y is smooth over S. We say that (h, f) is C-acyclic over S if

$$\begin{aligned}&(h^*C\times_W f^*FT^*Y)\cap \operatorname{Ker}\big((FT^*X\times_X W)\times_W (FT^*Y\times_Y W)\to FT^*W\big)\\ &\subset \operatorname{Ker}\big((FT^*X\times_X W)\times_W (FT^*Y\times_Y W)\to FT^*(X\times_S Y)\times_{X\times_S Y} W\big).\end{aligned}$$

2. We say that C is S-saturated if C is stable under the action of FT^*S.

Examples 4.7. 1. If (h, f) is C-acyclic over S and if $Y \to S$ is C'-acyclic, then, (h, f) is $\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C'$-transversal.

Since the right hand side is the image of $FT^*S\times_S W$, the C'-acyclicity of $Y \to S$ implies that the intersection of $(FT^*X\times_X W)\times_W f^*C'$ with the right hand side is a subset of the 0-section. Since $\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C'$ is the image of $C\times_S C'$, the intersection $(\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C')\cap \operatorname{Ker}\big(FT^*(X\times_S Y)\times_{X\times_S Y} W \to T^*W\big)$ is the image of $(C\times_S C')\cap \operatorname{Ker}\big((FT^*X\times_X W)\times_W (FT^*Y\times_Y W)\to T^*W\big)$. By the C-acyclicity of (h, f) the image of the intersection is a subset of the 0-section.

2. Assume that $h\colon W \to X$ is C-transversal and let $f\colon W \to S$ be the canonical morphism. If C is S-saturated, then (h, f) is C-acyclic over S.

Assume that (a, b) is in the kernel. Then, since C is S-saturated, $(a + b, 0)$ is in the kernel. Since h is C-transversal, we have $a + b = 0$. This means the inclusion.

Relation between the absolute micro support and S-relative micro support.

Expectation 4.8. Assume $\mathcal{F}$ is micro supported on C and $\mathcal{G}$ is micro supported on C'. If $\operatorname{supp}(C\cap C')$ is empty, then $\mathcal{F}\otimes\mathcal{G}$ is micro supported on $C + C'$.

More weakly, assume $\mathcal{F}$ on X is micro supported on C and $\mathcal{G}$ on Y smooth over S is micro supported on S-acyclic C'. Then $\mathcal{F}\boxtimes\mathcal{G}$ is micro supported on $\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C'$.

Lemma 4.9. 1. *Assume that $\mathcal{F}$ is ms on C. If* Expectation 4.8 *holds, then $\mathcal{F}$ is S-ms on C.*

2. *Assume that $\mathcal{F}$ is S-ms on S-saturated C. Then $\mathcal{F}$ is ms on C.*

Proof. 1. Suppose (h, f) is C-acyclic over S. Let $\mathcal{G}$ be a sheaf on Y micro supported on S-acyclic C'. Then, by Expectation, $\mathcal{F} \boxtimes \mathcal{G}$ is micro supported on $\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C'$. Since (h, f) is $\mathrm{pr}_1^\circ C + \mathrm{pr}_2^\circ C'$-transversal by Example 4.7.1, it is $\mathcal{F} \boxtimes \mathcal{G}$-transversal.

2. Assume that $h\colon W \to X$ is C-transversal. Let $f\colon W \to S$ be the canonical morphism. Then, since C is assumed S-saturated, (h, f) is C-acyclic over S by Example 4.7.2. Hence $(h, f)\colon W \to X \times_S S$ is $\mathcal{F} \boxtimes \Lambda$-transversal. □

If the singular supports exist, Lemma means the inclusions $SS_S\mathcal{F} \subset SS\mathcal{F} \subset SS_S^{\mathrm{sat}}\mathcal{F}$. Thus, $SS_S\mathcal{F}$ can be too small and $SS_S^{\mathrm{sat}}\mathcal{F}$ can be too large. Their difference is at most dimension 1. If $FT^*S \times_S X \to FT^*X$ is the 0-mapping, then we have $SS_S\mathcal{F} = SS_S^{\mathrm{sat}}\mathcal{F}$ and $SS\mathcal{F}$ should be also the same. If X is smooth over S, S-saturated closed conical subsets of FT^*X corresponds to closed conical subsets of $T^*(X/S)|_{\mathbf{F}_p}$.

Theorem 4.10. *$SS_S^{\mathrm{sat}}\mathcal{F}$ exists.*

We may reduce to the case $X = \mathbf{P}^n$. Then, we may work with closed conical subsets of $T^*(X/S)|_{\mathbf{F}_p}$ and the method of Beilinson works.

5 Appendix. Local acyclicity and transversality

Proof of Proposition 3.9. By devissage, the condition (2) is equivalent to the condition where $\mathcal{G}$ is restricted to $\mathcal{G} = i_*\mathcal{L}$ for immersions $i\colon V \to Y$ of regular subschemes and locally constant constructible sheaves $\mathcal{L}$ on V as in Proposition 5.1 (2′) since Y is excellent.

Let $h\colon W = X \times_Y V \to X$ be the base change of $i\colon V \to Y$ and consider the cartesian diagram

(5.1)
$$\begin{array}{ccccc} W & \xrightarrow{\gamma'} & X \times_S V & \xrightarrow{\mathrm{pr}_2} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle 1\times i} & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{\gamma} & X \times_S Y & \xrightarrow{\mathrm{pr}_2} & Y. \end{array}$$

The condition (2′) in Proposition 5.1 means that the morphism

$$\gamma^*(\mathcal{F} \boxtimes i_*\mathcal{L}) = \mathcal{F} \otimes f^*i_*\mathcal{L} \to h_*(h^*\mathcal{F} \otimes g^*\mathcal{L}) = h_*\gamma'^*(\mathcal{F} \boxtimes \mathcal{L})$$

is an isomorphism. Hence this for every V and $\mathcal{L}$ as above is equivalent to (1) by Proposition 5.1 (2′)⇔(1). Thus it suffices to show that the following conditions are equivalent:

(1′) For every immersion $i\colon V \to Y$ of regular subscheme and every locally constant constructible sheaf $\mathcal{L}$ on V, the morphism

(5.2)
$$\gamma^*(\mathcal{F} \boxtimes i_*\mathcal{L}) \to h_*\gamma'^*(\mathcal{F} \boxtimes \mathcal{L})$$

is an isomorphism.

(2′) For every immersion $i\colon V \to Y$ of regular subscheme and every locally constant constructible sheaf $\mathcal{L}$ on V, the morphism

(5.3)
$$c_{\gamma, \mathcal{F}\boxtimes i_*\mathcal{L}}\colon \gamma^*(\mathcal{F} \boxtimes i_*\mathcal{L}) \otimes \gamma^!\Lambda \to \gamma^!(\mathcal{F} \boxtimes i_*\mathcal{L})$$

is an isomorphism.

For an immersion $i\colon V \to Y$ of subscheme and a constructible sheaf $\mathcal{G}$ on V, we construct a commutative diagram

$$\begin{array}{ccc} \gamma^*(\mathcal{F}\boxtimes i_*\mathcal{G})\otimes\gamma^!\Lambda & \xrightarrow{c_{\gamma,\mathcal{F}\boxtimes i_*\mathcal{G}}} & \gamma^!(\mathcal{F}\boxtimes i_*\mathcal{G}) \\ {\scriptstyle (5.2)'}\downarrow & & \downarrow \\ h_*(\gamma'^*(\mathcal{F}\boxtimes\mathcal{G})\otimes h^*\gamma^!\Lambda) & \longrightarrow & h_*\gamma'^!(\mathcal{F}\boxtimes\mathcal{G}) \end{array} \tag{5.4}$$

similarly as (5.17). The lower horizontal arrow is obtained by applying h_* to the composition

$$\gamma'^*(\mathcal{F}\boxtimes\mathcal{G})\otimes h^*\gamma^!\Lambda \to \gamma'^!(\mathcal{F}\boxtimes\mathcal{G}) \tag{5.5}$$

of $c_{\gamma',\mathcal{F}\boxtimes\mathcal{G}}$ with the morphism induced by $h^*\gamma^!\Lambda \to \gamma'^!\Lambda$. The right vertical arrow is the composition of $\gamma^!(\mathcal{F}\boxtimes i_*\mathcal{G}) \to \gamma^!(1\times i)_*(\mathcal{F}\boxtimes\mathcal{G})$ and the base change isomorphism $\gamma^!(1\times i)_* \to h_*\gamma'^!$. By the assumption that the morphism $X \to S$ is universally $\mathcal{F}$-acyclic and by Corollary 5.2, the morphism $\mathcal{F}\boxtimes i_*\mathcal{G} \to (1\times i)_*(\mathcal{F}\boxtimes\mathcal{G})$ is an isomorphism. Hence the right vertical arrow is an isomorphism. The left vertical arrow is the adjoint of the isomorphism $h^*\gamma^*(\mathcal{F}\boxtimes i_*\mathcal{G})\otimes h^*\gamma^!\Lambda \to \gamma'^*(\mathcal{F}\boxtimes\mathcal{G})\otimes h^*\gamma^!\Lambda$. By the assumption that $Y \to S$ is smooth, we have an isomorphism $\Lambda(-d)[-2d] \to \gamma^!\Lambda$ for the relative dimension d of Y over S. Hence the left vertical arrow $(5.2)'$ is an isomorphism if and only if (5.2) is an isomorphism. By applying Lemma 5.3.2 to the left square of (5.1), we see that (5.4) is commutative.

We show that the following conditions are equivalent:

(1°) (5.5) and the left vertical arrow $(5.2)'$ are isomorphisms.

(2°) (5.3) is an isomorphism,

(3°) All arrows in (5.4) are isomorphisms.

(1°)⇒(3°): If (5.5) is an isomorphism, then the lower horizontal arrow in (5.4) is an isomorphism. Further if the left vertical arrow $(5.2)'$ is an isomorphism, (3°) is satisfied.

(2°)⇒(1°): If (5.3) is an isomorphism, then the composition of (5.4) via upper right is an isomorphism. Since h is an immersion, this implies that (5.5) is an isomorphism. The rest is similar to the proof of (1°)⇒(3°).

(3°)⇒(2°) is clear.

The implication (2°)⇒(1°) implies $(2')$⇒$(1')$. Thus it suffices to show that (5.5) is an isomorphism assuming $(1')$ for an immersion $i\colon V \to Y$ of regular schemes and a locally constant sheaf $\mathcal{L}$ on V. Since (5.5) for locally constant $\mathcal{L}$ is the tensor product of (5.5) for Λ with the identity of the pull-back of $\mathcal{L}$, it suffices to show the isomorphism for a single rank 1 sheaf $\mathcal{L}$. Thus, we may further assume that $\mathcal{L} = i^!\Lambda$ by purity, [7, Exposé XVI, Théorème 3.1.1].

First, we prove (5.5) is an isomorphism in the case where i is an open immersion and $\mathcal{L} = \Lambda$. Since Y is smooth over S, the morphism γ is a section of a smooth morphism $\mathrm{pr}_1\colon X\times_S Y \to X$. Hence γ is $\mathcal{F}\boxtimes\Lambda$-transversal. Since i is an open immersion, its restriction $\gamma'\colon W \to X\times_S V$ is also $\mathcal{F}\boxtimes\Lambda$-transversal. This means that $\gamma'^*(\mathcal{F}\boxtimes\Lambda)\otimes\gamma'^!\Lambda \to \gamma'^!(\mathcal{F}\boxtimes\Lambda)$ (5.5) is an isomorphism, Since we assume $(1')$, γ is $\mathcal{F}\boxtimes i_*\Lambda$-transversal by (1°)⇒(2°). Namely, (5.3) is an isomorphism for $\mathcal{L} = \Lambda$.

Before proving the general case, we prove (5.3) is an isomorphism for a closed immersion i and $\mathcal{G} = i^!\Lambda$. Let $j\colon Y - V \to Y$ be the open immersion of the complement and consider the distinguished triangle $\mathcal{F}\boxtimes i_*i^!\Lambda \to \mathcal{F}\boxtimes\Lambda \to \mathcal{F}\boxtimes j_*\Lambda \to$. As we have already shown above, $\gamma\colon X \to X \times_S Y$ is transversal for the last two terms. Hence the morphism γ is also $\mathcal{F} \boxtimes i_* i^! \Lambda$-transversal and (5.3) for $\mathcal{G} = i^!\Lambda$ is an isomorphism. By (2°)⇒(1°), (5.5) is an isomorphism for $\mathcal{G} = i^!\Lambda$.

We show (5.5) is an isomorphism in the general case where V is regular and $\mathcal{L} = i^!\Lambda$. Let $\bar{i}\colon \overline{V} \to Y$ be the closed immersion of the closure. Then we obtain an isomorphism (5.5) for $\mathcal{L} = i^!\Lambda$ as the restriction of that for $\mathcal{G} = \bar{i}^!\Lambda$. □

Proposition 5.1. (cf. [6, Theorem 7.6.9]) *Let $f\colon X \to Y$ be a morphism of schemes and $\mathcal{F}$ be a sheaf on X. The following conditions are equivalent:*

(1) *f is $\mathcal{F}$-acyclic.*

(2) *For every cartesian diagram*

$$\begin{array}{ccc} X & \xleftarrow{\;h\;} & W \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow{\;i\;} & V \end{array} \tag{5.6}$$

such that i is an immersion and for every sheaf $\mathcal{G}$ on V, the morphism

$$\mathcal{F} \otimes f^* i_* \mathcal{G} \to h_*(h^*\mathcal{F} \otimes g^*\mathcal{G}) \tag{5.7}$$

is an isomorphism.

(2′) *For every cartesian diagram* (5.6) *such that i is an immersion and for every locally constant constructible sheaf $\mathcal{L}$ on V, the morphism*

$$\mathcal{F} \otimes f^* i_* \mathcal{L} \to h_*(h^*\mathcal{F} \otimes g^*\mathcal{L}) \tag{5.8}$$

is an isomorphism.

If Y is an excellent noetherian scheme, we may assume that V is regular in (2′)*.*

Proof. (1)⇒(2): The immersion i is the composition of an open immersion and a closed immersion. For an open immersion, the isomorphism (5.7) is [8, Proposition 2.10], [6, Lemma 7.6.7]. If i is a closed immersion, the isomorphism (5.7) follows from the projection formula and the proper base change theorem.

(2)⇒(2′) is clear.

(2′)⇒(1): Let y be a geometric point of Y and t be a geometric point of $Y_{(y)}$. We show that the morphism

$$\mathcal{F}|_{X_y} \to i^* j_* \mathcal{F}|_{X_t} \tag{5.9}$$

is an isomorphism. Let v be the image of t by $Y_{(y)} \to Y$ and $V \subset Y$ be the closure of $\{v\}$. Then, there exists a projective system $p_\lambda\colon U_\lambda \to V_\lambda$ of finite étale morphisms of integral schemes and open immersions $j_\lambda\colon V_\lambda \to V$ and an isomorphism $\varinjlim (j_\lambda p_\lambda)_* \Lambda \to j_*\Lambda$. If Y is excellent, we may assume that V_λ are regular. Let $j\colon t \to V$ be the canonical morphism and consider the cartesian diagram

$$\begin{array}{ccccccc} X & \xleftarrow{\;h\;} & W & \xleftarrow{\;h_\lambda\;} & W \times_V V_\lambda & \xleftarrow{\;k_\lambda\;} & W \times_V U_\lambda \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g_\lambda} & & \downarrow \\ Y & \xleftarrow{\;i\;} & V & \xleftarrow{\;j_\lambda\;} & V_\lambda & \xleftarrow{\;p_\lambda\;} & U_\lambda \end{array}$$

extending (5.6). As the isomorphism (5.8) for the immersions $ij_\lambda\colon V_\lambda \to Y$ and the locally constant sheaves $p_{\lambda *}\Lambda$ on V_λ, we obtain an isomorphism

$$\mathcal{F} \otimes f^* i_* j_{\lambda *} p_{\lambda *}\Lambda \to (hh_\lambda)_*((hh_\lambda)^*\mathcal{F} \otimes g_\lambda^* p_{\lambda *}\Lambda) \tag{5.10}$$

on X. The target of (5.10) is identified with $(hh_\lambda k_\lambda)_*(hh_\lambda k_\lambda)^*\mathcal{F}$ by the projection formula.

We consider the cartesian diagram

$$\begin{array}{ccccccccc}
Y & \xleftarrow{i} & V & \xleftarrow{j_\lambda} & V_\lambda & \xleftarrow{p_\lambda} & U_\lambda & \longleftarrow & t \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
Y_{(y)} & \xleftarrow{i_{(y)}} & V_{(y)} & \xleftarrow{j_{\lambda(y)}} & V_{\lambda(y)} & \xleftarrow{p_{\lambda(y)}} & U_{\lambda(y)} & \longleftarrow & t_{(y)}
\end{array}$$

and let ${}_{(y)}$ denote the base change by $Y \leftarrow Y_{(y)}$. Then the pull-back of (5.10) to $X \times_Y Y_{(y)}$ gives an isomorphism

$$\mathcal{F}|_{X\times_Y Y_{(y)}} \otimes f_{(y)}^*(i_{(y)} p_{\lambda(y)} j_{\lambda(y)})_*\Lambda \to (hh_\lambda k_\lambda)_{(y)*}(hh_\lambda k_\lambda)_{(y)}^*\mathcal{F}|_{X\times_Y Y_{(y)}}. \tag{5.11}$$

Let $T_{\lambda(y)} \subset U_{\lambda(y)}$ be the connected component containing the image of $t \to U_{\lambda(y)}$ induced by $t \to Y_{(y)}$ and consider the cartesian diagram

$$\begin{array}{ccc}
X \times_Y Y_{(y)} & \xleftarrow{h_{T\lambda(y)}} & X \times_Y T_{\lambda(y)} \\
{\scriptstyle f_{(y)}}\big\downarrow & & \big\downarrow{\scriptstyle f_{T_{\lambda(y)}}} \\
Y_{(y)} & \xleftarrow{j_{T\lambda(y)}} & T_{\lambda(y)}
\end{array}$$

whose limit gives the cartesian diagram

$$\begin{array}{ccc}
X \times_Y Y_{(y)} & \xleftarrow{j} & X_t \\
{\scriptstyle f_{(y)}}\big\downarrow & & \big\downarrow{\scriptstyle f_t} \\
Y_{(y)} & \xleftarrow{j_Y} & t.
\end{array} \tag{5.12}$$

Then, we obtain an isomorphism

$$\mathcal{F}|_{X\times_Y Y_{(y)}} \otimes f_{(y)}^* j_{T\lambda(y)*}\Lambda \to h_{T\lambda(y)*}h_{T\lambda(y)}{}^*\mathcal{F}|_{X\times_Y Y_{(y)}}. \tag{5.13}$$

as a direct summand of (5.11). Since $t = \varprojlim T_{\lambda(y)}$, by taking the limit of (5.13), we obtain the isomorphism

$$\mathcal{F}|_{X\times_Y Y_{(y)}} \otimes f_{(y)}^* j_{Y*}\Lambda \to j_* j^*(\mathcal{F}|_{X\times_Y Y_{(y)}}). \tag{5.14}$$

□

Corollary 5.2. ([9, Proposition 2.2]) *Assume that $X \to S$ is universally $\mathcal{F}$-acyclic. Then, for any immersion $i\colon V \to Y$ of schemes over S and for any sheaf $\mathcal{G}$ on V, the morphism $\mathcal{F} \boxtimes i_*\mathcal{G} \to (1 \times i)_*(\mathcal{F} \boxtimes \mathcal{G})$ is an isomorphism.*

Lemma 5.3. *Let*

$$\begin{array}{ccc} W' & \xrightarrow{h'} & X' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ W & \xrightarrow{h} & X, \end{array} \tag{5.15}$$

be a cartesian diagram of noetherian schemes and assume that the horizontal arrows are separated morphisms of finite type.

1. *For a sheaf $\mathcal{F}$ on X, the diagram*

$$\begin{array}{ccc} g^*(h^*\mathcal{F} \otimes h^!\Lambda) & \xrightarrow{g^*c_{h,\mathcal{F}}} & g^*h^!\mathcal{F} \\ \downarrow & & \downarrow \\ h'^*f^*\mathcal{F} \otimes h'^!\Lambda & \xrightarrow{c_{h',f^*\mathcal{F}}} & h'^!f^*\mathcal{F} \end{array} \tag{5.16}$$

is commutative.

2. *Let $\mathcal{F}'$ be a sheaf on X'. Then, the following diagram is commutative:*

$$\begin{array}{ccc} h^*f_*\mathcal{F}' \otimes h^!\Lambda & \xrightarrow{c_{h,f_*\mathcal{F}'}} & h^!f_*\mathcal{F}' \\ \downarrow & & \downarrow \\ g_*(h'^*\mathcal{F}' \otimes h'^!\Lambda) & \xrightarrow{g_*(c_{h',\mathcal{F}'})} & g_*h'^!\mathcal{F}' \end{array} \tag{5.17}$$

where the left vertical arrow is the adjoint of

$$g^*(h^*f_*\mathcal{F}' \otimes h^!\Lambda) = g^*h^*f_*\mathcal{F}' \otimes g^*h^!\Lambda \xrightarrow{\mathrm{can}\otimes 1} h'^*f^*f_*\mathcal{F}' \otimes g^*h^!\Lambda \longrightarrow h'^*\mathcal{F}' \otimes h'^!\Lambda.$$

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