Characteristic cycle

X smooth $/k$ perfect $\qquad \dim X = n$.
Λ finite field $\quad$ char inv. in k
$\mathcal{F}$ constructible complex of Λ-mod. on X.

$\operatorname{Supp} \mathcal{F} \subset X$ $\quad$ closed subset
$SS\mathcal{F} \subset T^*X$. $\quad$ singular support $\quad$ Beilinson.
closed conical subset $\quad$ stable under $\mathbb{G}_m$-action of the cotangent bundle $\dim 2n$

$SS\mathcal{F} = \bigcup C_a \quad C_a$ irred $\dim n$. $\quad SS\mathcal{F} \cap T^*_x X = \operatorname{Supp}\mathcal{F}$
$CC\mathcal{F} = \sum m_a C_a$ $\quad$ characteristic cycle.
≥ 0 if perverse & $\operatorname{Supp}\, SS\mathcal{F} = CC\mathcal{F}$

1. Examples
2. Direct image.
3. Characteristic class.

Example. 1. $\dim X = 1$. $D \subset X$ smallest. $\mathcal{F}|_{X-D}$ loc. const., $\neq 0$
$$SS\mathcal{F} = T^*_X X \cup \bigcup_{x \in D} T^*_x X$$
0-section $\qquad$ fiber.
$$CC\mathcal{F} = -\left(\operatorname{rk} \mathcal{F}|_{X-D} \cdot T^*_X X + \sum_{x \in D} a_x \mathcal{F} \cdot T^*_x X\right)$$
Artin conductor $a_x\mathcal{F} = \operatorname{rk}\mathcal{F} - \operatorname{rk}\mathcal{F}_{\bar{x}} + \operatorname{Sw}_x \mathcal{F}$
Swan conductor

2 $\dim X = n$ any $D \subset X$ div. with s.n.c. $j: U = X - D \to X$
$\mathcal{F} = j_! \mathcal{G}[n]$ $\quad \mathcal{G} \neq 0$ loc. const $/U$. tamely ramified along D.
$SS\mathcal{F} = \bigcup_I T^*_{X_I} X$. $\quad$ conormal bundles $X_I = \bigcap_{i \in I} D_i \quad D = \bigcup D_i$
$CC\mathcal{F} = \sum_I \operatorname{rk} \mathcal{G} \cdot T^*_{X_I} X$. $\quad$ including $X_\emptyset = X$. $T^*_X X$ 0-section

1.2 $SS\mathcal{F}$ Lagrangean. $\quad$ not in general.

Direct image. or more generally proper on supp $\mathcal{F}$

$f: X \to Y$ proper morphism of smooth schemes $/k$

$\dim X = n$. $\dim Y = m$.

$f_* CC\mathcal{F}$ $\quad T^*X \leftarrow X \times_Y T^*Y \to T^*Y$

$f_\circ SS\mathcal{F}$

$f_* CC\mathcal{F} \in CH_m(f_\circ SS\mathcal{F})$

$\in Z_m(f_\circ SS\mathcal{F})$ if $\dim f_\circ SS\mathcal{F} = m$.

Conjecture. $CCRf_*\mathcal{F} = f_* CC\mathcal{F}$ in $CH_m(f_\circ SS\mathcal{F})$

Theorem. If $f: X \to Y$ is proj and $\dim f_\circ SS\mathcal{F} = m$,

$CCRf_*\mathcal{F} = f_* CC\mathcal{F}$ in $Z_m(f_\circ SS\mathcal{F})$

Example. 1. $Y = \operatorname{Spec} k$. $\dim f_\circ SS\mathcal{F} = 0$ always.

$\chi(X_{\bar{k}}, \mathcal{F}) = (CC\mathcal{F}, T^*_X X)_{T^*X}$.

In particular if $\dim X = 1$

$\chi(X_{\bar{k}}, \mathcal{F}) = \operatorname{rk}\mathcal{F} \cdot \chi(X_{\bar{k}}) - \sum_{x \in D} a_x\mathcal{F} \cdot \deg x$.

2. $\dim Y = 1$. $\dim f_\circ SS\mathcal{F} = 1$ is satisfied after b.c by iteration of F.

$-a_y Rf_*\mathcal{F} = (CC\mathcal{F}, df)_{T^*X, y} + 1$.

$\chi(X_{\bar{\eta}}, \mathcal{F}) - \chi(X_{\bar{y}}, \mathcal{F}) + \operatorname{Sw}_y H^*(X_{\bar{\eta}}, \mathcal{F})$

Ex $1 + 2 \Rightarrow$ Theorem.

Idea of pf of $1 \Rightarrow 2$. ~~Resp [illegible]~~

O.p.s Y, $X \to Y$ proj smooth

$1 \Rightarrow$ Sum of LHS = Sum of RHS.

fix $y \in Y$ $\exists Y' \to Y$ étale at y. Killing other terms.

kill ramification on $R\Phi\mathcal{F}$.

Characteristic class.

X possibly singular $i: X \to M$ ~~smooth~~ closed imm. M smooth, $\dim M$

$C(X)$ class of the closure $\overline{CC(i)}$ regarded as elt of
$$CH_N(\mathbb{P}(X\times_M T^*M \oplus \mathbb{A}^1_X)) = CH_\bullet(X)$$

Question: Relation to coh. c.c. & Swan class

$$CC_X : K(X,\Lambda) \to CH_\bullet(X)$$

char $k=0$

$$\begin{tikzcd} K(X,\Lambda) \arrow[rr] \arrow[dr, "\mathrm{rk}"'] & & CH_\bullet(X) \\ & F(X) \arrow[ur, "(-1)^\bullet c_M"'] & \end{tikzcd}$$

$(-1)^\bullet c_M$ MacPherson Chern class

$F(X)$ $\mathbb{Z}$-valued constructible fn.

char $k>0$. $\mathcal{V}_X = (\text{Sep. of ft}/k \,/\, X)$ $\quad K_0(\mathcal{V}_X)$

$$\hat{F}(X) = \mathrm{Hom}(K_0(\mathcal{V}_X), \mathbb{Z})$$

$$K(X,\Lambda)\times K_0(\mathcal{V}_X) \to \mathbb{Z}.\quad (\mathcal{F}, g: Z\to X) \mapsto \chi_c(Z_{\bar{k}}, g^*\mathcal{F})$$

$$K(X,\Lambda) \to \hat{F}(X). \qquad \mathrm{Ker} = K(X,\Lambda)_0$$

$$\begin{tikzcd} CC_X : K(X,\Lambda) \arrow[rr] \arrow[dr] & & CH_\bullet(X) \\ & K(X,\Lambda)/K(X,\Lambda)_0 \arrow[ur] & \end{tikzcd}$$

. Ref S $\quad f^{X\to Y}_{\text{proper}} \Rightarrow$

$$\begin{tikzcd} K(X,\Lambda) \arrow[r] \arrow[d] & CH_\bullet(X) \arrow[d] \\ K(X,\Lambda) \arrow[r] & CH_\bullet(Y) \end{tikzcd}$$

? counter example

. Conjecture $\Rightarrow$ $CH_\bullet$ replaced by CH_0.

Proof of Example 1 ⇒ 2.

Proposition $f : X \to Y$ smooth. Y curve. $y \in Y$ closed pt
$\mathcal{F}$ constructible cx
$\Rightarrow \exists\, Y' \to Y$ f.flat finite curve. $\exists\, y' \mapsto y$
$\exists\, \mathcal{F}'$ modification at $X'_{y'}$ of the pull-back of $\mathcal{F}$ to $X' = X \times_Y Y'$
s.t. $f' : X' \to Y'$ is C'-trans. at $X'_{y'}$
i.e. $(X'_{y'}, T^*Y') \cap SS\mathcal{F}' \subset T^*_{X'}X'$ 0-section in T^*X'

Pf.

(1) $\mathcal{F}$ perverse

(2) after base change inertia action on $R\Phi\mathcal{F}$ trivial

(3) replace $\mathcal{F}'$ by $j'_{!*}j'^*\mathcal{F}'$ $\quad j' : X' - X'_{y'} \to X' \xleftarrow{i'} X'_{y'}$

(4) (2) + (3) $\Rightarrow$ $f' : X' \to Y'$ loc acyclic rel. to $\mathcal{F}'$.
$i'^*\mathcal{F}' \xrightarrow{\sim} R\Phi\mathcal{F}'$

(5) semi-stable case $W \to Y$ semi-stable
$i : W \to X$ closed immersion $\mathcal{F} = i_*\Lambda$. induction on dim.

(1), (4), (5) $\Rightarrow$ conclusion
alteration + chain rule "