Characteristic Cycle of an l-adic sheaf on an alg. surface

k: alg. clos. char $p > 0$ $l \neq p$

X: smooth alg. var $/k$. $d = \dim X$

$\mathcal{F}$: l-adic sheaf on X.

Analogie with micro local analysis on $\mathcal{D}$-mod $/\mathbb{C}$

Expectation

middle dim (total dim $= 2d$)

One can define the char cycle $\mathrm{Char}(\mathcal{F})$

as an alg. cycle of dim d on the cotangent bdle T^*X

vector bdl of rk d ass to Ω_X^1

1) additivity

$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ exact seq

$\Rightarrow \mathrm{Char}(\mathcal{F}) = \mathrm{Char}(\mathcal{F}') + \mathrm{Char}(\mathcal{F}'')$

suffices to consider $j_!\mathcal{F}$ $j: U \hookrightarrow X$ dense open imm

$\mathcal{F}$: smooth on U.

2) étale local

3) Euler number

X: proper

$$\chi(X, \mathcal{F}) = \sum_{q=0}^{2d} (-1)^q \dim H_c^q(U, \mathcal{F}) = (\mathrm{Char}\,\mathcal{F}, T_X^*X)_{T^*X}$$

4) vanishing cycle $f : X \to C$ smooth morphism to a smooth curve C

$u \mapsto v$

$u \in X$ closed pt, isolated characteristic pt.

$$\underline{\dim\mathrm{tot}}\ \Phi_u(j_!\mathcal{F}, f) = \underline{(\mathrm{Char}(\mathcal{F}), f^*dt)_{T^*X}}$$

def'd

stalk at u

cx. vanishing cycle

each coh. is an ℓ-adic rep of K_v

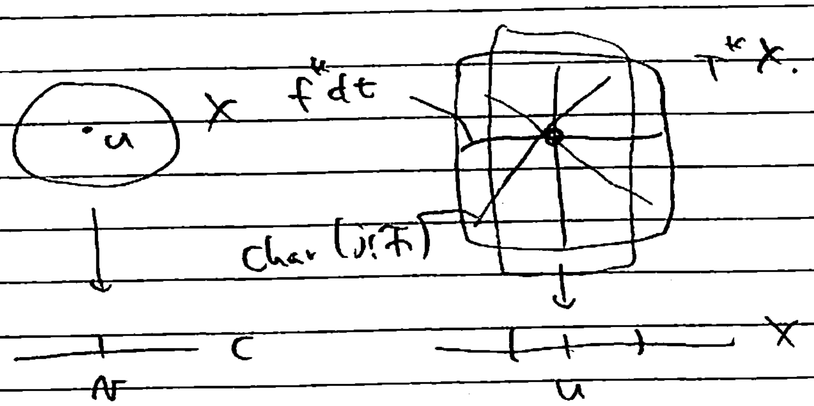
local field

dim tot = alternating sum of the dimension + Swan conductor of $\Phi_u^q(j_!\mathcal{F}, f)$

measure wild ramification of inertia

t: local parameter

f^*dt : section of T^*X on a nbd of u



Property 4) characterizes $\mathrm{Char}(j_!\mathcal{F})$

5) compatibility with the pull-back by non-characteristic morphism $Y \to X$.

Examples

1) tamely ramified case; $X \supset D = D_1 \cup \cdots \cup D_m$ divisor with simple normal crossing

$\mathcal{F}$: smooth on $U = X - D$ tamely ramified along D $\quad j_!\mathcal{F}$

$j : U \hookrightarrow X$

$I \subset \{1, \dots, m\}$ $\quad D_I = \bigcap_{i \in I} D_i \quad T^*_{D_I}X \subset T^*X \times_X D_I \subset T^*X$

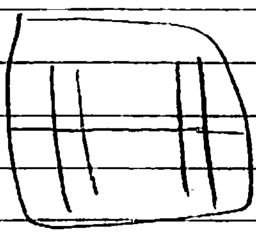
codim $= d - |I|$ $\qquad N_{D_I/X} \hookrightarrow \Omega^1_X \otimes \mathcal{O}_{D_I}$

$$\mathrm{Char}(j_!\mathcal{F}) = (-1)^d \operatorname{rank}\mathcal{F} \cdot \sum_{\emptyset \subset I \subset \{1, \dots, m\}} [T^*_{D_I}X]$$

2) X smooth curve $\quad D \subset X$ div. $\quad \mathcal{F}$ smooth on $U = X - D$

no condition on ramification

$$\mathrm{Char}(j_!\mathcal{F}) = -(\operatorname{rank}\mathcal{F})\underbrace{[T^*_X X]}_{\text{0-section}} + \sum_{x \in D} \mathrm{dimtot}_x \mathcal{F} \cdot \underbrace{[T^*_x X]}_{\text{fiber at } x}$$



$$\to \chi_c(U, \mathcal{F}) = (\mathrm{Char}(j_!\mathcal{F}), T^*_X X)_{T^*X}$$

G-O-S formula

<u>Theorem</u>

X : smooth surface $j: U \hookrightarrow X$ dense open imm

$\mathcal{F}$: smooth on U.

$\Rightarrow$ $Char(j_! \mathcal{F})$ is defined and satisfies the <u>expected properties</u>

$C \to X$ $D = X - U = \bigcup D_i$

How to define it :

codim 1 ① Ramification theory ... restriction to curve

codim 2 ② Radon transform ... vanishing cycle.

$$Char(j_! \mathcal{F}) = \mathrm{rank}\,\mathcal{F} \cdot [T^*_X X] \quad \text{codim } 0$$

$$+ \sum_{D_i:\dim 1} \sum_j n_{ij} [C_{ij}] \quad \text{codim } 1$$

C_{ij} "sub line bdle" of $T^*X \times_X D_i$ (dim 2, dim 3)

$$+ \sum_{\substack{x \in D \\ \text{closed pt}}} n_x [T^*_x X] \quad \text{codim } 2$$

fiber.

① filtration by ramification groups of the Galois group of a local field w. imperfect residue field (Abbes-S)

X : smooth dim d

D : smooth irr. div. ξ gen. pt.

K_ξ : local field at ξ = $\mathrm{Frac}\, \hat{\mathcal{O}}_{X,\xi}$ d.v.r.

F : residue field $F = \kappa(\xi)$ function field of D.

$G_K = \mathrm{Gal}(K^{sep}/K) \supset (G_K^r)$ decreasing filtration by closed normal subgroups indexed by $r \in \mathbb{Q}_{\geq 1}$

$G_K^1 = I$ = inertia group $\quad G_K^{r+} = \overline{\bigcup_{s>r} G_K^s} \subset G_K^r$

$\cup$

$G_K^{1+} = P$ = wild inertia : p-sylow of I

Prop

$r>1$ $\quad \mathrm{Gr}^r G_K = G_K^r / G_K^{r+}$ is abelian and killed by p

$\mathrm{Gr}^r G_K^\vee = \mathrm{Hom}(\mathrm{Gr}^r G_K, \mathbb{F}_p)$

$\exists$ a canonical inj.

$\{a \in K_{sep} \mid \mathrm{ord}(a) \geq -r\}$

$$\mathrm{Gr}^r G_K^\vee \hookrightarrow m_{K_{sep}}^{-r} \otimes_{O_{K_{sep}}} \bar{F} \otimes_{O_{X,x}} \Omega^1_{X,x}$$

$\bar{F}$-v. sp. of dim 1 $\qquad$ $\bar{F}$-vect. sp of dim d.

alg. clos. of F

$(\bar{F} = O_{K^{sep}}/m_{O_{K^{sep}}} \supset F)$

$U = X - D$ $\quad \mathcal{F}$ smooth on U

$\rightsquigarrow \bar{\eta} = \mathrm{Spec}\, K_{sep}$ $\quad \mathcal{F}_{\bar{\eta}}$: l-adic rep'n G_K

$\quad\quad \| $

$\quad\quad M$

filtration $\Rightarrow$ slope decomposition $\quad M^{G_K^{r+}} = \bigoplus_{s \leq r} M^{(s)}$

$r>1$ $\quad M^{(r)}$ as rep'n of $\mathrm{Gr}^r G_K^\vee$ (abelian) $\quad M^{(r)} = \bigoplus_{\chi \in \mathrm{Gr}^r G_K^\vee} \chi^{n(\chi)}$

Date

$\chi \in G_r^\vee G_K^\vee$ can. inj. = char.

x_0 $\bar{\mathcal{F}}$ char $(\chi) \subset L^{(r)} \otimes \Omega^1$ → line in $T^*X \times_X D$

inj. ⇒ $\neq 0$ 1-dim sub vector space $\| \; m_E \otimes \bar{F}$

② Radon transform X proj. smooth $\mathcal{L}$ very ample inv. $\mathcal{O}_X$-mod

$$X \hookrightarrow \mathbb{P} = \mathbb{P}(E^\vee) = (E^\vee - \{0\})/k^\times$$

$$E = \Gamma(X, \mathcal{L})$$

$\mathbb{P}^\vee = \mathbb{P}(E) = \{H \mid H \subset \mathbb{P} \text{ hyperplane}\}$ dual of $\mathbb{P}$

$\mathbb{H} = \{(x, H) \mid x \in H\} \subset \mathbb{P} \times \mathbb{P}^\vee$ univ. hyperplane

$$\begin{tikzcd} X \times_{\mathbb{P}} \mathbb{H} \arrow[r, "p"] \arrow[d, "q"] & \mathbb{P}^\vee \\ X & \end{tikzcd}$$

universal family of hyperplane section

$j_!\mathcal{F}$

$R_{\mathcal{L}}(j_!\mathcal{F}) = Rp_* q^* j_!\mathcal{F}$ Radon transform

"To define u_x using the ramification

$R_{\mathcal{L}}(j_!\mathcal{F})$ along the dual hyperplane

$\boxed{\mathbb{P}^\vee \supset H_x} = \{H \in \mathbb{P}^\vee \mid x \in \mathbb{P}\}$ of x