The Euler numbers of ℓ-adic sheaves of rank 1 in positive characteristic.

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One of the most important themes in ramification theory is the formula for the Euler characteristic of ℓ-adic sheaves. Although we have the Grothendieck-Ogg-Shafarevich formula [G] in one dimensional case, we don't have a general formula in higher dimension even in the form of a conjecture. However for sheaves of rank 1, K.Kato formulated a conjecture in arbitrary dimension and actually proved it in dimension 2 in [K2]. In this paper, we will prove it in arbitrary dimension under a certain hypothesis, which is hoped to hold when the variety is sufficiently blowed up.

We consider a smooth ℓ-adic sheaf $\mathcal{F}$ of rank 1 on a dense open subscheme U of a proper smooth variety X over an algebraically closed field k of characteristic $p \neq \ell$ such that the complement $X - U$ is a divisor with normal crossing. At each generic point of the complement divisor, we have the ramification theory of $\mathcal{F}$ by Kato's theory on the abelianized absolute Galois group of complete discrete valuation field in [K3]. When $\mathcal{F}$ is clean, which means the ramification of $\mathcal{F}$ is understood by those at codimension 1 points, the characteristic variety $Ch(\mathcal{F})$ in the cotangent bundle with logarithmic poles and the characteristic 0-cycle

$$c_{\mathcal{F}} = (-1)^{\dim(X)-1} \cdot (Ch(\mathcal{F}),\ 0\text{-section}) \in CH_0(X)$$

are defined in [K1]. The definition will be reviewed in the text. The main result of this paper is

THEOREM (VAGUE). *Assume $\mathcal{F}$ as above is clean and something arising from $\mathcal{F}$ are also clean. Then we have*

$$\chi_c(U, \mathcal{F}) - \chi_c(U) = -\deg\ c_{\mathcal{F}}.$$

We didn't give the exact form of the theorem here because to write down the precise assumption will need some preparation. The formula is conjectured by Kato in [K2] without redundant assumption. If dim $X = 1$, the assumption is automatically satisfied and the formula is the rank 1 case of the Grothendieck-Ogg-Shafarevich formula. In general, if X, U and $\mathcal{F}$ are given, it is hoped that there is a resolution $\pi : X' \to X$

such that $\pi^*U \simeq U$ and that $\pi^*\mathcal{F}$ satisfies the assumption of the theorem and also hoped that $\pi_*(c_{\pi^*\mathcal{F}})$ does not depend on the choice of X'. If this is true, we can eliminate the redundant assumption. Actually, if $\dim X = 2$, it is proved to be true in [K1] and the formula is shown in [K2] by Kato. Although the theorem is not a final definitive one, the author believes that the argument used in the proof will be a critical step in a full proof of the conjecture.

Recently, P.Berthelot has made a great progress in p-adic ramification theory by constructing a theory of p-adic $\mathcal{D}$-modules. The author hopes that this will shed some light on ℓ-adic theory.

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First we briefly review the ramification theory of sheaves of rank 1 by Kato ([K1] and [K3]) and fix the notations. Let K be a complete discrete valuation field with arbitrary residue field $\overline{K}$ and let A be the integer ring of K. We define a $\overline{K}$-vector space ω_K to be $\Omega^1_A(\log \overline{K}) \otimes_A \overline{K}$ where $\Omega^1_A(\log \overline{K})$ is the A-module of differential forms with logarithmic poles $(\Omega^1_A \oplus (A \otimes K^\times))/(da - a \otimes a, a \in A, \neq 0)$ (cf. [K1] (2.2)). For $a \in A, \neq 0$, the class of $1 \otimes a$ is denoted by $d\log a$. Then ω_K is generated by $\Omega^1_{\overline{K}}$ and $d \log \pi$, where π is a uniformizer of K, and fits in an exact sequence

$$0 \to \Omega^1_{\overline{K}} \to \omega_K \xrightarrow{\text{residue}} \overline{K} \to 0 \quad (\text{res} \cdot d\log \pi = 1).$$

Let $\overline{G}_K$ be the abelianized absolute Galois group $Gal(K^{ab}/K)$ and $X_K = Hom(\overline{G}_K, \mathbf{Q}/\mathbf{Z})$. Define the pairing $\{\ ,\ \}_K : X_K \times K^\times \to Br(K)$ by the cup product $H^2(K, \mathbf{Z}) \times H^0(K, \mathbf{G}_m) \to H^2(K, \mathbf{G}_m)$. Let K' be the completion of the field $K(T)$ with respect to the discrete valuation corresponding to the prime ideal $m_A \cdot A[T]$ of $A[T]$. For $n \in \mathbf{N}, > 0$, let $U^n_K = 1 + m^n_K$. Then the filtration by ramification on X_K is defined by

$$X^n_K = \{\chi \in X_K;\ \{\chi_{K'}, U^{n+1}_{K'}\}_{K'} = 0\}$$

for $n \in \mathbf{N}$, where $\chi_{K'}$ is the image of χ in $X_{K'}$. The set of at most tamely ramified characters is X^0_K and $X_K = \bigcup X^n_K$. There is a canonical morphism $\omega_K \to Br(K)$ extending the Artin-Schreier theory $\overline{K} \to X_{\overline{K}}$

and $\Omega^1_{\overline{K}} \to Br(\overline{K})$. We will give its definition in the proof of Lemma 5. For $n > 0$, we have a commutative diagram

$$\begin{array}{ccc} X_K^n \times U_K^n & \longrightarrow & \omega_K \\ \downarrow & & \downarrow \\ X_K \times K^\times & \longrightarrow & Br(K). \end{array}$$

The upper horizontal map induces a pairing $\{\ ,\ \}_K^n : gr^n(X_K) \times N_K^n \to \omega_K$ where $N_K^n = m_K^n/m_K^{n+1} \simeq U_K^n/U_K^{n+1}$. This pairing is $\overline{K}$-linear with respect to N_K^n and characterized by the property that the pairing $X_K^n \times U_{K'}^n \to Br(K')$ is given by the composite

$$X_K^n \times U_{K'}^n \to gr^n(X_K) \times N_{K'}^n \simeq$$
$$gr^n(X_K) \times N_K^n \otimes_{\overline{K}} \overline{K}' \to \omega_K \otimes_{\overline{K}} \overline{K}' \hookrightarrow \omega_{K'} \to Br(K')$$

since $\omega_K \xrightarrow{\times T} \omega_{K'} \to Br(K')$ is injective. For a character $\chi \in X_K$, we define the Swan conductor $sw(\chi)$ to be the minimum integer $n \in \mathbb{N}$ such that $\chi \in X_K^n$. We define the refined Swan conductor $rsw(\chi)$ of a character χ with $sw(\chi) = n > 0$ to be the $\overline{K}$-homomorphism

$$rsw(\chi) = \{\chi,\ \}_K^n : N_K^n \to \omega_K.$$

Let k be an algebraically closed field of characteristic p. Let X be a smooth variety over k of dimension d and U be an open subscheme of X whose complement is a divisor with (Zariski locally) normal crossings. We call such (X, U) an RN-pair over k. Let $\ell \neq p$ be a prime number and $\mathcal{F}$ be the smooth ℓ-adic sheaf on U of rank 1 corresponding to a character $\chi : \pi_1(U)^{ab} \to \overline{\mathbb{Q}}_\ell^\times$ of finite order. We fix an isomorphism $\mathbb{Q}/\mathbb{Z}(1)_{\overline{\mathbb{Q}}_\ell} \simeq \mathbb{Q}/\mathbb{Z}$. For each irreducible component C_i of $X - U$, let K_i be the completion of the function field of X with respect to the discrete valuation corresponding to C_i. Then by applying the theory reviewed above to the pull-back of χ to each K_i, we get $sw_i(\chi) \in \mathbb{N}$ and $rsw_i(\chi) : N_{K_i}^{sw_i(\chi)} \to \omega_{K_i}$ if $sw_i(\chi) > 0$. We define the Swan divisor D_χ of $\mathcal{F}$ by

$$D_\chi = \sum sw_i(\chi) \cdot C_i.$$

Let ω_X denote the sheaf $\Omega^1_{X/k}(\log(X - U))$ of differential 1-forms with logarithmic poles at $X - U$. Beware of the unusual notation. Generally, for a point x of a scheme X (resp. a closed immersion $i : Z \to X$) and a quasi-coherent $\mathcal{O}_X$-module $\mathcal{E}$, we put $\mathcal{E}(x) = \mathcal{E}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ (resp. $\mathcal{E}|_Z = i^*\mathcal{E}$). Then at each generic point ξ_i of C_i, we have $\omega_X(\xi_i) = \omega_{K_i}$ and hence $rsw_i(\chi)$ is a map $\mathcal{O}_X(-D_\chi)(\xi_i) \to \omega_X(\xi_i)$ for i with $sw_i(\chi) > 0$. By [K3] Theorem (7.1), this extends to a morphism $rsw_i(\chi) : \mathcal{O}_X(-D_\chi)|_{C_i} \to \omega_X|_{C_i}$. Further it is shown there that they extends to a morphism $rsw(\chi) : \mathcal{O}_X(-D_\chi)|_D \to \omega_X|_D$ where D is the support of D_χ with the reduced structure. The sheaf $\mathcal{F}$ is said to be clean if $rsw_i(\chi)$ is locally an isomorphism onto a direct summand for every i with $sw_i(\chi) > 0$. Assume $\mathcal{F}$ is clean. Then the characteristic variety $Ch(\mathcal{F})$ of $\mathcal{F}$ is a dimension d cycle of the cotangent bundle $\mathsf{V}(\omega_X) = \mathsf{V}(\Omega^1_{X/k}(\log(X-U)))$ with logarithmic poles

$$Ch(\mathcal{F}) = \sum sw_i(\mathcal{F}) \cdot [\mathsf{V}(\mathcal{O}_X(-D_\chi)|_{C_i})].$$

Here V denotes the covariant vector bundle associated to a locally free sheaf and $\mathsf{V}(\mathcal{O}_X(-D_\chi)|_{C_i})$ is regarded as a subvariety of $\mathsf{V}(\omega_X)$ by $rsw_i(\chi)$. The characteristic 0-cycle $c_{\mathcal{F}}$ is

$$c_{\mathcal{F}} = (-1)^{d-1} s^* Ch(\mathcal{F}) \in CH_0(X)$$

where $s : X \to \mathsf{V}(\omega_X)$ is the 0-section. By an elementary calculation, we have

$$c_{\mathcal{F}} = \{c^*(\omega_X) \cdot (1 + D_\chi)^{-1} \cdot D_\chi\}_{\dim 0}$$

where $c(\omega_X)$ is the total chern class of $\omega_X = \Omega^1_{X/k}(\log(X-U))$ and $*$ denotes the operator multiplying $(-1)^i$ on codimension i part.

CONJECTURE (KATO). *Let (X, U) be an RN-pair over k and $\mathcal{F}$ be the smooth ℓ-adic sheaf of rank 1 corresponding to a character of $\pi_1(U)^{ab}$ of finite order as above. Assume X is proper and $\mathcal{F}$ is clean. Then*

$$\chi_c(U, \mathcal{F}) - \chi_c(U) = - \deg c_{\mathcal{F}}.$$

Here $\chi_c(U, \mathcal{F}) = \sum (-1)^i \dim H^i_c(U, \mathcal{F})$ and $\chi_c(U) = \chi_c(U, \mathbb{Q}_\ell)$. If dim $X = 2$, this conjecture is proved by Kato in [K2]. The Euler number $\chi_c(U)$ itself is the degree of a 0-cycle $c_{X,U} = c^*(\omega_X)_{\dim 0}$ (cf. Lemma 0 below).

We give the precise statement of the assumption of our main theorem. Let (X, U) be an RN-pair over k as above. Let θ be a character of $\pi_1(U)^{ab}$ of order p. It is called s-clean if it is clean and for each C_i with $sw_i(\theta) > 0$, the composite $\mathcal{O}(-D_\theta)|_{C_i} \xrightarrow{rsw_i(\theta)} \omega_X|_{C_i} \xrightarrow{res} \mathcal{O}_{C_i}$ is either an isomorphism or a zero map (depending on C_i). Let $\pi : Y \to X$ be the integral closure of X in the etale covering V of U of degree p trivializing θ and $j : V \to Y$ be the inclusion. Then as we will show in Lemma 1, the logarithmic structure $M = \mathcal{O}_Y \cap j_*\mathcal{O}_V^\times$ on Y is regular. We will briefly review the theory of logarithmic structure later (or see [K4]). Hence we have the resolution $\tilde{Y} \to Y$ associated to a suitable proper subdivision of the fan associated to (Y, M) by the procedure of [K4] (10.4). We refer to [K4] Sections 5, 9 and 10 for the theory of fan and associated resolution (cf. Proof of Lemma 2). This $\tilde{Y} \to Y$ is an isomorphism on V and $(\tilde{Y}, V)$ is an RN-pair over k.

Let (X, U) be an RN-pair over k as above and χ be a character of $\pi_1(U)^{ab}$ of order n. We consider the following condition $*$ on (X, U, χ) which says the above construction works inductively.

$*$ There is a sequence $(X_i,\ U_i,\ \chi_i)$ for $0 \leqslant i \leqslant e = \mathrm{ord}_p n$, satisfying the following conditions

1). $(X_0, U_0, \chi_0) = (X,\ U,\ \chi)$.

2). For $0 \leqslant i < e$, the character $\theta_i = \frac{n}{p^{i+1}} \cdot \chi_i$ is s-clean and of order p, the pair (X_{i+1}, U_{i+1}) is $(\tilde{Y}, V)$ constructed by the above procedure from (X_i, U_i, θ_i) and χ_{i+1} is the pull-back of χ_i.

Then our main result is

THEOREM. *Let (X, U) be an RN-pair over k and χ be a character of $\pi_1(U)^{ab}$ of order n. Let $\mathcal{F}$ be the smooth ℓ-adic sheaf of rank 1 on U corresponding to χ. Assume that X is proper, the condition $*$ is satisfied and that (X_i, U_i, χ_i) is clean for every $0 \leqslant i \leqslant e = \mathrm{ord}_p n$. Then we have*

$$\chi_c(U, \mathcal{F}) - \chi_c(U) = -\deg c_{\mathcal{F}}.$$

REMARK: If dim $X = 2$, by [K1] Theorem (4.1), there is an RN-pair (X', U') with a proper morphism $\pi : X' \to X$ such that $\pi : U' \simeq U$ and that $(X', U', \pi^*\mathcal{F})$ satisfies the assumption of Theorem and $\pi_*(c_{\pi^*\mathcal{F}}) = c_{\mathcal{F}}$ by Theorem (5.2) loc. cit. Therefore if dim $X = 2$, Theorem implies Conjecture above as is shown in [K2].

PROOF: First we reduce it to the case where n is a power of p. Let χ_0 be the character of order p^e such that $\chi \cdot \chi_0^{-1}$ is of order prime to p. Then it is clear that χ_0 satisfies the assumption of Theorem and $c_\chi = c_{\chi_0}$. Let V be the etale covering of U of degree n trivializing χ and $G = Aut_U(V) \cong \mathbb{Z}/n$. For $\sigma \in G$, we put $Tr_V(\sigma) = \sum(-1)^i Tr(\sigma; H^i_c(V,\ \mathbb{Q}_\ell))$. Then

THEOREM DL ([D-L] THEOREM 3.2 AND PROPOSITION 3.3). *For all $\sigma \in G$, $Tr_V(\sigma)$ is a rational integer independent of ℓ and, if the order of σ is not a power of p, it is zero.*

Since $\chi_c(U,\mathcal{F}_X) = (\chi, Tr_V)_G$, we have $\chi_c(U,\mathcal{F}_X) = \chi_c(U,\mathcal{F}_{X_0})$. Therefore it is sufficient to prove Theorem in the case $n = p^e$ and $p \neq 0$.

Before continuing the proof, we need to review the theory of logarithmic structure and that of ramification of an automorphism by Kato [K4] and [K2]. A logarithmic structure on a scheme X is a morphism of sheaves of commutative monoids $\alpha : M \to \mathcal{O}_X$ with respect to the multiplication of $\mathcal{O}_X$ such that $\alpha^{-1}(\mathcal{O}_X^\times) \simeq \mathcal{O}_X^\times$. A log structure is called trivial if $M = \mathcal{O}_X^\times$. A scheme X with log structure M is called a log scheme (X, M). In this paper, we only consider such a log structure M that, locally on X, there is a finitely generated integral monoid P and $\alpha : P \to \mathcal{O}_X$ such that M is induced by P. This condition is slightly weaker than (S) in [K4] (1.5). Here M is induced by P means that M is the amalgamated sum of P and $\mathcal{O}_X^\times$ over $\alpha^{-1}(\mathcal{O}_X^\times)$. A commutative monoid P is said to be integral if the canonical morphism $P \to P^{gr}$ is injective and it is said to be saturated further if every $a \in P^{gr}$ such that $a^n \in P$ for some $n > 0$ is contained in P. A log structure M on a noetherian scheme X is called regular if it is locally induced by a finitely generated saturated monoid and if for every $x \in X$, the following condition is satisfied. If I_x denotes the ideal generated by the image of M_x in $\mathcal{O}_{X,x} - \mathcal{O}_{X,x}^\times$, then $\mathcal{O}_{X,x}/I_x$ is a regular local ring and $\dim\ \mathcal{O}_{X,x} = \dim(\mathcal{O}_{X,x}/I_x) + \mathrm{rank}(M_x^{gp}/\mathcal{O}_{X,x}^\times)$. A scheme X with a regular log structure M is said to be log regular and is normal ([K4] Theorem (4.1)). Further, for the largest open subscheme $j : U \hookrightarrow X$ such that M is trivial on U, we have $M = j_*\mathcal{O}_U^\times \cap \mathcal{O}_X$ and $M^{gp} = j_*\mathcal{O}_U^\times$ (loc. cit. Theorem (11.6)). We call this M the log structure associated to U. If (X, U) is an RN-pair, the log structure associated to U is regular and U is the largest open. For a log structure M on X, we define the sheaf $\omega_{X,M}$ of the differential forms with logarithmic poles in M to be the quasi-coherent $\mathcal{O}_X$-module

$$\omega_{X,M} = (\Omega^1_X \oplus (\mathcal{O}_X \otimes M^{gp}))/(d\alpha(a) - \alpha(a) \otimes a;\ a \in M).$$

If X is of finite type over k and M is regular, then $\omega_{X,M}$ is locally free of rank $\dim X$. In fact, for every $x \in X$, there is an exact sequence

$$0 \to \Omega^1_{\kappa(x)/k} \to \omega_{X,M}(x) \to \kappa(x) \otimes (M_x^{gp}/\mathcal{O}_{X,x}^\times) \to 0.$$

We put $c_{X,M} = c^*(\omega_{X,M})_{\dim 0}$. When (X, U) is an RN-pair over k and M is associated to U, the sheaf $\omega_{X,M}$ is equal to ω_X hence $c_{X,U} = c_{X,M}$.

LEMMA 0. *Let X be a proper k-scheme with a regular logarithmic structure M. Let U be the largest open subscheme of X where M is trivial. Then*

$$\chi_c(U) = \deg\ c_{X,M}.$$

PROOF: Take the resolution $\pi : X' \to X$ associated to a regular proper subdivision of the fan $F(X)$ associated to (X, M) ([K4] (10.4)). Then $U' = \pi^* U$ is isomorphic to U, (X', U') is an RN-pair and $\omega_{X'} \simeq \pi^* \omega_{X,M}$. Hence Lemma is reduced to the case where (X, U) is an RN-pair. The proof is easy in this case and left to the reader.

Let (X, M) be a logarithmic scheme and σ be an automorphism of (X, M). We define the fixed part $X^\sigma \subset X$ by the cartesian diagram

$$\begin{array}{ccc} X^\sigma & \longrightarrow & X \\ \downarrow & & \downarrow \text{graph of } \sigma \\ X & \underset{\text{diagonal}}{\longrightarrow} & X \times X \end{array}$$

and I_σ to be the ideal sheaf of $\mathcal{O}_X$ corresponding to X^σ. We say that σ is admissible if the action of σ on $M_x/\mathcal{O}_{X,x}^\times$ is trivial for all $x \in X^\sigma$. If σ is admissible, we define an ideal sheaf J_σ to be that generated by I_σ and $1 - (\sigma(a)/a)$ for $a \in M_x$ at $x \in X^\sigma$. The action of σ is called clean if J_σ is an invertible ideal. Then we let D_σ denote the Cartier divisor of J_σ. Assume X is of finite type over k, the log structure M is regular and that k-automorphism σ of (X, M) is clean. Then we define the 0-cycle c_σ by

$$c_\sigma = \{c^*(\omega_{X,M}) \cdot (1 + D_\sigma)^{-1} \cdot D_\sigma\}_{\dim 0} \in CH_0(X).$$

This 0-cycle has the following property which will not be used in the sequel.

PROPOSITION. *Let X be a proper k-scheme, M be a regular log structure on X and σ be a clean k-automorphism of (X, M). Let U be the largest open subscheme of X where M is trivial. Assume one of the followings*

1). There is an open covering of X by σ-stable affine subschemes.

2). (X, U) is an RN-pair.

Then we have

$$Tr_U(\sigma) = \deg c_\sigma.$$

This will be proved in [K2] at least in the case 2). In the present version of [K2], only a k-automorphism σ of an RN-pair (X, U) is treated and the definitions given there coincide with those here. We can reduce the case 1) to case 2) but we omit the detail (cf. Proof of Lemmas 0 and 2).

We return to the proof of Theorem.

LEMMA 1. *Let (X,U) be an RN-pair over k and θ be an s-clean character of $\pi_1(U)^{ab}$ of order p. Let $\pi : Y \to X$ be the integral closure of X in the etale covering V of U of degree p trivializing θ and $G = Aut_X(Y)$. Then*

1). The log structure M associated to $V \hookrightarrow Y$ is regular.

Let $\sigma \in G$ be a non-trivial element. Then

2). The action of σ on Y is clean with respect to M and we have $D_\theta = \pi_* D_\sigma$, $\pi^* D_\theta = pD_\sigma$ and $\pi_* c_\sigma = c_\theta$.

3). The sheaf $\omega_{Y/X} = Coker(\pi^*\omega_X \to \omega_{Y,M})$ is an invertible $\mathcal{O}_{D_{Y/X}}$-module where $D_{Y/X}$ is the divisor $\pi^* D_\theta - D_\sigma$. The map $\varphi_\sigma : \omega_{Y/X}|_{D_\sigma} \to \mathcal{O}(-D_\sigma)|_{D_\sigma}$ defined by $a \cdot d\log b \mapsto a(1-\sigma(b)/b)$ is an isomorphism. The total chern class $c(\omega_{Y/X})$ is equal to $(1-D_\sigma)\cdot(1-\pi^*D_\theta)^{-1}$ and $c_{Y/X} = c_{Y,M} - \pi^* c_{X,U}$ is

$$c_{Y/X} = -\{c^*(\omega_{Y,M})\cdot(1+D_\sigma)^{-1}\cdot D_{Y/X}\}_{\dim 0} = -(p-1)\cdot c_\sigma.$$

4). Let D be the support of D_σ with reduced structure. Then the sequence

$$0 \to \pi^*\mathcal{O}_X(-D_\theta)|_D \xrightarrow{\pi^*(rsw(\theta))|_D} \pi^*\omega_X|_D \to \omega_{Y,M}|_D \xrightarrow{\varphi_\sigma|_D} \mathcal{O}(-D_\sigma)|_D \to 0$$

is exact hence locally homotope to 0.

We show that Lemma 1 implies Theorem in the case $n = p$. By Lemma 0, we have $\chi_c(V) - p\cdot\chi_c(U) = \deg\ c_{Y/X}$. Since $Tr_V(\sigma) \in \mathbf{Z}$ for $\sigma \in G$ by Theorem DL, we have $(p-1)\cdot\chi_c(U,\mathcal{F}_\theta) = Tr_{\mathbf{Q}(\zeta_p)/\mathbf{Q}}(\theta, Tr_V)_G = \chi_c(V)-\chi_c(U)$. Hence $(p-1)\cdot(\chi_c(U,\mathcal{F}_\theta)-\chi_c(U)) = \deg\ c_{Y/X}$. Therefore by 2) and 3) of Lemma 1, we have

$$\chi_c(U,\mathcal{F}_\theta) - \chi_c(U) = -\deg c_\sigma = -\deg c_\theta.$$

PROOF OF LEMMA 1: Let $D_\theta = \sum n_i C_i$ and E be the union of the divisors with $p \nmid n_i$. Let $x \in X$ be a point of X, I be the set of indices i such that $x \in C_i$ and for every $i \in I$, π_i be a section of $\mathcal{O}_X$ defining C_i at x. Then at a neighborhood of x, the character θ corresponds to an Artin-Schreier extension

$$t^p - t = \frac{u}{\prod \pi_i^{n_i}}$$

such that

1). If $x \notin E$, every n_i is divisible by p and du is everywhere non-vanishing in $\Omega^1_{C_J}$ for $C_J = \cap_{i\in J} C_i$ where $J \subset I$ runs every subset $J \subset I$ containing at least one element i with $n_i \neq 0$.

2). If $x \in E$, one of n_i is not divisible by p and u is a unit.

Outside E, the integral closure (Y, V) is an RN-pair since Y is given by $Y = X[s]/(s^p - \prod \pi_i^{(p-1)n_i/p} \cdot s - u)$ where $s = (\prod \pi_i^{n_i/p}) \cdot t$. We consider at $x \in E$. Let Y_1 be the scheme

$$Y_1 = X[s, w]/(s^p w - \Pi, s^{p-1} w - w + u)$$

where Π denotes $\prod \pi_i^{n_i}$. It is easy to check that $Y_1 \times_X U = V$ by putting $s = t^{-1}$ and $w = u + \Pi \cdot t$. The scheme Y_1 is finite over X since w satisfies $w^p - \Pi^{p-1} \cdot w - u^p = 0$ and is a unit.

Let Q_1 be the integral submonoid of $\mathbf{Q}^I \times \mathbf{Q} \cdot e_w$ generated by $\mathbf{N}^I \times \mathbf{Z} \cdot e_w$ and $e_s = \frac{1}{p}(\sum n_i e_i - e_w)$. Let M_1 be the log structure of Y_1 induced by $Q_1 \to \mathcal{O}_{Y_1}$ defined by $e_i \mapsto \pi_i, e_s \mapsto s$ and $e_w \mapsto w$. Let $Q = \{a \in Q^{gp}; \exists n > 0, n \cdot a \in Q_1\}$ be the saturation of Q_1.

CLAIM. *The scheme $Y_2 = Y_1 \otimes_{k[Q_1]} k[Q]$ with the logarithmic structure M_2 induced by $Q \to \mathcal{O}_{Y_2}$ is log regular at the inverse image of x.*

PROOF OF CLAIM: Let y be a unique point of Y_1 lying on x. Let I_y be the ideal of $\mathcal{O}_{Y_1,y}$ generated by the image of $Q_1 - \mathcal{O}^\times_{Y_1,y}$. By [K4] Proposition (12.2), it is sufficient to check that

0). $Q_1^{gp}/Q_1^\times$ is torsion free

1). $\mathcal{O}_{Y_1,y}/I_y$ is regular

2). $\dim \mathcal{O}_{Y_1,y} = \dim (\mathcal{O}_{Y_1,y}/I_y) +$ rank$(Q_1^{gp}/Q_1^\times)$.

The condition 0) is clear. By definition of Y_1, it is clear that $\mathcal{O}_{Y_1,y}/I_y \simeq \mathcal{O}_{X,x}/(\pi_i, i \in I)$ and is regular. Since Y_1 is finite over X, we see that Y_1 is locally of complete intersection of dimension $d = \dim X$. Thus Claim is proved.

Since a log regular scheme is normal, $Y_2 = Y$ at y. Further it is easy to see that V is the largest open set where M_2 is trivial. Hence the log structure M associated to V coincides with M_2 and is regular.

Next we prove 2). Outside E, we have $\sigma(s) = s + \prod_i \pi_i^{n_i/p}$. Using this, it is easy to check that σ is admissible and clean and that J_σ is generated by $\prod_i \pi_i^{n_i/p}$. We consider at E. The log scheme (Y_1, M_1) above has an admissible action of G. In fact, $\sigma(s)/s = (s+1)^{-1}$ is invertible since $s^{p-1} - 1 = -u \cdot w^{-1}$ and $\sigma(w) = w + \Pi$ is also invertible at E. Using this and the fact that s divides Π, we can also easily check that σ is clean and J_σ is generated by s. By the definition of Y_2, it is easy to see that the action of σ is admissible and clean on Y_2 and that $J_{\sigma,Y_2} \simeq \varphi_1^* J_{\sigma,Y_1}$, where $\varphi_1 : Y_2 \to Y_1$. Thus we have the cleanness of σ and the equalities $\pi^* D_\theta = pD_\sigma$ and $\pi_* D_\sigma = D_\theta$.

We prove 3) and the equality $\pi_* c_\sigma = c_\theta$. Outside E, $\omega_{Y/X}$ is generated by ds and the relation is $(\prod \pi_i^{n_i/p})^{p-1} ds = 0$. On the other hand at E, $\omega_{Y_1/X}$ is generated by $d\log s$ and $d\log w$ and the relations are $s^{p-1} \cdot d\log s = d\log w = 0$. By the definition of Y_2, we see that $\omega_{Y_2/X} \simeq \varphi_1^* \omega_{Y_1/X}$. Thus we have the assertion on the structure of $\omega_{Y/X}$. Since φ_σ is a surjection of invertible sheaves, it is an isomorphism. From these fact, it is easy to check the formula for $c(\omega_{Y/X})$ and that for $c_{Y/X}$. The equality $\pi_* c_\sigma = c_\theta$ also immediately follows from the definition, the formula for $c(\omega_{Y/X})$ and from $\pi_* D_\sigma = D_\theta$.

We prove 4). In 3) we have already shown the exactness at $\omega_{Y,M}|_D$ and at $\mathcal{O}(-D_\sigma)|_D$. By ranks and the cleanness of θ, it is sufficient to show that the composite $\pi^* \mathcal{O}_X(-D_\theta)|_D \to \omega_{Y,M}|_D$ is zero. Outside E, $rsw\ \theta$ is given by $\prod \pi_i^{n_i} \mapsto du$. Since $u \equiv s^p \bmod \prod \pi_i^{n_i}$ on Y, the composite is zero. At E, $rsw\ \theta$ is given by $\prod \pi_i^{n_i} \mapsto du - u \sum n_i d\log \pi_i$. Since $u \equiv w \bmod s$ and $d\log w = \sum n_i d\log \pi_i$ on Y_1, the composite is also zero. Thus we have completed the proof of Lemma 1 and therefore that of Theorem in the case $n = p$.

We prove Theorem in the case $n = p^e$ by induction on e. Assume $e > 1$. Let $(\tilde{Y}, V, \chi')$ be (X_1, U_1, χ_1) in Theorem and let $\pi : \tilde{Y} \to X$ and $\varphi : \tilde{Y} \to Y$. Let W be the etale covering of U of degree $n = p^e$ trivializing χ and $G = Aut_U(W) \cong \mathbb{Z}/p^e$. Since $Tr_W(\sigma) \in \mathbb{Z}$ for $\sigma \in G$ by Theorem DL, we have $p \cdot \chi_c(U, \mathcal{F}_\chi) = Tr_{\mathbb{Q}(\zeta_{p^e})/\mathbb{Q}(\zeta_{p^{e-1}})}(\chi, Tr_W)_G = \chi_c(V, \mathcal{F}_{\chi'})$. Therefore by the assumption of induction and the equality $\deg c_{Y/X} = \chi_c(V) - p \cdot \chi_c(U)$ of Lemma 0, it is sufficient to show that

$$- \pi^* c_\chi = - c_{\chi'} + \varphi^* c_{Y/X}.$$

LEMMA 2. *Let (X, U) be an RN-pair over k and let χ and θ be clean characters of $\pi_1(U)^{ab}$ of finite order. Assume that θ is of order p and s-clean and that $D_\theta = 0$ (resp. $D_\theta < D_\chi$) where $D_\chi = 0$ (resp. $D_\chi \neq 0$). Let (Y, M) be as in Lemma 1, $\varphi : \tilde{Y} \to Y$ be the resolution associated to a regular proper subdivision $F' \to F(Y)$ and π denote the map $\tilde{Y} \to X$. If $\chi' = \pi^*(\chi)$ is also clean, we have*

$$\pi^* c_\chi = c_{\chi'} - \varphi^* c_{Y/X}.$$

We see Lemma 2 implies the induction step. In fact, it is clear that $\theta = \frac{n}{p} \cdot \chi$ satisfies $D_\theta = 0$ where $D_\chi = 0$ and $D_\theta < D_\chi$ where $D_\chi \neq 0$. Therefore we will complete the proof of Theorem by showing Lemma 2.

PROOF OF LEMMA 2: We show that everything appeared in Lemma 1 has its counterpart on $\tilde{Y}$ and that it is a pull-back to $\tilde{Y}$ of that on Y. By the definition [K4] (10.4), locally on $\tilde{Y}, \varphi : \tilde{Y} \to Y$ is described as follows. Let $Y = \mathrm{Spec}A$ and $P \to A$ be a morphism of monoids such that $P \times A^{\times} \simeq M$. Then locally on $\tilde{Y}$, there is a submonoid $\tilde{P}$ of P^{gp} containing P and isomorphic to $\mathbf{N}^r \times \mathbf{Z}^s$ for some r and $s \in \mathbf{N}$ such that $\tilde{Y} \to Y$ is given by $A \to A \otimes_{k[P]} k[\tilde{P}] = B$. By loc. cit., we see that φ induces an isomorphism on V, that $(\tilde{Y}, V)$ is an RN-pair, and that the log structure induced by $\tilde{P} \to B$ coincides with that associated to V. Here we identified V and $\varphi^* V$. From this description, it is easy to show that $\varphi^* \omega_{Y,M} = \omega_{\tilde{Y}}$. We show that the action of G extends to $\tilde{Y}$. Since Y is affine over X, we may assume G acts on A above. For all $a \in M, \neq 0$, we have $\sigma(a)/a \in A^{\times}$, since the action of σ on (Y, M) is admissible. It is straight forward to check that the action of G extends to $B = A \otimes_{k[P]} k[\tilde{P}]$ hence to $\tilde{Y}$. (More canonically, $B = A \otimes_{k[M]} k[M \times_{P^{gp}} \tilde{P}]$ and hence G acts on it.) Once we have the action of G, it is quite easy to check that the action of each $\sigma \in G$ is admissible ane clean and that $J_{\sigma,\tilde{Y}} = \varphi^* J_{\sigma,Y}$. Hence every statement in Lemma 1 applies to $\tilde{Y}$.

Now by definition and 3) of Lemma 2, $\pi^* c_X - (c_{X'} - \varphi^* c_{Y/X})$ is equal to the dimension 0 part of

$$c^*(\omega_{\tilde{Y}}) \left(\frac{(1 + \pi^* D_\theta) \cdot \pi^* D_X}{(1 + D_\sigma)(1 + \pi^* D_X)} - \left(\frac{D_{X'}}{1 + D_{X'}} + \frac{D_{\tilde{Y}/X}}{1 + D_\sigma} \right) \right).$$

By an elementary calculation, this is equal to

$$c^*(\omega_{\tilde{Y}}) \left(\frac{\pi^* D_X - (D_{X'} + D_{\tilde{Y}/X})}{(1 + D_{X'})(1 + D_\sigma)} - \frac{(\pi^* D_X - \pi^* D_\theta) \cdot (\pi^* D_X - D_{X'})}{(1 + \pi^* D_X)(1 + D_{X'})(1 + D_\sigma)} \right).$$

LEMMA 3. *Let the notation be as in Lemma 2 except that we do not assume the relations between D_θ and D_X. Then*

1). We have $D_{X'} \leqslant \pi^* D_X$. If C is a component of $\tilde{Y} - V$ such that $D_{X'} < \pi^* D_X$ at C, then $\pi^* D_\theta > 0$ at C and the sequence

$$0 \to \pi^* \mathcal{O}(-D_X)|_C \xrightarrow{\pi^*(rsw_X)} \pi^* \omega_X|_C \to \omega_{\tilde{Y}}|_C$$

is exact.

2). Assume $\pi^* D_\theta < \pi^* D_X$ at a component C of $\tilde{Y} - V$. Then we have $\pi^* D_X \leqslant D_{X'} + D_{\tilde{Y}/X}$ and $D_{X'} > 0$ at C. Further if $\pi^* D_X < D_{X'} + D_{\tilde{Y}/X}$ at C, then $\pi^* D_\theta > 0$ at C and the sequence

$$0 \to \mathcal{O}(-D_{X'})|_C \xrightarrow{rsw_{X'}} \omega_{\tilde{Y}}|_C \xrightarrow{\varphi_\sigma|_C} \mathcal{O}(-D_\sigma)|_C \to 0$$

is a complex. The cohomology sheaves are zero except at $\omega_{\tilde{Y}}|_C$ and it is locally free of rank dim $X - 2$ there.

We show Lemma 3 implies Lemma 2. It is sufficient to show the following equalities.

1). $(\pi^* D_X - \pi^* D_\theta)(\pi^* D_X - D_{X'}) = 0.$

2). $\left(\frac{c^*(\omega_{\tilde{Y}})}{(1 + D_{X'})(1 + D_\sigma)}(\pi^* D_X - (D_{X'} + D_{\tilde{Y}/X}))\right)_{\dim 0} = 0.$

To prove 1), it is sufficient to show that at each component C where $D_{X'} \neq \pi^* D_X$, there is an isomorphism $\pi^* \mathcal{O}_X(-D_X)|_C \simeq \pi^* \mathcal{O}_X(-D_\theta)|_C$. By 1) of Lemma 3 and 4) of Lemma 1, they are both equal to the kernel of $\pi^* \omega_X|_C \to \omega_Y|_C$. To prove 2), it is sufficient to show

$$\left(\frac{c(\omega_{\tilde{Y}})}{(1 - D_{X'})(1 - D_\sigma)} \cdot C\right)_{\dim 0} = 0$$

for each component C where $\pi^* D_X \neq D_{X'} + D_{Y/X}$. By the assumption, we have $\pi^* D_\theta < \pi^* D_X$ at C and 2) of Lemma 3 applies. Therefore it is equal to the (dim $X - 1$)-th chern class of the cohomology sheaf there, which is of rank dim $X - 2$, and is zero.

Proof of Lemma 3: The assertions are reduced to those at the generic point of each irreducible component of $\tilde{Y} - V$. In fact, this is clear for the inequalities and, for the rest, it follows from the cleaness of χ and χ' and 4) of Lemma 1. Furthur by 4) of Lemma 1, for the assertions concerning on the sequences and the cohomology sheaves, it is sufficient to show that the sequences are complexes i.e. the composites of the maps are zero at each generic point. We will show that we may assume $\tilde{Y} = Y$ i.e. Y is finite over X.

We need a lemma on fans ([K4] Section 9) as below. Let F and G be fans satisfying (S^{fan}) (loc. cit (9.4)). We call a morphism $f : G \to F$ an isogeny if the following conditions are satisfied.

1). f is a homeomorphism of underlying spaces.

2). $M^{gp}_{F,t} \to M^{gp}_{G,f^{-1}(t)}$ is injective for all $t \in F$.

3). There is an integer $n > 0$ such that $n(M_{G,f^{-1}(t)}) \subset Image\ M_{F,t}$ for all $t \in F$.

Lemma 4. *Let $f : G \to F$ be an isogeny and $G' \to G$ be a subdivision.*

Then there is a cartesian diagram of fans

$$\begin{array}{ccc} G' & \longrightarrow & F' \\ \downarrow & & \downarrow \\ G & \longrightarrow & F. \end{array}$$

Here $F' \to F$ is a subdivision and $G' \to F'$ is an isogeny. When $G' \to G$ is proper, $F' \to F$ is also proper.

The proof of Lemma 4 is easy by using $F \to G$ such that the composite $F \to F$ is induced by the multiplication by n and is left to the reader.

Let F and G be the fans associated to X and Y respectively. By the construction of Y given in the proof of Lemma 1, there is a natural map $G \to F$ induced by $Y \to X$. Let $G' \to G$ be the proper subdivision to which $\tilde{Y} \to Y$ is associated. We apply Lemma 4 to $G' \to G \to F$. Then by [K4] Proposition (9.9) and (9.10), we have $\varphi : X_1 \to X$ associated to the proper subdivision F' of $F = F(X)$. It is clear that $\tilde{Y}$ is the integral closure of X_1 in V. Let $U_1 = \varphi^* U$ and $\chi_1 = \varphi^* \chi$. To reduce Lemma 3 to the case where $\tilde{Y} = Y$, it is sufficient to show the following. At the generic point ξ of every component C of $X_1 - U_1$, if χ_1 is not unramified, then χ_1 is clean and we have $D_{\chi_1} = \varphi^* D_\chi$ and $rsw\ \chi_1 = \varphi^* rsw\ \chi$. By the definition of X_1, there is an open neighborhood $X^{(m)}$ of ξ and a sequence

$$X^{(m)} \to \cdots \to X^{(i+1)} \to X^{(i)} \to \cdots \to X^{(0)} \subset X.$$

Here $X^{(0)}$ is an open subscheme of X and $X^{(i+1)}$ is an open subscheme of the blowing-up of $X^{(i)}$ at the closure $C^{(i)}$ of the image of ξ. Further for each i, $C^{(i)}$ is the intersection of some irreducible components of the divisor $X^{(i)} - U^{(i)}$ with normal crossing, where $U^{(i)}$ is the inverse image of U. If χ is unramified at $C^{(0)}$, there is nothing to prove. Assume χ is ramified at $C^{(0)}$. Then χ is strongly clean at the generic point of $C^{(0)}$ ([K3] Definition (7.4)). In fact, for the generic point of the intersection of some components of the divisor, the cleaness is equivalent to the strong cleaness. Hence by applying inductively [K3] Theorem (8.1), we see that χ_1 is clean, $D_{\chi_1} = \varphi^* D_\chi$ and $rsw\ \chi_1 = \varphi^* rsw\ \chi$ at ξ. Thus we have reduced Lemma 3 to the case $\tilde{Y} = Y$. Namely Lemma 3 has been reduced to

LEMMA 3'. *Let K be a complete discrete valuation field with residue field $\overline{K}$ of $ch = p$. Let χ and θ be characters of $G_K = Gal\ (K^{ab}/K)$ of* *finite order. Assume that the order of θ is p. Let L be the extension of K of degree p trivializing θ and χ' be the restriction of χ to L. Then*

1). We have $sw(\chi') \leqslant e_{L/K} \cdot sw(\chi)$. If $sw(\chi') < e_{L/K} \cdot sw(\chi)$, then L is ramified over K and the sequence

$$0 \to N_K^{sw(\chi)} \otimes_{\overline{K}} \overline{L} \xrightarrow{rsw\chi\otimes 1_{\overline{L}}} \omega_K \otimes_{\overline{K}} \overline{L} \to \omega_L$$

is exact.

2). Assume $sw(\theta) < sw(\chi)$. Then we have $e_{L/K} \cdot sw(\chi) \leqslant sw(\chi') + d_{L/K}$ and $sw(\chi') > 0$. Here $d_{L/K}$ denotes $e_{L/K} \cdot sw(\theta) - s_\sigma$ and s_σ is the integer n such that m_L^n is generated by $1 - \sigma(a)/a, a \in \mathcal{O}_L, \neq 0$. Further assume $e_{L/K} \cdot sw(\chi) < sw(\chi') + d_{L/K}$. Then L is ramified over K and the composite

$$N_L^{sw\chi'} \xrightarrow{rsw\ \chi'} \omega_L \xrightarrow{\varphi_\sigma} N_L^{s_\sigma}$$

is zero, where $\varphi_\sigma : \omega_L \to N_L^{s_\sigma}$ is defined by $a \cdot d\log b \mapsto a \cdot (1 - \sigma(b)/b)$.

REMARK: The integer $d_{L/K}$ is equal to the length of $\Omega^1_{\mathcal{O}_L}(\log \overline{L}) / \mathcal{O}_L \otimes_{\mathcal{O}_K} \Omega^1_{\mathcal{O}_K}(\log \overline{K})$ (cf. 3) of Lemma 1). It is also equal to $\delta_{L/K} - (e_{L/K} - 1)$ where $\delta_{L/K}$ is the valuation of the different of L over K.

PROOF: Let K' be the completion of $K(t)$ appeared in the definition of the Swan conductor reviewed before and $L' = L \otimes_K K'$. We put $s = sw(\chi), s' = sw(\chi'), s_0 = sw(\theta), e = e_{L/K}$ and $d = d_{L/K}$ for short.

First we show 1). By [K3] Proposition (6.3), we see $\chi_{L'}$ annihilates $U_{L'}^{es+1}$ which means $sw(\chi') \leqslant e_{L/K} \cdot sw(\chi)$. Assume $s' < es$. Then L is ramified over K by [K3] Lemma (6.2). For the exactness, it is sufficient to show that the composite $N_K^s \to \omega_L$ is zero since the kernel of $\omega_K \otimes \overline{L} \to \omega_L$ is of dimension 1. By commutativity of the diagram

$$\begin{array}{ccc} K'^\times & \xrightarrow{\{\chi_{K'},\ \}_{K'}} & Br(K') \\ \downarrow & & \downarrow \\ L'^\times & \xrightarrow{\{\chi_{L'},\ \}_{L'}} & Br(L'), \end{array}$$

we see the composite $U_{K'}^s \xrightarrow{\{\chi,\ \}_{K'}} Br(K') \to Br(L')$ is zero. This implies the composite $N_K^s \xrightarrow{rsw\ \chi} \omega_K \to \omega_L \xrightarrow{\times T} \omega_{L'} \to Br(L')$ is zero. Since $\omega_L \xrightarrow{\times T} \omega_{L'} \to Br(L')$ is injective, we have $N_K^s \to \omega_L$ is zero.

We prove 2). By [K3] Proposition (6.8), we have $d = (p-1)es_0/p$. For $n > s_0$, we have $en - d > es_0$ and, by a similar computation as in [S] Chap.V § 3, we have $U^n_{K'} = N_{L'/K'}(U^{en-d}_{L'})$. The diagram

$$\begin{array}{ccc} L'^{\times} & \xrightarrow{\{\chi_{L'},\ \}_{L'}} & Br(L') \\ {\scriptstyle N_{L'/K'}}\downarrow & & \downarrow{\scriptstyle Cor_{L'/K'}} \\ K'^{\times} & \xrightarrow{\{\chi_{K'},\ \}_{K'}} & Br(K'), \end{array}$$

is commutative. Hence by taking $s = sw\chi > s_0$ as n above, we see that $\chi_{L'}$ does not annihilate $U^{es-d}_{L'}$. Therefore we have $sw(\chi') \geqslant es - d > es_0/p \geqslant 0$.

To complete the proof, we need the trace map $Tr_{L/K} : \omega_L \to \omega_K$. If L is unramified over K, it is simply $Tr_{\overline{L}/\overline{K}} : \omega_L \simeq \omega_K \otimes_{\overline{K}} \overline{L} \to \omega_K$. We assume L is ramified over K. Then it is defined as follows. It is easily seen that the exterier differential $d : \omega_K \to \Lambda^2\omega_K$ and the Cartier operator $C : \omega_{K,d=0} \to \omega_K$ are defined in the same way as in the usual case. Namely, d is defined by $d(a \cdot d\log b) = da \wedge d\log b$, the kernel $\omega_{K,d=0}$ of d is generated by da and $a^p \cdot d\log b$ as an abelian group and C is defined by $C(da) = 0$ and $C(a^p \cdot d\log b) = a \cdot d\log b$. We define the trace map $Tr_{L/K} : \omega_L \to \omega_K$ by $Tr_{L/K}(a \cdot d\log b) = C(a^p \cdot d\log N_{L/K}b)$. By an elementary computation, we check it is well-defined. It is also easily checked that it is non zero, $\overline{K}$-linear and annihilates the image of $\omega_K \otimes_{\overline{K}} \overline{L}$.

LEMMA 5. *Let K, θ and L be as in Lemma 3'. Then*

1). The diagram below is commutative.

$$\begin{array}{ccc} \omega_L & \longrightarrow & Br(L) \\ {\scriptstyle Tr_{L/K}}\downarrow & & \downarrow{\scriptstyle Cor_{L/K}} \\ \omega_K & \longrightarrow & Br(K). \end{array}$$

Here the horizontal arrows are the canonical maps, whose definition is reviewed below.

2). Let χ and χ' be as in Lemma 3' and assume $sw(\theta) < sw(\chi)$. Then

$rsw(\chi')$ is defined and there is a commutative diagram

$$\begin{array}{ccc} N_L^{s'} & \xrightarrow{rsw\ \chi'} & \omega_L \\ \downarrow & & \downarrow Tr_{L/K} \\ N_K^{s} & \xrightarrow{rsw\ \chi} & \omega_K \end{array}$$

where the left vertical map is the one induced by $Tr_{L/K}$ if $s' = es - d$ and 0 if $s' > es - d$.

PROOF: First we give the definition of the canonical map $\omega_K \to Br(K)$ used in the definition of the refined Swan conductor. The kernel $Br(K_{nr}/K)$ of $Br(K) \to Br(K_{nr})$ is isomorphic to $H^2(\overline{K}, K_{nr}^\times/U^1)$. It is easy to see that we have an exact sequence of $Gal(\overline{K}_{sep}/\overline{K})$-module

$$0 \to (K_{nr}^\times/U^1)/p \xrightarrow{d\log} \omega_{K_{nr},d=0} \xrightarrow{1-C} \omega_{K_{nr}} \to 0$$

extending the Artin-Schreier sequence of $\overline{K}$. Hence by taking cohomology, we have an isomorphism

$$\omega_K/(1-C)\omega_{K,d=0} \simeq {}_pBr(K_{nr}/K)$$

and the canonical homomorphism $\omega_K \to Br(K)$.

We show 1). If L is unramified over K, it immediately follows from the definition of the canonical map. If L is ramified over K, it also follows from the definition and the commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (L_{nr}^\times/U^1)/p & \xrightarrow{d\log} & \omega_{L_{nr},d=0} & \xrightarrow{1-C} & \omega_{L_{nr}} & \longrightarrow & 0 \\ & & \downarrow Norm & & \downarrow Tr & & \downarrow Tr & & \\ 0 & \longrightarrow & (K_{nr}^\times/U^1)/p & \xrightarrow{d\log} & \omega_{K_{nr},d=0} & \xrightarrow{1-C} & \omega_{K_{nr}} & \longrightarrow & 0. \end{array}$$

We prove 2). Since we have already shown $sw(\chi') > 0$, the refined swan conductor $rsw(\chi')$ is defined. By definition of rsw and by 1), we have a commutative diagram

$$\begin{array}{ccccccc} U_L^{s'} & \longrightarrow & N_L^{s'} & \xrightarrow{rsw\ \chi'} & \omega_L & \longrightarrow & Br(L) \\ \downarrow N_{L/K} & & & & \downarrow Tr_{L/K} & & \downarrow Cor_{L/K} \\ U_K^{s} & \longrightarrow & N_K^{s} & \xrightarrow{rsw\ \chi} & \omega_K & \longrightarrow & Br(K). \end{array}$$

By a similar computation as in [S] Chap. V §3, if $n > s_0$, we have $N_{L/K}(U_L^{en-d+1}) \subset U_K^{n+1}, Tr_{L/K}(m_L^{en-d+1}) \subset m_K^{n+1}$ and a commutative diagram

$$\begin{array}{ccc} U_L^{en-d}/U_L^{en-d+1} & \xrightarrow{\sim} & N_L^{en-d} \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle Tr_{L/K}} \\ U_K^n/U_K^{n+1} & \xrightarrow{\sim} & N_K^n. \end{array}$$

Now it is easy to see the diagram of 2) is commutative. Thus Lemma 5 is proved.

We complete the proof of 2) of Lemma 3'. Assume $s' > es - d$. Then by 2) of Lemma 5, the composite $N_L^{s'} \xrightarrow{rsw\ \chi'} \omega_L \xrightarrow{Tr_{L/K}} \omega_K$ is zero. By [K3] Lemma (6.2), L is ramified over K. Hence $\varphi_\sigma : \omega_L \to N_L^{s_\sigma}$ is well-defined. It is clear that the composite $\omega_K \otimes \overline{L} \to \omega_L \xrightarrow{\varphi_\sigma} N_L^{s_\sigma}$ is zero and the cokernel of $\omega_K \otimes \overline{L} \to \omega_L$ is one dimensional. Therefore, if the composite $N_L^{s'} \xrightarrow{rsw\ \chi'} \omega_L \xrightarrow{\varphi_\sigma} N_L^{s_\sigma}$ was not zero, ω_L would be the sum of the image of $rsw\ \chi'$ and that of $\omega_K \otimes_{\overline{K}} \overline{L}$. But this is a contradiction since $Tr_{L/K}(\omega_L) \neq 0$. Thus we have completed the proof of Lemma 3' and hence of Theorem. Q.E.D.

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