# Characteristic cycles of constructible sheaves and microlocalization

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**Abstract**

For a constructible sheaf on a smooth scheme over a field of positive characteristic, the singular support and the characteristic cycle are defined on the cotangent bundle. We study if a construction using the microlocalization similar to that in a transcendental setting due to Kashiwara–Schapira works in the algebraic setting and prove partial positive results.

For a constructible sheaf $\mathcal{F}$ on a smooth scheme $X$ over a field $k$ of characteristic $p > 0$, the singular support $SS\mathcal{F}$ and the characteristic cycle $CC\mathcal{F}$ are defined on the cotangent bundle $T^*X$ [2], [16]. However, unlike modules over the ring of microdifferential operators in the context of $\mathcal{D}$-modules, a direct construction of an object on $T^*X$ giving rise to them is not available. In a transcendental context, the singular supports and the characteristic cycles allow a construction using the microlocalization $\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$ defined on the cotangent bundle [11]. We study if a similar construction works in the algebraic setting.

We define a closed conical subset $SS_\mu\mathcal{F} \subset T^*X$ in Definition 2.1.1 and a cohomology class $CC_\mu\mathcal{F}$ supported on $SS_\mu\mathcal{F}$ in Definition 2.1.4 using the microlocalization $\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$. We state their possible relations with $SS\mathcal{F}$ and $CC\mathcal{F}$ as Question 2.1.9. We give partial positive answers to the question. If $\dim X \leqq 1$, we prove in Proposition 2.5.1 that $CC_\mu\mathcal{F}$ equals the cycle class of $CC\mathcal{F}$. For a general $X$, we prove a conditional result in Proposition 2.5.2 that the inclusion $SS_\mu\mathcal{F} \subset SS\mathcal{F}$ implies the same equality between $CC_\mu\mathcal{F}$ and $CC\mathcal{F}$ as above.

We sketch an outline of the proof. First, we prove the equality in Corollary 2.2.5 for a tamely ramified sheaf $\mathcal{F}$ on a curve by computing $\nu\mathcal{H}om(\mathcal{F},\mathcal{F})$. Then, using a global argument based on a theorem of Katz–Gabber [13], we prove Proposition 2.5.1 for curves. For the proof, we use an index formula for $CC_\mu$ proved in Corollary 2.3.4 and compare it with the Grothendieck–Ogg–Shafarevich formula. The index formula Corollary 2.3.4 is a special case of the compatibility of $CC_\mu$ with proper push-forward Proposition 2.3.3.

To prove Proposition 2.5.2 in higher dimension, we prove that $CC_\mu\mathcal{F}$ also satisfies the characterization of $CC\mathcal{F}$ by the Milnor formula. By the assumption $SS_\mu\mathcal{F} \subset SS\mathcal{F}$, the theorem of Beilinson $\dim SS\mathcal{F} = \dim X$ implies $\dim SS_\mu\mathcal{F} \leqq \dim X$. By the weak cohomological purity, this makes it suffice to compare the coefficients in $CC_\mu\mathcal{F}$ and in $CC\mathcal{F}$ of irreducible components of dimension $\dim X$. By the compatibility with push-forward for a closed immersion, it is reduced to the case where $X$ is projective. Then, by taking a Lefschetz pencil and by the compatibility with proper push-forward Proposition

2.3.3, the Milnor formula is reduced to the equality $CC_\mu = \mathrm{cl}\, CC$ for the direct image on $\mathbf{P}^1$ proved in Proposition 2.5.1.

We briefly sketch the contents of each section. We introduce in Section 1.2 the specialization functor following Verdier [19]. The specialization is defined by applying the nearby cycles functor to the deformation to normal bundle. Since the microlocalization is defined as the Fourier transform of the specialization, we recall basic properties of Fourier transforms in Section 1.1.

In Section 2.1, we define $SS_\mu\mathcal{F}$ and $CC_\mu\mathcal{F}$ using the microlocalization $\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$ and state their possible relations with $SS\mathcal{F}$ and $CC\mathcal{F}$ as Question 2.1.9. In Sections 2.3 and 2.4, we study the compatibilities with proper push-forward and with smooth pullback respectively. We prove some positive results on the relation between $CC_\mu$ and $CC$ in Section 2.5 by reducing to a computation of $CC_\mu$ in Section 2.2 for a tamely ramified sheaf on a curve. We give an explicit and independent computation of $CC_\mu$ for a wildly ramified Artin–Schreier sheaf on a curve in Section 2.6.

When the author studied the question with Ahmed Abbes in 2004, we thought that it would not work because another approach using the blow-up taken in [1] worked well. The author recently computed the example in Section 2.6 and noticed that we had reached the negative conclusion too quickly. The same question was also considered independently by D. Cisinski, A. Khan and E. Yang.

In this article, $k$ denotes a field of characteristic $p > 0$. If $X$ is a smooth scheme over $k$, $TX$ and $T^*X$ denote the tangent and the cotangent bundle. Let $\Lambda$ be a finite local ring over $\mathbf{Z}/\ell^n\mathbf{Z}$ for a prime number $\ell \neq p$ and an integer $n \geqq 1$. For a noetherian scheme $X$ over $k$, $D^b_c(X,\Lambda) \supset D_{\mathrm{ctf}}(X,\Lambda)$ denote the derived category of bounded constructible complexes of $\Lambda$-modules on $X$ and its full-subcategory of complexes of finite tor-dimension. The letters $R$ and $L$ to denote derived functors will be omitted. For a separated morphism $f\colon X \to \operatorname{Spec} k$ of finite type, let $K_X \in D_{\mathrm{ctf}}(X,\Lambda)$ denote $f^!\Lambda$.

# Contents

# 1 Fourier transforms and the specialization

## 1.1 Complements on Fourier transforms

Let $k$ be a field of characteristic $p > 0$. Let $\Lambda$ be a finite local ring over $\mathbf{Z}/\ell^n\mathbf{Z}$ for a prime number $\ell \neq p$ and an integer $n \geqq 1$. Fix a non-trivial character $\psi\colon \mathbf{F}_p \to \Lambda^\times$. Let $\psi^\vee\colon \mathbf{F}_p \to \Lambda^\times$ be the character defined by $\psi^\vee(x) = \psi(-x) = \psi(x)^{-1}$.

**Definition 1.1.1.** Let $k$ be a field of characteristic $p > 0$ and $\psi\colon \mathbf{F}_p \to \Lambda^\times$ be a non-trivial character.

1. Let $\mathbf{A}^1 = \operatorname{Spec} k[t] \to \mathbf{A}^1 = \operatorname{Spec} k[x]$ be the Artin–Schreier covering defined by $t^p - t = x$ and identify the Galois group with $\mathbf{F}_p$ by the action of $a \in \mathbf{F}_p$ defined by $t \mapsto t + a$. Let $\mathcal{L}_\psi$ be the locally constant constructible sheaf of free $\Lambda$-modules of rank 1 on $\mathbf{A}^1 = \operatorname{Spec} k[x]$ trivialized by the Artin–Schreier covering $\mathbf{A}^1 \to \mathbf{A}^1$ corresponding to the character $\psi\colon \mathbf{F}_p \to \Lambda^\times$.

2. Let $X$ be a noetherian scheme over $k$ and let $E$ be a vector bundle over $X$. Let $E^\vee$ be the dual vector bundle and

$$\begin{tikzcd} E & E \times_X E^\vee \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mu"'] & E^\vee \\ & \mathbf{A}^1 & \end{tikzcd}$$

be the projections and the canonical pairing. We define the (naive) Fourier transform

$$F = F_\psi\colon D^b_c(E, \Lambda) \to D^b_c(E^\vee, \Lambda) \tag{1.1}$$

by $F_\psi\mathcal{F} = \mathrm{pr}_{2!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mu^*\mathcal{L}_\psi)$ and the (naive) dual Fourier transform

$$F^\vee = F_{\psi^\vee}\colon D^b_c(E^\vee, \Lambda) \to D^b_c(E, \Lambda) \tag{1.2}$$

by $F_{\psi^\vee}\mathcal{G} = \mathrm{pr}_{2!}(\mathrm{pr}_1^*\mathcal{G} \otimes \mu^*\mathcal{L}_{\psi^\vee})$.

Let $0\colon X \to E$ and $0^\vee\colon X \to E^\vee$ be the 0-sections and let $e\colon E \to X$ and $e^\vee\colon E^\vee \to X$ denote the canonical morphisms.

**Lemma 1.1.2.** *Let $e\colon E \to X$ be a vector bundle of rank $n$ over a noetherian scheme $X$ over $k$ and let $e^\vee\colon E^\vee \to X$ be the dual vector bundle.*

1. *For $\mathcal{G} \in D_{\mathrm{ctf}}(E, \Lambda)$, we have a canonical isomorphism*

$$F_\psi(e^*(-) \otimes \mathcal{G}) \to e^{\vee *}(-) \otimes F_\psi\mathcal{G} \tag{1.3}$$

*of functors $D^b_c(X, \Lambda) \to D^b_c(E^\vee, \Lambda)$.*

2. *We have canonical isomorphisms*

$$F_\psi 0_* \to e^{\vee *}, \tag{1.4}$$

$$F_\psi e^* \to 0^\vee_*(- \otimes e_!\Lambda) \tag{1.5}$$

*of functors $D^b_c(X, \Lambda) \to D^b_c(E^\vee, \Lambda)$. The isomorphism (1.5) is equivalently formulated as an isomorphism*

$$F_\psi e^! \to 0^\vee_*. \tag{1.6}$$

3. *Let* $\mathrm{adj}_e\colon 0_! \to e^!$ *be the adjoint of* $e_!0_! \to 1$ *and let* $\mathrm{res}_{e^\vee}\colon e^{\vee *} \to 0_*$ *be the adjoint of* $0^*e^* \to 1$. *Then the diagram*

$$\begin{tikzcd}
F_\psi e^! \arrow[r, "(1.6)"] & 0^\vee_* \\
F_\psi 0_! \arrow[u, "\mathrm{adj}_e"] \arrow[r, "(1.4)"] & e^{\vee *} \arrow[u, "\mathrm{res}_{e^\vee}"']
\end{tikzcd}\tag{1.7}$$

*is commutative.*

*Proof.* Let $\mathcal{F} \in D^b_c(X, \Lambda)$.

1. We have an isomorphism $F_\psi(e^*\mathcal{F} \otimes \mathcal{G}) = \mathrm{pr}_{2!}(\mathrm{pr}_2^* e^{\vee *}\mathcal{F} \otimes \mathrm{pr}_1^*\mathcal{G} \otimes \mu^*\mathcal{L}_\psi) \to e^{\vee *}\mathcal{F} \otimes F_\psi\mathcal{G}$ by the projection formula.

2. Since the restriction $\mu^*\mathcal{L}_\psi|_{X \times_X E^\vee}$ is canonically identified with $\Lambda$, we have an isomorphism $F_\psi 0_*\mathcal{F} = \mathrm{pr}_{2!}(\mathrm{pr}_1^* 0_*\mathcal{F} \otimes \mu^*\mathcal{L}_\psi) \to e^{\vee *}\mathcal{F}$.

We have an isomorphism $F_\psi\Lambda = \mathrm{pr}_{2!}\mu^*\mathcal{L}_\psi \to 0_*e_!\Lambda$ by the proper base change theorem. Since $e^{\vee *}-\otimes 0^\vee_* e_!\Lambda = 0^\vee_*(-\otimes e_!\Lambda)$, (1.3) for $\mathcal{G} = \Lambda$ implies (1.5).

By the canonical isomorphism $e^*(-)\otimes e^!\Lambda \to e^!$ of Poincaré duality, (1.5) is equivalently formulated as (1.6).

3. Let $\mathcal{F} \in D^b_c(E, \Lambda)$ and we consider the commutative diagram

$$\begin{tikzcd}
\mathrm{pr}_1^* e^!\mathcal{F} \arrow[r, "\mathrm{res}"] & e^!\mathcal{F} \boxtimes 0^\vee_!\Lambda \\
\mathrm{pr}_1^* 0_!\mathcal{F} \arrow[u, "\mathrm{adj}_e"] \arrow[r, equal] & 0_!\mathcal{F} \boxtimes \Lambda \arrow[u, "\mathrm{adj}_e \boxtimes \mathrm{res}_{e^\vee}"']
\end{tikzcd}\tag{1.8}$$

on $E \times_X E^\vee$. The support $\mathrm{supp}\, 0^\vee_!\Lambda = X \subset E^\vee$ annihilates $\mathrm{supp}\, e^!\mathcal{F} \subset E$ and $\mathrm{supp}\,\Lambda = E^\vee$ annihilates $\mathrm{supp}\, 0_!\mathcal{F} \subset X \subset E$ respectively. Hence by taking the tensor product $\otimes\mu^*\mathcal{L}$ and $\mathrm{pr}_{2!}$, we obtain the left commutative square in the diagram

$$\begin{tikzcd}
Fe^!\mathcal{F} \arrow[r] & e^{\vee *}e_!e^!\mathcal{F} \otimes 0^\vee_!\Lambda \arrow[r] & 0^\vee_!\mathcal{F} \\
F0_!\mathcal{F} \arrow[u, "\mathrm{adj}_e"] \arrow[r, "(1.4)"] & e^{\vee *}e_!0_!\mathcal{F} \arrow[u, "e^{\vee *}e_!(\mathrm{adj}_e)\otimes \mathrm{res}_{e^\vee}"'] \arrow[r] & e^{\vee *}\mathcal{F}. \arrow[u, "\mathrm{res}_{e^\vee}"']
\end{tikzcd}\tag{1.9}$$

The right horizontal arrows are induced by the canonical morphisms $e_!e^! \to 1$ and $e_!0_! \to 1$ and the compositions of the horizontal arrows are the isomorphisms (1.6) and (1.4). Since the diagram

$$\begin{tikzcd}
e_!e^! \arrow[r, "\mathrm{can}"] & 1 \\
e_!0_! \arrow[u, "e_!\mathrm{adj}_a"] \arrow[ur, "\mathrm{can}"'] &
\end{tikzcd}$$

is commutative, the right square of (1.9) is commutative and the diagram (1.9) gives (1.7). □

The following is a fundamental result of Katz and Laumon.

**Proposition 1.1.3.** 1. ([12, Théorème (2.4.4)]). *Let* $j\colon \mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1 \times \mathbf{P}^1$ *be the open immersion over* $k$ *and let* $\mathcal{L}_\psi(y/x)$ *denote the pull-back of* $\mathcal{L}_\psi$ *by the morphism* $y/x\colon \mathbf{G}_m \times$

$\mathbf{A}^1 = \operatorname{Spec} k[x^{\pm 1}, y] \to \mathbf{A}^1$. *Then, the projection* $\mathrm{pr}_1 \colon \mathbf{A}^1 \times \mathbf{P}^1 \to \mathbf{A}^1$ *is universally locally acyclic relatively to* $j_!\mathcal{L}_\psi(y/x)$.

2. ([12, (a) in the proof of Théorème (2.4.1)]). *Let* $X$ *be a noetherian scheme and consider the cartesian diagram*

$$\begin{tikzcd}
\mathbf{A}^2_X \arrow[r, "j_2"] \arrow[d, "\mathrm{pr}_1"'] & \mathbf{P}^1_X \times_X \mathbf{A}^1_X \arrow[d, "\overline{\mathrm{pr}}_1"] \\
\mathbf{A}^1_X \arrow[r, "j"] & \mathbf{P}^1_X .
\end{tikzcd}$$

*Then, the morphism* $j_{2!}(\mathrm{pr}_1^*(-) \otimes \mu^*\mathcal{L}) \to j_{2*}(\mathrm{pr}_1^*(-) \otimes \mu^*\mathcal{L})$ *is an isomorphism.*

*Proof.* 1. First, we show the universal local acyclicity on $\mathbf{A}^2 = \operatorname{Spec} k[x, y] \subset \mathbf{A}^1 \times \mathbf{P}^1$. It suffices to show the universal local acyclicity after the base change by the purely inseparable morphism $\mathbf{A}^1 \to \mathbf{A}^1$ sending $x$ to $x^p$. The normalization of the covering of $\mathbf{A}^2$ defined by $t^p - t = y/x^p$ is $\mathbf{A}^2 = \operatorname{Spec} k[x, s] \to \mathbf{A}^2 = \operatorname{Spec} k[x, y]$ given by $s^p - x^{p-1}s = y$ for $s = xt$ and the projection $\mathbf{A}^2 = \operatorname{Spec} k[x, s] \to \mathbf{A}^1 = \operatorname{Spec} k[x]$ is smooth. Hence the assertion follows from the local acyclicity of smooth morphisms.

Next, we show the universal local acyclicity on $\mathbf{A}^1 \times (\mathbf{P}^1 - \{0\})$. Changing the coordinates and notation, it suffices to show that the projection $\mathbf{A}^2 = \operatorname{Spec} k[x, y] \to \mathbf{A}^1 = \operatorname{Spec} k[x]$ is universally locally acyclic with respect to $j_!\mathcal{L}(1/xy)$ for the open immersion $j \colon \mathbf{G}_m^2 \to \mathbf{A}^2$. Let $g \colon \mathbf{A}^1 \times \mathbf{G}_m = \operatorname{Spec} k[x, u^{\pm 1}] \to \mathbf{A}^2 = \operatorname{Spec} k[x, u]$ be the open immersion. By the generic universal local acyclicity [5, Corollaire 2.16], the projection $p_1 \colon \mathbf{A}^2 = \operatorname{Spec} k[x, u] \to \mathbf{A}^1 = \operatorname{Spec} k[x]$ is universally local acyclicity relatively to $g_!\mathcal{L}_\psi(1/u)$. The blow-up $q \colon P \to \mathbf{A}^2$ at the origin is the union of two affine planes $A = \mathbf{A}^2 = \operatorname{Spec} k[x, y]$ and $B = \mathbf{A}^2 = \operatorname{Spec} k[u, v]$ where $u = xy$ and $x = uv$. Since $q$ is an isomorphism outside the exceptional divisor $E$, the composition $p_1 \circ q \colon P \to \mathbf{A}^1$ is locally acyclicity on the complement $P - E$. Further, by the first step above, the composition $p_1 \circ q \colon P \to \mathbf{A}^1$ is locally acyclicity relatively to $q^*g_!\mathcal{L}_\psi(1/u)$ except possibly at the origins of $A$ and $B$. Since $q \colon P \to \mathbf{A}^2$ is proper, by proper base change theorem, the composition $p_1 \circ q \colon P \to \mathbf{A}^1$ is everywhere locally acyclicity relatively to $q^*g_!\mathcal{L}_\psi(1/u)$. Since the restriction of $q^*g_!\mathcal{L}_\psi(1/u)$ on $A$ is the same as $j_!\mathcal{L}_\psi(1/xy)$, the assertion follows.

2. Let $\mathcal{F} \in D_c^b(\mathbf{A}^1_X, \Lambda)$. By 1, the projection $\overline{\mathrm{pr}}_1 \colon \mathbf{P}^1_X \times_X \mathbf{A}^1_X \to \mathbf{P}^1_X$ is locally acyclic relatively to $j_{2!}\mu^*\mathcal{L}$. Hence by [9, Proposition 2.10], the morphism $j_{2!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mu^*\mathcal{L}) = \overline{\mathrm{pr}}_1^* j_*\mathcal{F} \otimes j_{2!}\mu^*\mathcal{L} \to j_{2*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mu^*\mathcal{L})$ is an isomorphism. □

Proposition 1.1.3.2 implies that the canonical morphism

$$F_\psi\mathcal{F} = \mathrm{pr}_{2!}(\mathrm{pr}_1^*\mathcal{F} \otimes \mu^*\mathcal{L}_\psi) \to \mathrm{pr}_{2*}(\mathrm{pr}_1^*\mathcal{F} \otimes \mu^*\mathcal{L}_\psi) \tag{1.10}$$

is an isomorphism [12, Théorème (2.4.1)].

**Corollary 1.1.4.** *Let* $E$ *be a line bundle on* $X$ *and* $E^\vee$ *be the dual. Let* $g \colon E^\times = E - X \to E$ *and* $g^\vee \colon E^{\vee\times} = E^\vee - X \to E^\vee$ *be the open immersions of the complements of the* 0*-sections. Let* $0 \colon X \to E$ *be the* 0*-section and define a morphism* $0_*\Lambda_X \to g_!\Lambda_{E^\times}[1]$ *by the exact sequence* $0 \to g_!\Lambda_{E^\times} \to \Lambda_E \to 0_*\Lambda_X \to 0$. *Then, there exists a unique isomorphism*

$$F_\psi g_!\Lambda_{E^\times}[1] \to g^\vee_*\Lambda_{E^{\vee\times}} \tag{1.11}$$

*fitting in the commutative diagram*

$$\begin{tikzcd}
F_\psi 0_*\Lambda_X \arrow[r, "(1.4)"] \arrow[d] & \Lambda_{E^\vee} \arrow[d] \\
F_\psi g_!\Lambda_{E^\times}[1] \arrow[r, "(1.11)"] & g^\vee_*\Lambda_{E^{\vee\times}}.
\end{tikzcd} \tag{1.12}$$

*Proof.* By the isomorphism $F_\psi\Lambda_E \to 0^\vee_*\Lambda_X(-1)[-2]$, the exact sequence $0\to g_!\Lambda_{E^\times}\to \Lambda_E\to\Lambda_X\to 0$ induces an isomorphism $(F_\psi 0_*\Lambda_X)|_{E^{\vee\times}} = \Lambda_{E^{\vee\times}} \to F_\psi g_!\Lambda_{E^\times}[1]|_{E^{\vee\times}}$. By adjunction, there exists a unique morphism $F_\psi g_!\Lambda_{E^\times}[1]\to g^\vee_*\Lambda_{E^{\vee\times}}$ making (1.12) commutative. Its restriction on $E^{\vee\times}$ is an isomorphism.

It suffices to show that the morphism $Fg_!\Lambda \to g^\vee_* g^{\vee *}Fg_!\Lambda$ is an isomorphism. We consider the commutative diagram

$$\begin{tikzcd}
Fg_!\Lambda \arrow[r] \arrow[d] & \mathrm{pr}_{2*}(\mathrm{pr}_1^*g_!\Lambda\otimes\mu^*\mathcal{L}) \arrow[d] \\
g^\vee_*g^{\vee *}Fg_!\Lambda \arrow[r] & g^\vee_*\mathrm{pr}_{2*}((\mathrm{pr}_1^*g_!\Lambda\otimes\mu^*\mathcal{L})|_{E\times_X E^{\vee\times}}).
\end{tikzcd}$$

By the isomorphism (1.10), the horizontal arrows are isomorphisms. It suffices to show that the right vertical arrow is an isomorphism. Since $e_*g_!\Lambda=0$, the morphism $\mathrm{pr}_{2*}(\mathrm{pr}_1^*g_!\Lambda\otimes\mu^*\mathcal{L})\to\mathrm{pr}_{2*}((g_!\Lambda\boxtimes g^\vee_*\Lambda)\otimes\mu^*\mathcal{L})$ is an isomorphism. Let $g^\vee_2\colon E\times_X E^{\vee\times}\to E\times_X E^\vee$ be the open immersion. Then $\mathrm{pr}_{2*}((g_!\Lambda\boxtimes g^\vee_*\Lambda)\otimes\mu^*\mathcal{L})\to \mathrm{pr}_{2*}g^\vee_{2*}((\mathrm{pr}_1^*g_!\Lambda\otimes\mu^*\mathcal{L})|_{E\times_X E^{\vee\times}})$ is an isomorphism and the assertion follows. □

**Proposition 1.1.5** ([15, Théorème (1.2.2.1)])**.** *Let $e\colon E\to X$ be a vector bundle over a noetherian scheme $X$ over $k$ and let $e^\vee\colon E^\vee\to X$ be the dual vector bundle. There exists a canonical isomorphism*

$$i\colon F_\psi F_{\psi^\vee}\to -\otimes e^{\vee *}e_!\Lambda \tag{1.13}$$

*of functors $D^b_c(E^\vee,\Lambda)\to D^b_c(E^\vee,\Lambda)$.*

*Proof.* Let $\mathcal{L}_\psi(x(y^\vee-x^\vee))$ denote the pull-back of $\mathcal{L}_\psi$ by the morphism $E^\vee\times_X E\times_X E^\vee\to \mathbf{A}^1$ sending $(x^\vee,x,y^\vee)$ to $\mu(x,y^\vee-x^\vee)$. We consider the cartesian diagram

$$\begin{tikzcd}
E & E\times_X E^\vee \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & E^\vee \\
E^\vee\times_X E \arrow[u, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & E^\vee\times_X E\times_X E^\vee \arrow[l, "\mathrm{pr}_{12}"'] \arrow[r, "\mathrm{pr}_{13}"] \arrow[u, "\mathrm{pr}_{23}"'] \arrow[ur, "\mathrm{pr}_3"] \arrow[dl, "\mathrm{pr}_1"'] & E^\vee\times_X E^\vee \arrow[u, "\mathrm{pr}_2"'] \\
E^\vee & E^\vee\times_X E \arrow[u, "\delta\times 1"] \arrow[r, "\mathrm{pr}_1"] & E^\vee. \arrow[u, "\delta"']
\end{tikzcd} \tag{1.14}$$

By the proper base change theorem applied to the lower right square, the pull-back by the lower middle vertical arrow induces isomorphisms

$$\mathrm{pr}_{13!}\mathcal{L}_\psi(x(y^\vee-x^\vee))\to\mathrm{pr}_{13!}(\delta\times 1)_*(\delta\times 1)^*\mathcal{L}_\psi(x(y^\vee-x^\vee))\to\delta_*\mathrm{pr}_{1!}\Lambda\to\delta_*e^{\vee *}e_!\Lambda. \tag{1.15}$$

By proper base change theorem, the projection formula and by (1.15), we obtain isomorphisms

$$\begin{aligned} F_\psi F_{\psi^\vee} \to \mathrm{pr}_{3!}(\mathrm{pr}_1^* - \otimes \mathcal{L}_\psi(x(y^\vee - x^\vee))) &\to \mathrm{pr}_{2!}(\mathrm{pr}_1^* - \otimes \mathrm{pr}_{13!}\mathcal{L}_\psi(x(y^\vee - x^\vee))) \\ &\to \mathrm{pr}_{2!}(\delta_*(- \otimes e^{\vee *} e_! \Lambda)) \to - \otimes e^{\vee *} e_! \Lambda. \end{aligned}$$

□

In the following, we fix $\psi$ and drop it from the notation. The isomorphism (1.13) is equivalently reformulated as an isomorphism

$$i\colon F(F^\vee(-) \otimes e^! \Lambda) \to 1. \tag{1.16}$$

**Proposition 1.1.6.** *Let $e\colon E \to X$ be a vector bundle of rank $n$ over a noetherian scheme $X$ over $k$ and let $e^\vee\colon E^\vee \to X$ be the dual vector bundle. The diagram*

$$\begin{tikzcd} F(F^\vee e^{\vee !}(-) \otimes e^! \Lambda) \arrow[r, "i"] \arrow[d, "(1.5)"'] & e^{\vee !} \\ F(0_*(-) \otimes e^! \Lambda) \arrow[r, "(1.4)"] & e^{\vee *}(-) \otimes e^{\vee *} 0^* e^! \Lambda \arrow[u, "(-1)^n \mathrm{can}"'] \end{tikzcd} \tag{1.17}$$

*of functors $D_c^b(X, \Lambda) \to D_c^b(E, \Lambda)$ is commutative. The upper horizontal arrow is defined by* (1.16) *and the right vertical arrow is $(-1)^n$-times the morphism induced by the canonical isomorphism $e^{\vee *} 0^* e^! \Lambda \to e^{\vee !} \Lambda$.*

*Proof.* We consider closed immersions

$$E^\vee \times_X E^\vee \xrightarrow{(1,0,1)} E^\vee \times_X E \times_X E^\vee \xleftarrow{\delta \times 1} E^\vee \times_X E \tag{1.18}$$

defined by $(\delta \times 1)(x^\vee, x) = (x^\vee, x, x^\vee)$. Let $p_1, p_3\colon E^\vee \times_X E^\vee \to E^\vee$ and $q\colon E^\vee \times_X E \to E^\vee$ be the restrictions of the projections $\mathrm{pr}_1, \mathrm{pr}_3\colon E^\vee \times_X E \times_X E^\vee \to E^\vee$. By the proof of Proposition 1.1.5, the upper horizontal morphism $i\colon F(F^\vee e^{\vee !} \mathcal{F} \otimes e^! \Lambda) \to e^{\vee !} \mathcal{F}$ is induced by the composition via upper right in the diagram

$$\begin{tikzcd} \mathrm{pr}_{3!}(\mathrm{pr}_1^* e^{\vee !} \mathcal{F} \otimes \mathcal{L}((y^\vee - x^\vee)x) \otimes \mathrm{pr}_2^* e^! \Lambda) \arrow[r] \arrow[d] & q_!(e^{\vee !} \mathcal{F} \boxtimes e^! \Lambda) \arrow[d] \\ p_{3!}(\mathrm{pr}_1^* e^{\vee !} \mathcal{F} \otimes (e^\vee \times e^\vee)^* 0^* e^! \Lambda) \arrow[r] & e^{\vee !} \mathcal{F}. \end{tikzcd} \tag{1.19}$$

The upper horizontal arrow and the left vertical arrows are induced by the pull-back by the immersions in (1.18) respectively. The left vertical arrow is induced by the isomorphism $\mathrm{Tr}_{E/X}\colon e_! e^! \to 1$. The lower horizontal arrow is induced by the isomorphism $\mathrm{Tr}_{E^\vee/X}\colon e^\vee_! e^{\vee !} \to 1$ for the first factor $E^\vee$ and the canonical isomorphism $e^{\vee *} 0^* e^! \Lambda \to e^{\vee !} \Lambda$. The arrows in (1.17) via lower left is induced by the corresponding arrows in (1.19). Hence it suffices to show that (1.19) is $(-1)^n$-commutative. Similarly as Lemma 1.1.2.1, we may assume $\mathcal{F} = \Lambda$.

After changing the coordinate $(x^\vee, x, y^\vee)$ by $(y^\vee - x^\vee, x, y^\vee)$, the immersions in (1.18) are the base change by $E^\vee \to X$ of the immersions $E^\vee \to E \times_X E^\vee \leftarrow E$ and $\mathcal{L}((y^\vee - x^\vee)x)$ is the pull-back of $\mathcal{L}(z^\vee x)$. Hence, it suffices to show Lemma 1.1.7.1 below. □

**Lemma 1.1.7.** 1. *Let* $e\colon E \to X$ *be a vector bundle of rank* $n$ *and* $e^\vee\colon E^\vee \to X$ *be the dual. Then, the diagram*

(1.20)

$$\begin{tikzcd}
(e\times e^\vee)_!\mu^*\mathcal{L}_\psi(n)[2n] \arrow[r, equal] \arrow[d, equal] & e^\vee_! F_\psi e^! \Lambda \arrow[r, "(1.5)"] & e^\vee_! 0^\vee_* \Lambda \arrow[d] \\
e_! F^\vee_\psi e^{\vee !}\Lambda \arrow[r, "(1.5)"] & e_! 0_* \Lambda \arrow[r] & \Lambda
\end{tikzcd}$$

*of isomorphisms of sheaves on* $X$ *is* $(-1)^n$*-commutative.*

2. *Let* $\mu\colon \mathbf{A}^2 = \operatorname{Spec} k[x,y] \to \mathbf{A}^1 = \operatorname{Spec} k[t]$ *be the morphism defined by* $t = xy$ *and let* $\Phi_0\Lambda$ *denote the vanishing cycles complex at* $0 \in \mathbf{A}^2$ *with respect to* $\mu$*. Then, we have an isomorphism* $R^1\mu_!\Lambda \to \Lambda$*, an exact sequence* $0 \to \Phi^1_0\Lambda \to R^2\mu_!\Lambda \to \Lambda(-1) \to 0$ *and* $R^q\mu_!\Lambda = 0$ *for* $q \neq 1, 2$*.*

3. *Let* $k$ *be an algebraically closed field and let* $\sigma$ *be the involution of* $\mathbf{A}^2 = \operatorname{Spec} k[x,y]$ *switching* $x$ *and* $y$*. Then* $\sigma$ *acts on* $\mathrm{H}^2_c(\mathbf{A}^2, \mathcal{L}_\psi(xy))$ *by* $-1$*.*

*Proof.* 1. Since the assertion is local on $X$, we may assume that $E = \mathbf{A}^n_X$. By the Künneth formula, we may assume that $n = 1$. Further, we may assume that $X = \operatorname{Spec} k$ for an algebraically closed field $k$. Hence, the assertion is reduced to 3.

2. We construct a relative compactification $\overline{\mu}\colon P \to \mathbf{A}^1$. The second projection $\mathbf{P}^1 \times \mathbf{A}^1 \to \mathbf{A}^1 = \operatorname{Spec} k[t]$ is proper. Let $P \to \mathbf{P}^1 \times \mathbf{A}^1$ be the blow-up at the origin of $\mathbf{A}^2 = \operatorname{Spec} k[y,t] \subset \mathbf{P}^1 \times \mathbf{A}^1$. Let $L_0$ be the proper transform of $(y = 0) \subset \mathbf{A}^2$ and $L_\infty$ be the inverse image of $(\mathbf{P}^1 \times \mathbf{A}^1) - \mathbf{A}^2$. Then, the complement $P - (L_0 \cup L_\infty)$ is identified with $\mathbf{A}^2 = \operatorname{Spec} k[x,y]$ by $xy = t$ and hence $P \to \mathbf{P}^1 \times \mathbf{A}^1 \to \mathbf{A}^1 = \operatorname{Spec} k[t]$ is a relative compactification of $\mu\colon \mathbf{A}^2 = \operatorname{Spec} k[x,y] \to \mathbf{A}^1$.

Let $j\colon \mathbf{A}^2 = \operatorname{Spec} k[x,y] \to P$ be the open immersion. The morphism $\overline{\mu}\colon P \to \mathbf{A}^1$ is smooth except at $0 \in \mathbf{A}^2$ and $L_0$ and $L_\infty$ are sections of $\overline{\mu}\colon P \to \mathbf{A}^1$. Hence the morphism $P \to \mathbf{A}^1$ is locally acyclic relatively to $j_!\Lambda$ except at the origin $0 \in \mathbf{A}^2$. Let $\bar\eta_0$ be the generic geometric point of the strict henselization at $0 \in \mathbf{A}^1$ and let $\Phi_0\Lambda$ denote the vanishing cycles complex at 0 with respect to $\mu$. By the Picard–Lefschetz formula, the complex $\Phi_0\Lambda$ is acyclic except at degree 1 and the $\Lambda$-module $\Phi^1_0\Lambda$ is free of rank 1. Let $+ = \mathbf{A}^2 \times_{\mathbf{A}^1} 0$ and $\mathbf{G}_{m,\bar\eta_0} = \mathbf{A}^2 \times_{\mathbf{A}^1} \bar\eta_0$ be the fibers of $\mu\colon \mathbf{A}^2 \to \mathbf{A}^1$. Then, the assertion follows from the distinguished triangle $\Gamma_c(+, \Lambda) \to \Gamma_c(\mathbf{G}_{m,\bar\eta_0}, \Lambda) \to \Phi_0\Lambda \to$.

3. By the projection formula, we have a canonical isomorphism $\mu_!\mathcal{L}_\psi(xy) \to \mathcal{L}_\psi \otimes \mu_!\Lambda$. Hence by 2. and $\Gamma_c(\mathbf{A}^1, \mathcal{L}_\psi) = 0$, we have a canonical isomorphism $\Phi^1_0\Lambda \to \mathrm{H}^0_c(\mathbf{A}^1, \mathcal{L}_\psi(xy) \otimes R^2\mu_!\Lambda) \to \mathrm{H}^2_c(\mathbf{A}^1, \mathcal{L}_\psi(xy))$. Since $\sigma$ acts on $\Phi^1_0\Lambda$ by $-1$ by the Picard–Lefschetz formula, the assertion follows. □

We study relations of Fourier transforms with pull-back and push-forward by linear morphisms of vector bundles.

**Proposition 1.1.8** ([15, Théorème (1.2.2.4)])**.** *Let* $a\colon E \to E'$ *be a linear morphism of vector bundles on a noetherian scheme* $X$ *over* $k$ *and* $a^\vee\colon E'^\vee \to E^\vee$ *be the dual. Let* $F$ *and* $F'$ *denote the Fourier transforms for* $E$ *and* $E'$ *respectively. We have a canonical isomorphism*

$$d\colon F'a_! \to a^{\vee *}F \tag{1.21}$$

*of functors* $D^b_c(E, \Lambda) \to D^b_c(E'^\vee, \Lambda)$*.*

The morphism (1.4) is the case where $a\colon X \to E$ is the 0-section.

*Proof.* By applying the proper base change theorem to the cartesian squares in the commutative diagram

$$\begin{tikzcd}
& E \times_X E^\vee \arrow[r] \arrow[dl] & E^\vee \\
E \arrow[d, "a"'] & E \times_X E'^\vee \arrow[l] \arrow[r] \arrow[u] \arrow[d] & E'^\vee \arrow[u, "a^\vee"'] \\
E' & E' \times_X E'^\vee \arrow[l] \arrow[ur] &
\end{tikzcd}$$

and by the projection formula, we obtain isomorphisms

$$\begin{aligned}
F'a_! &= \mathrm{pr}'_{2!}(\mathrm{pr}'^*_1(a_!-) \otimes \mathcal{L}_\psi(x'x'^\vee)) \to \mathrm{pr}'_{2!}(a \times 1_{E'^\vee})_!(\mathrm{pr}'^*_1(-) \otimes \mathcal{L}_\psi(a(x)x'^\vee)) \\
&\to \mathrm{pr}'_{2!}(1_E \times a^\vee)^*(\mathrm{pr}^*_1(-) \otimes \mathcal{L}_\psi(xx^\vee)) \to a^{\vee *}\mathrm{pr}_{2!}(\mathrm{pr}^*_1(-) \otimes \mathcal{L}_\psi(xx^\vee)) = a^{\vee *}F.
\end{aligned}$$

□

**Corollary 1.1.9.** *Let the notation be as in Proposition* 1.1.8 *and let* $F^\vee$ *and* $F'^\vee$ *denote the inverse Fourier transforms for* $E$ *and* $E'$ *respectively.*

1. *There exists a unique isomorphism*

$$d^\vee\colon F(a^*(-) \otimes e^!\Lambda) \to a^\vee_! F'(- \otimes e'^!\Lambda) \tag{1.22}$$

*of functors* $D^b_c(E', \Lambda) \to D^b_c(E^\vee, \Lambda)$ *such that the diagram*

$$\begin{tikzcd}
F(a^*F'^\vee(-) \otimes e^!\Lambda) \arrow[r, "d^\vee"] & a^\vee_! F'(F'^\vee(-) \otimes e'^!\Lambda) \arrow[d, "i_{E'}"] \\
F(F^\vee a^\vee_!(-) \otimes e^!\Lambda) \arrow[u, "d"] \arrow[r, "i_E"] & a^\vee_!
\end{tikzcd} \tag{1.23}$$

*is commutative. The left vertical arrow is an isomorphism induced by the isomorphism* $d$ *for* $a^\vee\colon E'^\vee \to E^\vee$.

*If* $a\colon E \to E'$ *is smooth,* (1.22) *is equivalently reformulated as an isomorphism*

$$d^\vee\colon Fa^! \to a^\vee_* F'. \tag{1.24}$$

2. *The isomorphism* $Fe^! \to 0^\vee_!$ (1.6) *is the special case of* $d^\vee$ (1.24) *where* $a\colon E \to X$ *is the structure morphism.*

*Proof.* 1. The functor $F'^\vee\colon D^b_c(E'^\vee, \Lambda) \to D^b_c(E', \Lambda)$ is an equivalence of categories by Proposition 1.1.5. Hence the commutative diagram (1.23) uniquely defines a functor $d^\vee$.

2. By the characterization of $d^\vee$, it suffices to show that $i_E\colon F(F^\vee 0^\vee_!(-) \otimes e^!\Lambda) \to 0^\vee_!$ equals the composition of $d\colon F(F^\vee 0^\vee_!(-) \otimes e^!\Lambda) \to F(e^*(-) \otimes e^!\Lambda)$ and $F(e^*(-) \otimes e^!\Lambda) \to Fe^! \to 0^\vee_!$ (1.6). By the proof of Proposition 1.1.6, $i_E\colon F(F^\vee 0^\vee_!(-) \otimes e^!\Lambda) \to 0^\vee_!$ equals the composition of the isomorphism $F(F^\vee 0^\vee_!(-) \otimes e^!\Lambda) \to \mathrm{pr}_{1!}(0^\vee_!(-) \boxtimes e^!\Lambda)$ induced by the pull-back by $\delta \times 1\colon E^\vee \times E \to E^\vee \times E \times E^\vee$ and the isomorphism $\mathrm{pr}_{1!}(0^\vee_!(-) \boxtimes e^!\Lambda) \to 0^\vee_!(-) \otimes e_! e^!\Lambda \to 0^\vee_!$ induced by $\mathrm{Tr}_{E/X}$. Since $F(e^*(-) \otimes e^!\Lambda) \to Fe^! \to 0^\vee_!$ (1.6) is also induced by $\mathrm{Tr}_{E/X}$, we have the equality as required. □

We show that the Fourier transforms of adjunction morphisms are identified with the restriction morphisms

**Lemma 1.1.10.** *Let $a\colon E \to E'$ be a linear morphism of vector bundles on a noetherian scheme $X$ over $k$ and $a^\vee\colon E'^\vee \to E^\vee$ be the dual. Let $F$ and $F'$ denote the Fourier transforms for $E$ and $E'$ respectively. Let $\mathrm{adj}_a\colon a_!e^! \to e'^!$ be the adjoint of $e^! \to a^!e'^!$ and $\mathrm{res}_{a^\vee}\colon a^{\vee *}0^\vee_* \to 0'^\vee_*$ be the adjoint of $0^\vee_* \to a^\vee_*0'^\vee_*$. Then the diagram*

(1.25)
$$\begin{tikzcd}
F'a_!e^! \arrow[r, "d"] \arrow[d, "\mathrm{adj}_a"'] & a^{\vee *}Fe^! \arrow[r, "(1.6)_E"] & a^{\vee *}0^\vee_* \arrow[dl, "\mathrm{res}_{a^\vee}"] \\
F'e'^! \arrow[r, "(1.6)_{E'}"] & 0'^\vee_* &
\end{tikzcd}$$

*is commutative.*

*Proof.* Let $\mathcal{F} \in D^b_c(X, \Lambda)$ and consider the commutative diagram

(1.26)
$$\begin{tikzcd}
\mathrm{pr}'^*_1 a_!e^!\mathcal{F} \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{adj}_a"'] & a_!e^!\mathcal{F} \boxtimes a^{\vee *}0^\vee_*\Lambda \arrow[d, "\mathrm{adj}_a \boxtimes \mathrm{res}"] \\
\mathrm{pr}'^*_1 e'^!\mathcal{F} \arrow[r, "\mathrm{res}"] & e'^!\mathcal{F} \boxtimes 0'^\vee_*\Lambda
\end{tikzcd}$$

on $E' \times_X E'^\vee$. The horizontal arrows are induced by the restriction morphisms $\Lambda \to a^{\vee *}0^\vee_*\Lambda$ and $\Lambda \to 0'^\vee_*\Lambda$. On the right column, the supports of $a^{\vee *}0^\vee_*\Lambda$ and $0'^\vee_*\Lambda$ annihilate the supports of $a_!e^!\mathcal{F}$ and $e'^!\mathcal{F}$ respectively. The canonical isomorphisms $e'_!a_!e^!\Lambda \to \Lambda$ and $e'_!e'^!\Lambda \to \Lambda$ are compatible with the $\mathrm{adj}_a$. Hence by taking the tensor product $\otimes \mu'^*\mathcal{L}$ and $\mathrm{pr}'_{2!} = e'_! \boxtimes 1$, we obtain (1.25) from (1.26). □

We show an analogue of Lemma 1.1.10 for $d^\vee$ and determine the sign appearing.

**Proposition 1.1.11.** *Let $a\colon E \to E'$ be a linear morphism of vector bundles on a noetherian scheme $X$ over $k$ and $a^\vee\colon E'^\vee \to E^\vee$ be the dual. Let $n = \operatorname{rank} E$, $n' = \operatorname{rank} E'$ and $r = n - n'$.*

1. *Let $\mathrm{res}_a\colon a^*0'_* \to 0_*$ be the adjoint of the isomorphism $0'_* \to a_*0_*$ and $\mathrm{adj}_{a^\vee}\colon a^\vee_! e'^{\vee !} \to e^{\vee !}$ be the adjoint of $e'^{\vee !} \to a^{\vee !}e^{\vee !}$. Then the diagram*

(1.27)
$$\begin{tikzcd}
F(a^*0'_*(-) \otimes e^!\Lambda) \arrow[r, "d^\vee"] \arrow[d, "\mathrm{res}_a"'] & a^\vee_! F'(0'_*(-) \otimes e'^!\Lambda) \arrow[r, "(1.4)_{E'}"] & a^\vee_! e'^{\vee !} \arrow[dl, "(-1)^r \mathrm{adj}_{a^\vee}"] \\
F(0_*(-) \otimes e^!\Lambda) \arrow[r, "(1.4)_E"] & e^{\vee !} &
\end{tikzcd}$$

*is commutative.*

2. *Assume that $a$ is smooth. Let $\mathrm{adj}_a\colon 0_! \to a^!0'_!$ be the adjoint of $a_!0_! \to 0'_!$ and $\mathrm{res}_{a^\vee}\colon e^{\vee *} \to a^\vee_* e'^{\vee *}$ be the adjoint of $a^{\vee *}e^{\vee *} \to e'^{\vee *}$. Then the diagram*

(1.28)
$$\begin{tikzcd}
Fa^!0'_! \arrow[r, "d^\vee"] & a^\vee_* F'0'_! \arrow[r, "(1.4)_{E'}"] & a^\vee_* e'^{\vee *} \\
F0_! \arrow[u, "\mathrm{adj}_a"] \arrow[r, "(1.4)_E"] & e^{\vee *} \arrow[ur, "\mathrm{res}_{a^\vee}"'] &
\end{tikzcd}$$

*is commutative.*

*Proof.* 1. We consider the diagram

$$\begin{tikzcd}
F(a^*0'_*(-)\otimes e^!\Lambda) \arrow[r, "d^\vee"] & a^\vee_! F'(0'_*(-)\otimes e'^!\Lambda) \\
F(a^*F'^\vee e'^{\vee!}(-)\otimes e^!\Lambda) \arrow[u, "(1.6)_{E'^\vee}"] \arrow[r, "d^\vee"] \arrow[d, "d"'] & a^\vee_! F'(F'^\vee e'^{\vee!}(-)\otimes e'^!\Lambda) \arrow[u, "(1.6)_{E'^\vee}"'] \arrow[d, "i_{E'}"] \\
F(F^\vee a^\vee_! e'^{\vee!}(-)\otimes e^!\Lambda) \arrow[r, "i_E"] \arrow[d, "\mathrm{adj}_{a^\vee}"'] & a^\vee_! e'^{\vee!} \arrow[d, "\mathrm{adj}_{a^\vee}"] \\
F(F^\vee e^{\vee!}(-)\otimes e^!\Lambda) \arrow[r, "i_E"] \arrow[d, "(1.6)_{E^\vee}"'] & e^{\vee!} \arrow[d, "(-1)^n"] \\
F(0_*(-)\otimes e^!\Lambda) \arrow[r, "(1.4)_E"] & e^{\vee!}.
\end{tikzcd}$$

By Lemma 1.1.10, the composition of the left column is the left vertical arrow $\mathrm{res}_a$ in (1.27). The top vertical arrows are induced by the isomorphism $F'^\vee e'^{\vee!} \to 0'_*$ $(1.6)_{E'}$ and the top square is commutative by the functoriality of $d^\vee$. The second square is (1.23) and is commutative by the definition of $d^\vee$. The vertical arrows in the third square are induced by $\mathrm{adj}_{a^\vee}\colon a^\vee_! e'^{\vee!} \to e^{\vee!}$ and the square is commutative by the functoriality of $i_E$. The bottom square is (1.17) and is commutative by Proposition 1.1.6.

The upper two arrows in the right column are obtained by applying $a^\vee_!$ to the upper left arrows in the commutative diagram (1.17)

$$\begin{tikzcd}
F'(F'^\vee e'^{\vee!}(-)\otimes e'^!\Lambda) \arrow[r, "i_{E'}"] \arrow[d] & e'^{\vee!} \arrow[d, "(-1)^{n'}"] \\
F'(0'_*(-)\otimes e'^!\Lambda) \arrow[r, "(1.4)_{E'}"] & e'^{\vee!}
\end{tikzcd}$$

for $E'$. Hence the diagram (1.27) is commutative.

2. Let $b\colon E'' \to E$ be the inclusion of the kernel $E'' = \mathrm{Ker}(a\colon E \to E')$ and $b^\vee\colon E^\vee \to E''^\vee$ be the dual. We consider the diagram

(1.29)
$$\begin{tikzcd}
Fa^!0'_! \arrow[r, "d^\vee"] \arrow[d] & a^\vee_* F'0'_! \arrow[r, "(1.4)_{E'}"] & a^\vee_* e'^{\vee *} \arrow[d] \\
Fb_!e''^! \arrow[r, "d"] & b^{\vee *}F''e''^! \arrow[r, "(1.6)_{E''}"] & b^{\vee *}0''^\vee_* \\
Fb_!0''_! \arrow[u] \arrow[r, "d"] & b^{\vee *}F''0''_! \arrow[u] \arrow[r, "(1.4)_{E''}"] & b^{\vee *}e''^{\vee *} \arrow[u, "\mathrm{res}"'] \arrow[dl] \\
F0_! \arrow[u] \arrow[r, "(1.4)_E"] & e^{\vee *} &
\end{tikzcd}$$

of functors $D^b_c(X,\Lambda) \to D^b_c(E^\vee,\Lambda)$. The bottom pentagon is commutative by the transitivity of the isomorphisms (1.21) and (1.4) and the remark after Proposition 1.1.8. The middle right square is obtained by applying $b^{\vee *}$ to (1.7) for $e''\colon E'' \to X$. The middle left square is commutative by the functoriality of $d$. By the commutative diagram

$$\begin{tikzcd}
a^! 0'_! \arrow[r, "\mathrm{bc}"] & b_! e''^! \\
0_! \arrow[u, "\mathrm{adj}_a"] \arrow[r, "\mathrm{bc}"] & b_! 0''_!, \arrow[u]
\end{tikzcd}$$

the left vertical arrow in (1.28) equals the composition of the left column. Similarly, the slant arrow in (1.28) equals the composition of the right column and the slant arrow. Since the top line is the same as that of (1.28), it suffices to show that the upper rectangle is commutative.

The morphism $d^\vee$ fits in the following commutative diagram

$$\begin{tikzcd}
F a^! 0'_! \arrow[r, "d^\vee"] & a^\vee_* F' 0'_! \\
F a^! F'^\vee e'^{\vee !} \arrow[u, "(1.5)_{E'}"] \arrow[r, "d^\vee"] & a^\vee_* F' F'^\vee e'^{\vee !} \arrow[u, "(1.5)_{E'}"'] \arrow[d, "i_{E'}"] \\
F(F^\vee a^\vee_* e'^{\vee !}(-) \otimes a^! \Lambda) \arrow[u, "d"] \arrow[r, "i_E"] & a^\vee_* e'^{\vee *}.
\end{tikzcd} \tag{1.30}$$

The upper square is commutative by the functoriality of $d^\vee$ and the lower square is (1.23). The composition of the right column is $(-1)^{n'}$-times the upper right horizontal arrow $(1.4)_{E'}$ in (1.29) By Proposition 1.1.6. Hence, it suffices to show that the top rectangle in (1.29) with the top line replaced by the composition $F a^! 0'_! \to a^\vee_* e'^!$ in (1.30) via lower right is $(-1)^{n'}$-commutative.

Let $\alpha$ and $\beta$ denote the compositions

$$\begin{aligned}
\alpha\colon F^\vee a^\vee_! e'^{\vee !} &\xrightarrow{d} a^* F'^\vee e'^{\vee !} \xrightarrow{(1.5)_{E'}} a^* 0'_* \xrightarrow{\mathrm{bc}} b_* e''^* \\
\beta\colon F b_! e''^! &\xrightarrow{d} b^{\vee *} F' e''^! \xrightarrow{(1.5)_{E''}} b^{\vee *} 0''^\vee_* \xrightarrow{\mathrm{bc}} a^\vee_* e'^{\vee *}.
\end{aligned}$$

The left vertical arrows in (1.30) and the top left vertical arrow in (1.29) are induced by the arrows in $\alpha$ and $\beta$ is the same as the arrows in the top rectangle in (1.29) via lower right. Hence it suffices to show that the bottom horizontal arrow $i_E\colon F(F^\vee a^\vee_* e'^!(-) \otimes a^! \Lambda) \to a^\vee_* e'^*$ in (1.30) equals $(-1)^{n'}$-times $\beta \circ F(\alpha \otimes a^! \Lambda)$. We consider the commutative diagram

$$\begin{tikzcd}
E & E \times E^\vee \arrow[l, "\mathrm{pr}_1"'] \arrow[r, "\mathrm{pr}_2"] & E^\vee \\
E'' \arrow[u, "b"] & E'' \times E'^\vee \arrow[u] \arrow[l, "\mathrm{pr}'_1"'] \arrow[r, "\mathrm{pr}'_2"] & E'^\vee. \arrow[u, "a^\vee"']
\end{tikzcd}$$

The vertical arrows are closed immersions and the image of $a^\vee$ is the exact annihilator of the image of $b$. Hence by the proper base change theorem, the pull-back by the middle vertical immersion induces an isomorphism $F b_! e''^! \to a^\vee_* \mathrm{pr}'_{2!} \mathrm{pr}'^*_1 e''^!$. Composing this with the isomorphism induced by $\mathrm{pr}'_{2!} \mathrm{pr}'^*_1 e''^! \to e'^{\vee *} e''_! e''^* \to e'^{\vee *}$, we obtain an isomorphism

$Fb_!e''^! \to a^\vee_* e'^{\vee *}$. This is equal to $\beta$ by the definition of $d$. Using the same diagram from the right to the left, we obtain a similar equality for $\alpha$. We consider the immersions

$$E'^\vee \times E'' \times E'^\vee \xrightarrow{1\times b\times 1} E'^\vee \times E \times E'^\vee \xleftarrow{\delta\times 1} E'^\vee \times E \tag{1.31}$$

defined by $(\delta \times 1)(x^\vee, x) = (x^\vee, x, x^\vee)$. Let $p_1, p_3 \colon E'^\vee \times E'' \times E'^\vee \to E'^\vee$, $p_2 \colon E'^\vee \times E'' \times E'^\vee \to E''$, and $q \colon E'^\vee \times E \to E'^\vee$ be the restrictions of the projections $\mathrm{pr}_1, \mathrm{pr}_3 \colon E'^\vee \times E \times E'^\vee \to E'^\vee$ and $\mathrm{pr}_2 \colon E'^\vee \times E \times E'^\vee \to E''$. Then, by the descriptions of $\alpha$ and $\beta$ above, the composition $\beta \circ F(\alpha \otimes a^!\Lambda)$ equals the composition via upper right in the diagram

$$\begin{tikzcd}
F(F^\vee a^\vee_* e'^{\vee !}(-) \otimes a^!\Lambda) \arrow[r] \arrow[d] & a^\vee_* p_{3!}(p_1^* e'^{\vee !}(-) \otimes p_2^* e''^!\Lambda) \arrow[d] \\
a^\vee_* q_!(e'^{\vee *}(-) \boxtimes e^!\Lambda) \arrow[r] & a^\vee_* e'^{\vee *}.
\end{tikzcd}$$

The upper horizontal arrow and the left vertical arrow are the isomorphisms induced by the pull-back by the immersions in (1.31) respectively. By the proof of Proposition 1.1.5, the morphism $i_E \colon F(F^\vee a^\vee_* e'^{\vee !}(-) \otimes a^!\Lambda) \to a^\vee_* e'^*$ equals the composition via lower left. Since the left vertical arrow is induced by $\mathrm{Tr}_{E'^\vee/X} \cdot \mathrm{Tr}_{E''/X}$ and the lower horizontal arrow is induced by $\mathrm{Tr}_{E/X}$, the diagram is $(-1)^{n'}$-commutative, similarly as in the proof of Proposition 1.1.6. □

In [14, Lemma 13.5], the diagram (1.27) is studied with an unspecified isomorphism.

Let $X \to Y$ be a separated morphism of finite type of noetherian schemes over $k$ and $E$ be a vector bundle on $Y$. Let $b \colon E \times_Y X \to E$ and $b \colon E^\vee \times_Y X \to E^\vee$ be the base change morphisms for $E$ and for the dual vector bundle. Let $F_X$ and $F_Y$ denote the Fourier transform for $E \times_Y X$ and $E$ respectively. Then we have canonical isomorphisms

$$b^\vee_! F_X \to F_Y b_!, \tag{1.32}$$

$$b^{\vee *} F_Y \to F_X b^* \tag{1.33}$$

by the proper base change theorem.

Let $X$ be a smooth curve over a field $k$ and $0 \in X$ be a $k$-rational point. We identify $R^1 j_* \Lambda(1)$ with $\Lambda_0$ by the isomorphism

$$\Lambda_0 \to R^1 j_* \Lambda(1) \tag{1.34}$$

defined by the image of a uniformizer by the boundary morphism $j_* \mathbf{G}_m \to R^1 j_* \Lambda(1)$. Let $e \colon E \to X$ be a line bundle over $X$ and $e^\vee \colon E^\vee \to X$ be the dual. Let $j \colon X - \{0\} \to X$ be the open immersion and let $E_0 = E \times_X 0$ and $E_0^\vee = E^\vee \times_X 0$ be the fibers. Let $j_E \colon E - E_0 \to E$ and $j_{E^\vee} \colon E^\vee - E_0^\vee \to E^\vee$ be the base change. Let $E_0^\times = E_0 - \{0\}$ and $E_0^{\vee\times} = E_0^\vee - \{0\}$ be the complement of the origin and let $g_0 \colon E_0^\times \to E_0$ and $g_0^\vee \colon E_0^{\vee\times} \to E_0^\vee$ be the open immersions. We identify the non-trivial cohomology sheaf $\mathcal{H}^{-1}(e^* j_* K_U) = e^* R^1 j_* \Lambda(1)$ at the top degree with $\Lambda_{E_0}$ by the isomorphism (1.34). Define a subcomplex $\mathcal{N} = \mathcal{N}_\Lambda$ of $e^* j_* K_U$ on $E$ by replacing $\mathcal{H}^{-1}(e^* j_* K_U) = e^* R^1 j_* \Lambda(1) = \Lambda_{E_0}$ by the subsheaf $g_{0!}\Lambda_{E_0^\times}$. Here and in the following, by abuse of notation, we identify a sheaf on a closed subscheme of $E$ with its direct image on $E$ and similarly for $E^\vee$. By the definition, we have a canonical isomorphism

$$0^* \mathcal{N}_\Lambda = 0^* e^* R^{-2} j_* K_U^* \to K_X. \tag{1.35}$$

**Proposition 1.1.12.** 1. *We have a distinguished triangle*

$$\mathcal{N}_\Lambda \to e^* j_* K_U \oplus g_{0!}\Lambda_{E_0^\times}[1] \to \Lambda_{E_0}[1] \to . \tag{1.36}$$

2. *Let* $+\colon 0^\vee(X) \cup E_0^\vee \to E^\vee$ *be the closed immersion of the union of the* 0*-section and the fiber* $E_0^\vee$*. Then, there exists a unique morphism*

$$F_\psi \mathcal{N}_\Lambda \to +_! +^! e^{\vee *} K_X \tag{1.37}$$

*fitting in the commutative diagram*

$$\begin{tikzcd} F_\psi \mathcal{N}_\Lambda \arrow[r, "(1.37)"] \arrow[d] & +_! +^! e^{\vee *} K_X \arrow[d, "\mathrm{adj}"] \\ F_\psi 0_* K_X \arrow[r, "(1.4)"] & e^{\vee *} K_X . \end{tikzcd} \tag{1.38}$$

*The left vertical arrow is induced by the adjoint of* $0^* \mathcal{N}_\Lambda \to K_X$ (1.35) *and the right vertical arrow is the adjunction morphism.*

3. *Let* $i\colon E_0^\vee \to E^\vee$ *be the closed immersion and let* $\Lambda_{0(X)} \to 0^! e^{\vee *} K_X$ *and* $\Lambda_{E_0^\vee} \to i^! e^{\vee *} K_X$ *be the canonical isomorphisms defined by the cycle classes* $[0(X)]$ *and* $[E_0^\vee]$*. Define* $+_! +^! e^{\vee *} K_X \to 0^\vee_* j_* \Lambda \oplus g_* \Lambda$ *to be that induced by the inverses of these isomorphisms. Then, the diagram*

$$\begin{tikzcd} F_\psi \mathcal{N}_\Lambda \arrow[r, "(1.37)"] \arrow[d, "(1.36)"'] & +_! +^! e^{\vee *} K_X \arrow[d] \\ F_\psi (e^* j_* K_U \oplus g_{0!}\Lambda_{E_0^\times}[1]) \arrow[r, "(1.5)\oplus(1.11)"] & 0^\vee_* j_* \Lambda \oplus g_* \Lambda \end{tikzcd} \tag{1.39}$$

*is* $(-1)$*-commutative and the morphism* (1.37) *is an isomorphism.*

*Proof.* 1. We have a canonical morphism $\mathcal{N}_\Lambda \to e^* j_* K_U \oplus g_{0!}\Lambda_{E_0^\times}[1]$ by construction. The morphisms of cohomology sheaves are isomorphisms except for degree $-1$. For degree $-1$, the morphism is an injection and the cokernel is $\Lambda_{E_0}$. Hence, we obtain (1.36).

2. By abuse of notation, we identify a sheaf on a closed subscheme of $E^\vee$ with its direct image on $E^\vee$. Applying Fourier transform to (1.36), we obtain a distinguished triangle

$$F_\psi \mathcal{N}_\Lambda \to 0^\vee_* j_* \Lambda \oplus g^\vee_{0*} \Lambda \to \Lambda_0(-1)[-1] \to . \tag{1.40}$$

Hence $F_\psi \mathcal{N}_\Lambda$ is supported on the union $+ = 0^\vee(X) \cup E_0^\vee$ and the composition $F_\psi \mathcal{N}_\Lambda \to F_\psi 0_* K_X \to e^{\vee *} K_X$ of the arrows in (1.38) via lower left induces $F_\psi \mathcal{N}_\Lambda \to +_! +^! e^{\vee *} K_X$ (1.37) making (1.38) commutative.

3. First, we show the $(-1)$-commutativity for the first component. It suffices to consider the restriction on $U = X - \{0\}$. The restriction of $0_! \Lambda \to \mathcal{N}_\Lambda \to 0_* K_X$ on $U$ equals that of the composition of the adjoint $0_! \Lambda \to e^* K_X$ and the restriction $e^* K_X \to 0_* K_X$. By Proposition 1.1.11.1, its Fourier transform is $(-1)$-times the composition of the restriction $\Lambda \to 0^\vee_* \Lambda$ and the adjoint $0^\vee_! \Lambda \to e^{\vee *} K_X$. Since the adjoint $0^\vee_! \Lambda \to e^{\vee *} K_X$ is defined by the cycle class $[0^\vee(X)]$, the $(-1)$-commutativity for the first component follows.

We show the $(-1)$-commutativity for the second component. Since the identification $\Lambda \to R^1 j_* \Lambda(1)$ (1.34) is defined by sending 1 to the class of a uniformizer $t$, its image by the boundary map $R^1 j_*\Lambda(1) \to \mathrm{H}^2_0(X, \Lambda(1))$ is the minus $-[0]$ of the cycle class by [7, 2.1.3]. Thus, the $(-1)$-commutativity for the second component follows.

We show that (1.37) is an isomorphism. By cohomological purity, $+_!+^! e^{\vee *}K_X$ fits in a distinguished triangle $+_!+^! e^{\vee *}K_X \to 0^\vee_* j_*\Lambda \oplus g^\vee_{0*}\Lambda \to \Lambda_0(-1)[-1] \to$. The lower horizontal arrow is an isomorphism by Corollary 1.1.4. Since the diagram (1.39) is $(-1)$-commutative, by comparing this with (1.40), we see that that (1.37) is an isomorphism. □

By purity, the $\Lambda$-module $\mathrm{H}^0_{0^\vee(X)\cup E^\vee_0}(E^\vee, e^{\vee *}K_X)$ is free of rank 2 generated by the cycle classes $[0^\vee(X)]$ and $[E^\vee_0]$.

**Corollary 1.1.13.** *Let $I$ denote the inertia group at $0 \in E_0$.*

1. *We have a distinguished triangle*

$$0^!\mathcal{N}_\Lambda \to j_*\Lambda \oplus \Gamma(I, \Lambda)_0 \to \mathrm{H}^1(I, \Lambda)_0[-1] \to . \tag{1.41}$$

*The last two terms are placed at $0 \in T \cap X \subset TX$.*

2. *Let $\Lambda \to 0^!\mathcal{N}_\Lambda$ be a morphism and let $\Lambda \to F_\psi\mathcal{N}_\Lambda \to +^! e^{\vee *}K_X$ be the composition of the Fourier transform of its adjoint $0_!\Lambda \to \mathcal{N}_\Lambda$ and (1.37). If the composition $\Lambda \to 0^!\mathcal{N}_\Lambda \to j_*\Lambda \oplus \Gamma(I, \Lambda)_0$ with the first arrow in (1.41) maps $1 \in \Lambda$ to $(a, b) \in \Lambda \oplus \mathrm{H}^0(I, \Lambda)$, then the composition $\Lambda \to +^! e^{\vee *}K_X$ defines an element $-a[0^\vee(X)] - b[E^\vee_0] \in \mathrm{H}^0_{0^\vee(X)\cup E^\vee_0}(E^\vee, e^{\vee *}K_X)$.*

*Proof.* 1. For terms in (1.36), we have canonical isomorphisms $0^! e^* j_* K_U \to j_*\Lambda$, $0^! g_{0!}\Lambda_{E^\times_0} \to \Gamma(I, \Lambda)_0[-1]$ and $0^!\Lambda_{E_0} \to \mathrm{H}^1(I, \Lambda)_0[-2]$. Hence (1.36) induces (1.41).

2. Let $g^\vee_{T^*}\colon T^* - \{0\} \to T^* = T^*_0 X$ be the open immersion to the fiber at 0. Then the Fourier transform of the first arrow in (1.41) defines the upper line in the commutative diagram

$$\begin{tikzcd}
F_\psi 0^!\mathcal{N}_\Lambda \arrow[r] \arrow[d] & F_\psi(j_*\Lambda \oplus \Gamma(I, \Lambda)_0) \arrow[r, "(1.4)"] \arrow[d] & e^{\vee *}j_*\Lambda \oplus e^{\vee *}\Gamma(I, \Lambda)_0 \arrow[d, "\mathrm{res}"] \\
F_\psi\mathcal{N}_\Lambda \arrow[r] & F_\psi(e^* j_* K_U \oplus g_{0!}\Lambda_{E^\times_0}[1]) \arrow[r] & 0^\vee_* j_*\Lambda \oplus g^\vee_{0*}\Lambda.
\end{tikzcd}$$

The lower line is (1.39) via lower left. Thus the asserion follows from the $(-1)$-commutativity of (1.39) □

## 1.2 Deformation to normal bundle and the specialization functor

First, we study dilatations of schemes smooth over an affine line $\mathbf{A}^1 = \operatorname{Spec} k[u]$.

**Definition 1.2.1.** Let $k$ be a field and let $Z \to X$ be a closed immersion of schemes smooth over the affine line $\mathbf{A}^1$. Let $\{0\} \subset \mathbf{A}^1 = \operatorname{Spec} k[u]$ be the closed subscheme defined by the ideal $(u)$ and define closed subschemes $Z_0 = Z \times_{\mathbf{A}^1} \{0\} \subset X_0 = X \times_{\mathbf{A}^1} \{0\} \subset X$. Let $\overline{D}_Z X \to X$ be the blow-up at $Z_0 \subset X$ and define the dilatation $D_Z X \subset \overline{D}_Z X$ to be the open subscheme obtained by removing the proper transform of $X_0 \subset X$.

If $X = \operatorname{Spec} A$ is affine and $Z \subset X$ is defined by an ideal $I \subset A$, then $D_Z X = \operatorname{Spec} A[I/u]$ for the subring $A[I/u] \subset A[1/u]$.

**Lemma 1.2.2.** 1. *The immersion* $Z \to X$ *is lifted uniquely to* $Z \to D_Z X$.

2. *The closed fiber* $D_Z X \times_{\mathbf{A}^1} \{0\}$ *is canonically identified with the normal bundle* $T_{Z_0} X_0$.

3. *The scheme* $D_Z X$ *is smooth over* $\mathbf{A}^1$.

*Proof.* 1. Since $Z_0 \subset Z$ is a Cartier divisor, the assertion follows from the universality of dilatations.

2. The exceptional divisor of the blow-up $\overline{D}_Z X \to X$ is canonically identified with the projective space bundle $\mathbf{P}(T_{Z_0} X)$. Since $T_{Z_0} X = T_{Z_0} X_0 \oplus \mathbf{A}^1_{Z_0}$, the intersection $\mathbf{P}(T_{Z_0} X) \cap D_Z X$ is identified with $T_{Z_0} X_0$.

3. We may assume that $X = \operatorname{Spec} A$ and that $Z \subset X$ is defined by an ideal $I \subset A$. Then $D_Z X = \operatorname{Spec} A[I/u]$ for the subring $A[I/u] \subset A[1/u]$ and is flat over $\mathbf{A}^1$. Since $D_Z X \times_{\mathbf{A}^1} 0 = T_{Z_0} X_0$ is smooth over $k$, $D_Z X$ is smooth over $\mathbf{A}^1$. $\square$

Let

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd} \tag{1.42}$$

be a commutative diagram of smooth schemes over $\mathbf{A}^1$ such that the horizontal arrows are closed immersions. Then, we have a canonical morphism $D_Z X \to D_W Y$. We say that the diagram (1.42) is transversal if it is cartesian and if the morphism $\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} \mathcal{O}_W \to \mathcal{O}_Z$ is an isomorphism. Further if the diagram (1.42) is transversal, then the diagram

$$\begin{tikzcd} D_Z X \arrow[r] \arrow[d] & X \arrow[d] \\ D_W Y \arrow[r] & Y \end{tikzcd} \tag{1.43}$$

is cartesian.

**Lemma 1.2.3.** *Let*

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ W \arrow[r] & Y \end{tikzcd} \tag{1.44}$$

*be a commutative diagram of smooth schemes over* $\mathbf{A}^1$ *such that the horizontal arrows are closed immersions.*

1. *Assume that* $X \to Y$ *is a closed immersion and that the diagram* (1.44) *is cartesian. Then the morphism* $D_Z X \to D_W Y$ *is a closed immersion.*

2. *Assume that* $X \to Y$ *and* $Z \to W$ *are smooth. Then the morphism* $D_Z X \to D_W Y$ *is smooth.*

*Proof.* 1. We may assume that $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ are affine and $Z \subset X$ and $W \subset Y$ are defined by ideals $I \subset A$ and $J \subset B$. By the assumption that the diagram (1.42) is cartesian, the surjection $B \to A$ induces a surjection $J \to I$. Hence the induced morphism $B[J/u] \to A[I/u]$ is a surjection.

2. By the assumption that $X \to Y$ is smooth, the base change of the morphism $D_Z X \to D_W Y$ by $\mathbf{G}_m \to \mathbf{A}^1$ is smooth. Further by the assumption that $Z \to W$ is smooth, the morphism $T_{Z_0}X_0 \to T_{W_0}Y_0$ is smooth. Hence the morphism $D_Z X \to D_W Y$ of smooth schemes over $\mathbf{A}^1$ is smooth. □

We apply the construction above to define the deformation to normal bundle.

**Definition 1.2.4.** Let $k$ be a field and let $i\colon Z \to X$ be a closed immersion of smooth schemes over $k$. Let $T_Z X$ be the normal bundle. We define the deformation to the normal bundle $A_Z X$ to be the dilatation $D_{Z\times\mathbf{A}^1}(X\times\mathbf{A}^1)$ for the closed immersion $Z\times\mathbf{A}^1 \to X\times\mathbf{A}^1$ of smooth schemes over the affine line $\mathbf{A}^1 = \operatorname{Spec} k[u]$.

The deformation to the normal bundle $A_Z X$ fits in a cartesian diagram

$$\begin{tikzcd}
T_Z X \arrow[r] \arrow[d] & A_Z X \arrow[d] & X \times \mathbf{G}_m \arrow[l] \arrow[d] \\
0 \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m. \arrow[l]
\end{tikzcd}$$

We consider the diagonal immersion $\delta\colon \Delta_X = X \to X \times X$. Then, the normal bundle $T_X(X \times X)$ is the tangent bundle $TX$. Hence, we have a cartesian diagram

$$\begin{tikzcd}
TX \arrow[r] \arrow[d] & A_X(X \times X) \arrow[d] & X \times X \times \mathbf{G}_m \arrow[l] \arrow[d] \\
0 \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m. \arrow[l]
\end{tikzcd}$$

If $X \to Y$ is a morphism of smooth schemes over $k$, we have a canonical morphism $A_X(X \times X) \to A_Y(Y \times Y)$.

**Lemma 1.2.5.** *Let $X \to Y$ be a morphism of smooth schemes over $k$.*

1. *Assume that $X \to Y$ is a closed immersion. Then the morphism $A_X(X \times X) \to A_Y(Y \times Y)$ is a closed immersion.*

2. *Assume that $X \to Y$ is smooth. Then the morphism $A_X(X \times X) \to A_Y(Y \times Y)$ is smooth.*

*Proof.* 1. Since the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times X \arrow[d] \\
Y \arrow[r] & Y \times Y
\end{tikzcd}$$

is cartesian, the assertion follows from Lemma 1.2.3.1.

2. The assertion follows from Lemma 1.2.3.2. □

We briefly recall the definition and functorialities of nearby cycles functor. Let $\mathcal{O}_K$ be a henselian discrete valuation ring. Let $S^{\mathrm{ur}} = \operatorname{Spec} \mathcal{O}_{K^{\mathrm{ur}}}$ be the maximal unramified extension of $S = \operatorname{Spec} \mathcal{O}_K$. Let $s, \eta \in S$ be the closed point and the generic point. Let $\bar{s} \in S^{\mathrm{ur}}$ be the closed point and $\bar{\eta} = \operatorname{Spec} K_{\mathrm{sep}} \to S^{\mathrm{ur}}$ be a geometric point above $\eta$. Let $\Lambda$ be a finite ring and assume that the cardinality is invertible in $\mathcal{O}_K$. For a scheme $X$ of finite type over $S$, let

$$\begin{tikzcd}
X_{\bar{s}} \arrow[r, "\bar{i}"] \arrow[d] & X \times_S S^{\mathrm{ur}} \arrow[d] & X_{\bar{\eta}} \arrow[l, "\bar{j}"'] \arrow[r, "j"] \arrow[d] & X_\eta \arrow[d] \\
\bar{s} \arrow[r] & S^{\mathrm{ur}} & \bar{\eta} \arrow[l] \arrow[r] & \eta
\end{tikzcd}$$

be a cartesian diagram. Let $D^b_c(X_s \times_s \eta, \Lambda)$ be the derived category of constructible sheaves on $X_{\bar{s}}$ with continuous action of $\mathrm{Gal}(\bar{\eta}/\eta)$ compatible with the action on $X_{\bar{s}}$ as in [3, Exposé XIII, Construction 1.2.4 (b)]. Then the nearby cycles functor

$$\Psi \colon D^b_c(X_\eta, \Lambda) \to D^b_c(X_s \times_s \eta, \Lambda)$$

is defined by $\Psi = \bar{i}^* \bar{j}_* j^*$. Its composition $D^b_c(X, \Lambda) \to D^b_c(X_s \times_s \eta, \Lambda)$ with the pull-back is also denoted by $\Psi$.

Let $f \colon X \to Y$ be a morphism of schemes separated of finite type over $S$. Then the base change morphisms define morphisms of functors

$$f_s^* \Psi_Y \to \Psi_X f_{\bar{\eta}}^*, \tag{1.45}$$

$$\Psi_Y f_{\bar{\eta}*} \to f_{s*} \Psi_X, \tag{1.46}$$

$$f_{s!} \Psi_X \to \Psi_Y f_{\bar{\eta}!}. \tag{1.47}$$

The morphisms (1.45) and (1.46) are adjoint to each other. If $f$ is proper, (1.46) and (1.47) are isomorphisms inverse to each other by the proper base change theorem. If $f$ is smooth, (1.46) is an isomorphism by the smooth base change theorem. A canonical morphism

$$\Psi_X f_{\bar{\eta}}^! \to f_s^! \Psi_Y \tag{1.48}$$

is defined as the adjoint of (1.47).

We recall the definition of monodromic sheaves from [19, Définition 3.1].

**Definition 1.2.6.** Let $X$ be a noetherian scheme over a field $k$ and $E$ be a vector bundle over $X$. For a geometric point $x$ of $E$, define a morphism $w_x \colon \mathbf{G}_{m,x} \to E$ by $w_x(t) = t \cdot x$.

We say that $\mathcal{F} \in D^b_c(E, \Lambda)$ is monodromic if for every geometric point $x$ of $E$ and for every integer $q \in \mathbf{G}$, the cohomology sheaf $\mathcal{H}^q(w_x^* \mathcal{F})$ on $\mathbf{G}_{m,x}$ is locally constant and tamely ramified along $0, \infty$.

We study a property of a locally constant sheaf on $\mathbf{G}_m$ tamely ramified along $0, \infty$. Let $k$ be a field of characteristic $p > 0$ and $\Lambda$ be a finite ring such that the cardinality is invertible in $k$.

**Lemma 1.2.7.** *Let $\mathcal{M}$ be a locally constant constructible sheaf on $\mathbf{G}_m = \operatorname{Spec} k[x^{\pm 1}]$ tamely ramified along $0, \infty \in \mathbf{P}^1$ and let $j \colon \mathbf{G}_m \to \mathbf{A}^1$ be the open immersion. Let $\bar{k}$ be a separable closure of $k$.*

1. *The canonical morphism $\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{M}) \to (j_* \mathcal{M})_{\bar{0}}$ is an isomorphism.*
2. $\Gamma_c(\mathbf{A}^1_{\bar{k}}, j_* \mathcal{M}) = 0$.

*Proof.* 1. Let $I_0^{\mathrm{t}}$ be the tame inertia group at 0 and let $M$ be the stalk of $\mathcal{M}$ at the generic geometric point of $\mathbf{G}_m$. Then, the canonical morphism $I_0^{\mathrm{t}} \to \pi_1(\mathbf{G}_{m,\bar{k}})^{\mathrm{t}}$ to the tame fundamental group is an isomorphism and we obtain a commutative diagram

$$\begin{tikzcd}
\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{M}) \arrow[r] & j_*\mathcal{M}_{\bar{0}} \\
\Gamma(\pi_1(\mathbf{G}_{m,\bar{k}})^{\mathrm{t}}, M) \arrow[r] \arrow[u] & \Gamma(I_0^{\mathrm{t}}, M) \arrow[u]
\end{tikzcd}$$

of isomorphisms.

2. Let $j_\infty \colon \mathbf{A}^1 \to \mathbf{P}^1$ be the open immersion. Then, we have a distinguished triangle $\Gamma_c(\mathbf{A}^1_{\bar{k}}, j_*\mathcal{M}) \to \Gamma(\mathbf{P}^1_{\bar{k}}, j_{\infty*}j_*\mathcal{M}) \to j_{\infty*}j_*\mathcal{M}|_{\bar{\infty}} \to$. The second morphism is identified with $\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{M}) \to j_{\infty*}j_*\mathcal{M}|_{\bar{\infty}}$ and is an isomorphism similarly as in 1. Hence we have $\Gamma_c(\mathbf{A}^1_{\bar{k}}, j_*\mathcal{M}) = 0$. □

We recall the definition of the specialization functor from [19]. Let $i \colon Z \to X$ be a closed immersion of smooth schemes over a field $k$. We consider the cartesian diagram

$$\begin{tikzcd}
T_Z X \arrow[r] \arrow[d] & A_Z X \arrow[d] & X \times \mathbf{G}_m \arrow[l] \arrow[d] \\
0 \arrow[r] & \mathbf{A}^1 & \mathbf{G}_m. \arrow[l]
\end{tikzcd} \tag{1.49}$$

Let $\eta_0$ be the generic point of the henselization of $\mathbf{A}^1$ at 0. Let $\Psi$ be the nearby cycles functor with respect to the middle vertical arrow in (1.49) at $0 \in \mathbf{A}^1$ and let $p_1 \colon X \times \mathbf{G}_m \to X$ denote the projection. We define the specialization functor

$$\nu_{Z/X} \colon D_c^b(X, \Lambda) \to D_c^b(T_Z X \times_k \eta_0, \Lambda) \tag{1.50}$$

by $\nu_Z = \Psi \circ p_1^*$.

Let

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
W \arrow[r] & Y
\end{tikzcd} \tag{1.51}$$

be a commutative diagram of smooth schemes over $k$ such that the horizontal arrows are closed immersions. Then, we have a commutative diagram

$$\begin{tikzcd}
T_Z X \arrow[r] \arrow[d, "T_f"'] & A_Z X \arrow[d, "A_f"'] & X \times \mathbf{G}_m \arrow[l] \arrow[d] \\
T_W Y \arrow[r] & A_W Y & X \times \mathbf{G}_m \arrow[l]
\end{tikzcd} \tag{1.52}$$

and the morphisms (1.45), (1.46), (1.47), (1.48) define morphisms

$$T_f^* \nu_{W/Y} \to \nu_{Z/X} f^*, \tag{1.53}$$

$$\nu_{W/Y} f_* \to T_{f*} \nu_{Z/X}, \tag{1.54}$$

$$T_{f!} \nu_{Z/X} \to \nu_{W/Y} f_!, \tag{1.55}$$

$$\nu_{Z/X} f^! \to T_f^! \nu_{W/Y}. \tag{1.56}$$

**Proposition 1.2.8.** *Let* $i\colon Z \to X$ *be a closed immersion of smooth schemes over* $k$ *and* $\mathcal{F} \in D^b_c(X, \Lambda)$.

1 ([19, Section 8 (SP1)]). *The specialization* $\nu_{Z/X}\mathcal{F}$ *on* $T_Z X$ *is monodromic.*

2 (cf. [19, Section 8 (SP5)]). *The morphisms* $0^*\nu_{Z/X}\mathcal{F} \to i^*\mathcal{F}$ (1.53) *and* $i^!\mathcal{F} \to 0^!\nu_{Z/X}\mathcal{F}$ (1.56) *for the immersion* $A_Z Z = Z \times \mathbf{A}^1 \to A_Z X$ *are isomorphisms.*

*Proof.* 2. The first isomorphism is [19, Section 8 (SP5)]. The following proof of the second isomorphism is similar to the proof for the first isomorphism loc. cit. It suffices to show that $0^!\nu_{Z/X}\mathcal{F} = 0$ assuming that $i^!\mathcal{F} = 0$. Let $j\colon U = X - Z \to X$ be the open immersion of the complement. Then, the assumption $i^!\mathcal{F} = 0$ means that the canonical morphism $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism. By blow-up, we may assume that $Z \subset X$ is a Cartier divisor of $X$ and that there exists a cartesian diagram

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & X \arrow[d] \\
0 \arrow[r] & \mathbf{A}^1
\end{tikzcd}$$

where the vertical arrows are smooth. We prove the assertion by induction on the dimension $d$ of the support of $\mathcal{F}$. If $d = 0$, then $\mathcal{F} = 0$ on a neighborhood of $Z$ and the assertion holds. Assume $d \geqq 1$.

First, we show $0^!\nu_{Z/X}\mathcal{F} = 0$ generically on $Z$. By replacing $X$ by an open neighborhood of the generic point of $Z$, we may assume that $j^*\mathcal{F}$ is locally constant. By further replacing $X$ by a finite scheme over $X$ and shrinking $X$ and using the isomorphism (1.54) for finite morphisms, we may assume that the cohomology sheaves of $j^*\mathcal{F}$ are constant and further $j^*\mathcal{F} = \mathcal{H}^0 j^*\mathcal{F}[0]$. Let $j_1\colon U \times \mathbf{G}_m \to A_Z X - A_Z Z$ be the open immersion. Then, since $A_Z X - A_Z Z \to \mathbf{A}^1$ is smooth, the direct image $R^0 j_{1*}\mathrm{pr}_1^* j^*\mathcal{F}$ is a constant sheaf. Let $j_2\colon A_Z X - A_Z Z \to A_Z X$ be the open immersion and let $j_{2,0}\colon T_Z X - Z \to T_Z X$ be its restriction on the closed fiber. Further, since $A_Z X$ and $A_Z Z$ are smooth over $\mathbf{A}^1$, we have $\nu_{Z/X}\mathcal{F} = (j_{2*}R^0 j_{1*}\mathrm{pr}_1^* j^*\mathcal{F})|_{T_Z X} = j_{2,0*}((R^0 j_{1*}\mathrm{pr}_1^* j^*\mathcal{F})|_{T_Z X - Z})$. Thus we have $0^!\nu_{Z/X}\mathcal{F} = 0$ generically on $Z$.

We show $0^!\nu_{Z/X}\mathcal{F} = 0$. Since the assertion is local on $X$, we may take an étale morphism $X \to \mathbf{A}^n$ such that $Z \subset X$ is the inverse image of $\mathbf{A}^{n-1} \subset \mathbf{A}^n$. By considering the normalization of $\mathbf{A}^n$ in $X$ and the compatibility with finite direct image, we may assume $Z = \mathbf{A}^{n-1} \subset X = \mathbf{A}^n$. Since the support of $0^!\nu_{Z/X}\mathcal{F}$ is of dimension $< n-1$, there exists a linear projection $X = \mathbf{A}^n \to X' = \mathbf{A}^{n-1}$ such that $Z = \mathbf{A}^{n-1} \to Z' = \mathbf{A}^{n-2}$ is finite on the support $S$ of $0^!\nu_{Z/X}\mathcal{F}$. Let $\overline{X} = X' \times \mathbf{P}^1 \to X'$ and $\overline{q}\colon \overline{Z} = Z' \times \mathbf{P}^1 \to Z'$ be projective completions and let $\overline{\mathcal{F}} = j_{X*}\mathcal{F}$ for the open immersion $j_X\colon X \to \overline{X}$. Then, we have $0^!\nu_{Z/X}\mathcal{F} = (0^!\nu_{\overline{Z}/\overline{X}}\overline{\mathcal{F}})|_Z$ and $\overline{q}\colon \overline{Z} \to Z'$ is finite on $\overline{S} \cup Z$ containing the support of $0^!\nu_{\overline{Z}/\overline{X}}\overline{\mathcal{F}}$. By the induction hypothesis and the compatibility with finite direct images, we have $\overline{q}_* 0^!\nu_{\overline{Z}/\overline{X}}\overline{\mathcal{F}} = 0$ and hence $0^!\nu_{\overline{Z}/\overline{X}}\overline{\mathcal{F}} = 0$. Thus, we obtain $0^!\nu_{Z/X}\mathcal{F} = 0$. □

We compute the nearby cycles complex for certain tamely ramified sheaves.

**Lemma 1.2.9.** *Let* $X$ *be a scheme smooth over the semi-stable curve* $\operatorname{Spec} k[s,t,u]/(st-u)$ *over* $\mathbf{A}^1 = \operatorname{Spec} k[u]$ *and let* $D_1, D_2 \subset X$ *be the divisors defined by* $s$ *and* $t$ *respectively. Let* $\mathcal{F}$ *be a locally constant constructible sheaf of* $\Lambda$*-modules on* $U = X - (D_1 \cup D_2)$ *tamely ramified along* $D_1 \cup D_2$ *and let* $\Psi = \Psi\mathcal{F}$ *be the nearby cycles complex on* $X \times_{\mathbf{A}^1} 0 = D_1 \cup D_2$ *with respect to the morphism* $X \to \mathbf{A}^1 = \operatorname{Spec} k[u]$ *at* $u = 0$. *Let* $j_1\colon D_1^\circ = D_1 - (D_1 \cap D_2) \to D_1$ *and* $j_2\colon D_2^\circ = D_2 - (D_1 \cap D_2) \to D_1$ *be the open immersions.*

1. *The restrictions* $\Psi_1^\circ = \Psi|_{D_1^\circ}$ *and* $\Psi_2^\circ = \Psi|_{D_2^\circ}$ *are locally constant and tamely ramified along* $D_1 \cap D_2$.

2. *The canonical morphisms* $\Psi|_{D_1} \to j_{1*}\Psi_1^\circ$ *and* $\Psi|_{D_2} \to j_{2*}\Psi_2^\circ$ *are isomorphisms.*

*Proof.* 1. Let $\mathbf{A}^{1\,\mathrm{t}} = \operatorname{Spec} k[u^{1/n}; p \nmid n]$ be the maximal tamely ramified covering and $j^{\mathrm{t}} \colon U \times_{\mathbf{A}^1} \mathbf{A}^{1\,\mathrm{t}} \to X \times_{\mathbf{A}^1} \mathbf{A}^{1\,\mathrm{t}}$ be the base change of the open immersion $j \colon U \to X$. Then, by the assumption that $\mathcal{F}$ is tamely ramified along $D_1 \cup D_2$, the extension $R^0 j^{\mathrm{t}}_* \mathcal{F}$ is locally constant outside the inverse image of $D_1 \cap D_2$ and the restrictions $(R^0 j^{\mathrm{t}}_*\mathcal{F})|_{D_1^\circ}$ and $(R^0 j^{\mathrm{t}}_*\mathcal{F})|_{D_2^\circ}$ are tamely ramified along $D_1 \cap D_2$. Since $X - (D_1 \cap D_2)$ is smooth over $\mathbf{A}^1$, we have isomorphisms $(R^0 j^{\mathrm{t}}_*\mathcal{F})|_{D_1^\circ} \to \Psi|_{D_1^\circ}$ and $(R^0 j^{\mathrm{t}}_*\mathcal{F})|_{D_2^\circ} \to \Psi|_{D_2^\circ}$ of locally constant sheaves of locally constant sheaves on $D_1^\circ$ and $D_2^\circ$.

2. It suffices to show the isomorphisms at each geometric point $x$ of $D_1 \cap D_2$. Let $X_x$ be the strict localization and $U_x = X_x \times_{\operatorname{Spec} k[u]} \operatorname{Spec} k(u)$ be the generic fiber. Let $U_x^{\mathrm{t}} = U_x \times_{\operatorname{Spec} k[s,t,u]/(st-u)} \operatorname{Spec} \varinjlim_{p\nmid n} k[s^{1/n}, t^{1/n}, u^{1/n}]/(s^{1/n}t^{1/n} - u^{1/n})$ be the universal tamely ramified covering. By cohomological purity, the canonical morphism $\Lambda \to \Gamma(U_x^{\mathrm{t}}, \Lambda)$ is an isomorphism (cf. [6, Théorème 3.3]). The canonical morphism $U \to \mathbf{A}^1$ is lifted to $U_x^{\mathrm{t}} \to \mathbf{A}^{1\,\mathrm{t}}$.

We identify the Galois group $\operatorname{Gal}(U_x^{\mathrm{t}}/U_x)$ with the product $I_s \times I_t$ of $I_s = I_t = \varprojlim_{p\nmid n} \mu_n$ acting on $s^{1/n}, t^{1/n}$ by multiplication respectively. The morphism $U_x^{\mathrm{t}} \to \mathbf{A}^{1\,\mathrm{t}}$ induces $I_s \times I_t \to I_u = \operatorname{Gal}(\mathbf{A}^{1\,\mathrm{t}}/\mathbf{A}^1)$. Let $I_0 = \operatorname{Ker}(I_s \times I_t \to I_u)$ be the kernel. The pull-back of $\mathcal{F}$ on $U_x^{\mathrm{t}}$ is a constant sheaf and the pull-back of $\mathcal{F}$ on $U_x$ is the locally constant sheaf corresponding to the representation $M = \mathrm{H}^0(U_x^{\mathrm{t}}, \mathcal{F})$ of $I_s \times I_t$. The isomorphism $\Lambda \to \Gamma(U_x^{\mathrm{t}}, \Lambda)$ induces an isomorphism $\Gamma(I_0, M) = \Gamma(I_0, \Gamma(U_x^{\mathrm{t}}, \mathcal{F})) \to \Psi_x$.

For $i = 1, 2$, let $D_{ix} = D_i \times_X X_x$ and $U_{ix} = D_{ix} - \{x\}$. Let $U_{ix}^{\mathrm{t}}$ be the maximal tamely ramified covering. We identify $\operatorname{Gal}(U_{1x}^{\mathrm{t}}/U_{1x}) = I_t$ and $\operatorname{Gal}(U_{2x}^{\mathrm{t}}/U_{1x}) = I_s$ with $I_0$ by the isomorphisms $I_0 \to I_t$ and $I_0 \to I_s$ induced by the projections. The canonical morphisms $\Lambda \to \Gamma(U_{ix}^{\mathrm{t}}, \Lambda)$ are also isomorphisms. The pull-backs of $\Psi$ on $U_{1x}$ and $U_{2x}$ correspond to the representation $M$ of $I_0$. The isomorphisms $\Lambda \to \Gamma(U_{ix}^{\mathrm{t}}, \Lambda)$ induce isomorphisms $\Gamma(I_0, M) = \Gamma(I_0, \Gamma(U_{1x}^{\mathrm{t}}, \Psi_1^\circ)) \to (j_{1*}\Psi_1^\circ)_x$ and $\Gamma(I_0, M) = \Gamma(I_0, \Gamma(U_{2x}^{\mathrm{t}}, \Psi_2^\circ)) \to (j_{2*}\Psi_2^\circ)_x$. Hence we have canonical isomorphisms $\Psi_x \to (j_{1*}\Psi_1)^\circ_x$ and $\Psi_x \to (j_{2*}\Psi_2^\circ)_x$. □

Let $X = \mathbf{A}^1 = \operatorname{Spec} k[x]$ and $Z = \{0\} \subset X$ be the reduced closed subscheme consisting of the origin. Let $(X \times \mathbf{A}^1)' \to X \times \mathbf{A}^1 = \operatorname{Spec} k[x, u]$ be the blow-up at $(0,0)$ and $E = \mathbf{P}^1$ be the exceptional divisor. Let $(\{0\} \times \mathbf{A}^1)', (X \times \{0\})' \subset (X \times \mathbf{A}^1)'$ be the proper transforms of $\{0\} \times \mathbf{A}^1$ and $X \times \{0\}$ respectively and let $0, \infty \in E$ be the intersection point with $(\{0\} \times \mathbf{A}^1)', (X \times \{0\})'$. The complement $(X \times \mathbf{A}^1)' - (X \times \{0\})'$ is the deformation to normal bundle $A_Z X$ and $E - \{\infty\}$ is canonically identified with the normal bundle $T = T_Z X$.

**Lemma 1.2.10.** *Let* $\mathcal{F}$ *be a locally constant sheaf on* $\mathbf{G}_m = \operatorname{Spec} k[x^{\pm 1}]$ *tamely ramified at* $0$ *and let* $j \colon \mathbf{G}_m \to X = \mathbf{A}^1 = \operatorname{Spec} k[x]$ *be the open immersion. Let* $\mathrm{pr}_1 \colon X \times \mathbf{G}_m \to X$ *be the projection and* $\Psi = \Psi \mathrm{pr}_1^* j_! \mathcal{F}$ *be the nearby cycles complex with respect to the morphism* $(X \times \mathbf{A}^1)' \to \mathbf{A}^1 = \operatorname{Spec} k[u]$ *at* $0 \in \mathbf{A}^1$. *Let* $\mathcal{M}$ *be the restriction* $\Psi|_{T^\times}$ *on the* $\mathbf{G}_m$*-torsor* $T^\times = T_Z X - \{0\} \subset T_Z X = \operatorname{Spec} k[v]$.

1. *Let* $\eta_x$ *be the generic point of the henselization at* $0 \in X$ *and let* $M$ *be the representation of the tame decomposition group* $G_x^{\mathrm{tame}} = \operatorname{Gal}(\eta_{x,\mathrm{tame}}/\eta_x)$. *Let* $\eta_u$ *be the generic point of the henselization of* $\mathbf{A}^1 = \operatorname{Spec} k[u]$ *at* $0$ *and define* $\pi_1^{\mathrm{tame}}(T^\times \times_k \eta_u)$ *to be the fiber product* $\pi_1^{\mathrm{tame}}(T^\times) \times_{\operatorname{Gal}(k_{\mathrm{sep}}/k)} \operatorname{Gal}(\eta_{u,\mathrm{tame}}/\eta_u)$. *Define a morphism* $\pi_1^{\mathrm{tame}}(T^\times \times_k \eta_u) \to G_x^{\mathrm{tame}}$

*induced by* $T^\times \times (\mathbf{A}^1 - \{0\}) = \operatorname{Spec} k[v^{\pm 1}, u^{\pm 1}] \to \mathbf{G}_m = \operatorname{Spec} k[x^{\pm 1}]$ *given by* $x = uv$. *Then the restriction* $\mathcal{M}$ *of* $\Psi$ *on* $T^\times \times_k \eta_u = \mathbf{G}_m \times_k \eta_u$ *is a monodromic sheaf defined by the pull-back of* $M$ *to* $\pi_1^{\mathrm{tame}}(T^\times \times_k \eta_u)$.

2. *The restriction of* $\Psi$ *on the proper transform* $(X \times \{0\})'$ *is* $j_*\mathcal{F}$. *Let* $j_0 \colon T^\times \to T$ *and* $j_\infty \colon T \to E$ *be the open immersion. Then, the restriction of* $\Psi$ *on* $T$ *is isomorphic to* $\overline{\mathcal{M}} = j_{0!}\mathcal{M}$. *The restriction of* $\Psi$ *on* $E$ *is isomorphic to* $j_{\infty *}\overline{\mathcal{M}}$.

*Proof.* 1. By Proposition 1.2.8.1, $\mathcal{M} = \Psi|_{T^\times}$ is monodromic. Let $\operatorname{Spec} k[u,v] = A_Z X \subset (X \times \mathbf{A}^1)'$ be the deformation to the normal bundle where $v = x/u$. Then, $\mathrm{pr}_1 \colon X \times \mathbf{G}_m \to X$ is defined by $x = uv$. Hence $\mathcal{M}$ corresponds to the pull-back of $M$ by $\pi_1^{\mathrm{tame}}(T^\times \times_k \eta_u) \to G_0^{\mathrm{tame}}$.

2. By Proposition 1.2.8.2, $\overline{\mathcal{M}} = j_{0!}\mathcal{M} \to \Psi|_T$ is an isomorphism. The isomorphisms $\Psi|_{(X \times \{0\})'} \to j_*\mathcal{F}$ and $\Psi|_E \to j_{\infty *}\overline{\mathcal{M}}$ follows from Lemma 1.2.9.2. □

# 2 Characteristic cycles and microlocalization

In this section, let $X$ be a separated smooth scheme of finite type over a field $k$ of characteristic $p > 0$ and $\Lambda$ be a finite local ring with residue characteristic $\ell \neq p$. We fix a non-trivial character $\psi \colon \mathbf{F}_p \to \Lambda^\times$.

## 2.1 Construction

We identify the normal bundle $T_X(X \times X)$ and the conormal bundle $T^*_X(X \times X)$ for the diagonal $\delta \colon X \to X \times X$ with the tangent bundle $TX$ and the cotangent bundle $T^*X$. Let $\eta$ denote the generic point of the henselization of $\mathbf{A}^1$ at 0. Let $F_\psi \colon D_{\mathrm{ctf}}(TX, \Lambda) \to D_{\mathrm{ctf}}(T^*X, \Lambda)$ denote the Fourier transform, Definition 1.1.1.2. We define bifunctors

$$\begin{aligned} \nu\mathcal{H}om &\colon D_{\mathrm{ctf}}(X,\Lambda)^{\mathrm{op}} \times D_{\mathrm{ctf}}(X,\Lambda) \to D_{\mathrm{ctf}}(TX \times_k \eta, \Lambda), \\ \mu\mathcal{H}om &\colon D_{\mathrm{ctf}}(X,\Lambda)^{\mathrm{op}} \times D_{\mathrm{ctf}}(X,\Lambda) \to D_{\mathrm{ctf}}(T^*X \times_k \eta, \Lambda) \end{aligned} \tag{2.1}$$

by $\nu\mathcal{H}om(\mathcal{F}_1, \mathcal{F}_2) = \nu_{X/X \times X}\mathcal{H}om(\mathrm{pr}_1^*\mathcal{F}_1, \mathrm{pr}_2^!\mathcal{F}_2)$ (1.50) and $\mu\mathcal{H}om = \mathcal{F}_\psi \circ \nu\mathcal{H}om$.

Let $a \colon X \to \operatorname{Spec} k$ be the canonical morphism and define $K_X = a^!\Lambda \in D_{\mathrm{ctf}}(X, \Lambda)$. Let $e \colon TX \to X$ and $e^\vee \colon T^*X \to X$ be the projections and let $0 \colon X \to TX$ and $0^\vee \colon X \to T^*X$ denote the 0-sections. For $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$, we define a closed subset $SS_\mu\mathcal{F} \subset T^*X$ in Definition 2.1.1 and an element $CC_\mu\mathcal{F} \in \mathrm{H}^0_{SS_\mu\mathcal{F}}(T^*X, e^{\vee *}K_X)$ in Definition 2.1.4.

**Definition 2.1.1.** Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. We define a closed subset $SS_\mu\mathcal{F} \subset T^*X$ by

$$SS_\mu\mathcal{F} = \operatorname{supp} \mu\mathcal{H}om(\mathcal{F}, \mathcal{F}). \tag{2.2}$$

We say that $\mathcal{F} \in D^b_c(X, \Lambda)$ is locally constant if every cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is locally constant.

**Lemma 2.1.2.** *Let* $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$.

1. *The closed subset* $SS_\mu\mathcal{F} \subset T^*X$ *is conical.*

2. *Assume that* $\mathcal{F}$ *is locally constant and that* $X$ *is irreducible. Then, we have* $SS_\mu\mathcal{F} = T^*_X X$ *if* $\mathcal{F} \neq 0$ *and* $SS_\mu\mathcal{F} = \varnothing$ *if* $\mathcal{F} = 0$.

3. *Assume* $\dim X = 1$. *Then, we have* $SS_\mu\mathcal{F} \subset SS\mathcal{F}$.

*Proof.* 1. The specialization $\nu\mathcal{H}om(\mathcal{F},\mathcal{F})$ and hence the microlocalization $\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$ are monodromic by Proposition 1.2.8.1 and [20, Proposition 2.5 4)]. Thus its support $SS_\mu\mathcal{F}$ is conical.

2. Since $\mathcal{H}om(\mathrm{pr}_1^*\mathcal{F},\mathrm{pr}_2^!\mathcal{F})$ is locally constant, we have isomorphisms $e^*(\mathcal{H}om(\mathcal{F},\mathcal{F})\otimes K_X)\to\nu\mathcal{H}om(\mathcal{F},\mathcal{F})$ and $0^\vee_*\mathcal{H}om(\mathcal{F},\mathcal{F})\to\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$ (1.5).

3. We may assume that $X$ is irreducible. Let $U\subset X$ be the maximal open subset on which $\mathcal{F}|_U$ is locally constant and identify $T^*U=T^*X\times_X U$. Then by 2, we have $SS_\mu\mathcal{F}\cap T^*U=T^*_UU$ or $=\varnothing$ according to $\mathcal{F}|_U\neq 0$ or $=0$. Since $SS\mathcal{F}$ is the union of $SS_\mu\mathcal{F}\cap T^*U$ and fibers $T^*X - T^*U$, the assertion follows. □

Let $\mathcal{H}$ denote $\mathcal{H}om(\mathrm{pr}_1^*\mathcal{F},\mathrm{pr}_2^!\mathcal{F})\in D_{\mathrm{ctf}}(X\times X,\Lambda)$. By [8, (3.2.1)], we have a canonical isomorphism

$$\mathcal{H}om(\mathcal{F},\mathcal{F})\to\delta^!\mathcal{H}. \tag{2.3}$$

Hence the identity $1_\mathcal{F}$ defines a canonical morphism

$$\Lambda\to\delta^!\mathcal{H}. \tag{2.4}$$

Let $\boxtimes\colon D_{\mathrm{ctf}}(X,\Lambda)\times D_{\mathrm{ctf}}(X,\Lambda)\to D_{\mathrm{ctf}}(X\times X,\Lambda)$ denote the bifunctor $\mathrm{pr}_1^*-\otimes\mathrm{pr}_2^*-$. The canonical isomorphism $D_X\mathcal{F}\boxtimes\mathcal{F}\to\mathcal{H}$ [8, (3.1.1)] on $X\times X$ induces an isomorphism

$$D_X\mathcal{F}\otimes\mathcal{F}\to\delta^*\mathcal{H}. \tag{2.5}$$

Hence the evaluation morphism $D_X\mathcal{F}\otimes\mathcal{F}\to K_X$ defines a canonical morphism

$$\delta^*\mathcal{H}\to K_X. \tag{2.6}$$

The morphisms (2.4), (2.6) and the canonical morphism $\delta^!\to\delta^*$ define

$$\Lambda\to\delta^!\mathcal{H}\to\delta^*\mathcal{H}\to K_X. \tag{2.7}$$

**Definition 2.1.3** ([1, Definition 2.1.1])**.** The canonical class $cc\,\mathcal{F}\in\mathrm{H}^0(X,K_X)$ is defined by the composition (2.7).

The adjoint of (2.7) defines

$$\delta_!\Lambda\to\mathcal{H}\to\delta_*K_X. \tag{2.8}$$

The specialization and the microlocalization define

$$0_!\Lambda\to\nu\mathcal{H}om(\mathcal{F},\mathcal{F})\to 0_*K_X, \tag{2.9}$$

$$\Lambda\to\mu\mathcal{H}om(\mathcal{F},\mathcal{F})\to e^{\vee*}K_X. \tag{2.10}$$

Alternatively, by Proposition 1.2.8.2, (2.4) and (2.6) define the first and the last arrows in

$$\Lambda\to 0^!\nu_{X/X\times X}\mathcal{H}\to 0^*\nu_{X/X\times X}\mathcal{H}\to K_X \tag{2.11}$$

on $T^*X$. The middle arrow is defined by the morphism $0^!\to 0^*$ of functors. Taking the adjoint, we obtain (2.9).

Let $i\colon SS_\mu\mathcal{F}\to T^*X$ be the closed immersion. Since $SS_\mu\mathcal{F}\subset T^*X$ is defined as the support of $\mu\mathcal{H}om(\mathcal{F},\mathcal{F})$, (2.10) define

$$\Lambda\to i^*\mu\mathcal{H}om(\mathcal{F},\mathcal{F})\to i^!e^{\vee*}K_X \tag{2.12}$$

on $SS_\mu\mathcal{F}$ by adjunction. This allows us to make the following definition.

**Definition 2.1.4** (cf. [11, Definition 9.4.1])**.** Let $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. We define

$$CC_{\mu}\mathcal{F} \in \mathrm{H}^0(SS_{\mu}\mathcal{F}, i^! e^{\vee *} K_X) = \mathrm{H}^0_{SS_{\mu}\mathcal{F}}(T^*X, e^{\vee *}K_X) \tag{2.13}$$

to be the class of the composition of (2.12).

**Lemma 2.1.5.** *Assume that $\mathcal{F}$ is locally constant of rank $r = \operatorname{rank} \mathcal{F}$ and that $X$ is of dimension $n$. Then, we have*

$$CC_{\mu}\mathcal{F} = (-1)^n \cdot \operatorname{rank} \mathcal{F} \cdot \mathrm{cl}(T^*_X X).$$

*Proof.* Since $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_1^* \mathcal{F}, \mathrm{pr}_2^! \mathcal{F})$ is locally constant, we have an isomorphism $e^*(\mathcal{H}om(\mathcal{F}, \mathcal{F}) \otimes K_X) \to \nu\mathcal{H}om(\mathcal{F}, \mathcal{F})$. The morphisms (2.10) and (2.9) give a diagram

$$\begin{tikzcd}
F0_!\Lambda \arrow[r, "F\mathrm{adj}"] \arrow[d, "(1.4)"'] & Fe^!\mathcal{H}om(\mathcal{F}, \mathcal{F}) \arrow[r, "F(\mathrm{res})"] \arrow[d, "(1.6)"] & F0_!(\mathcal{H}om(\mathcal{F}, \mathcal{F}) \otimes K_X) \arrow[r, "F(\mathrm{ev})"] \arrow[d, "(1.4)"] & F0_*K_X \arrow[d, "(1.4)"] \\
\Lambda \arrow[r, "\mathrm{res}"] & 0_*\mathcal{H}om(\mathcal{F}, \mathcal{F}) \arrow[r, "\mathrm{adj}"] & e^{\vee *}(\mathcal{H}om(\mathcal{F}, \mathcal{F}) \otimes K_X) \arrow[r, "\mathrm{Tr}"] & e^{\vee *}K_X.
\end{tikzcd}$$

By Lemma 1.1.2.1, the left square is commutative. Since (1.6) is $d^{\vee}$ by Corollary 1.1.9.2, the middle square is $(-1)^n$-commutative by Proposition 1.1.11.1. The right square is commutative by the functoriality of (1.4). Since the class of the composition of the lower line is $r \cdot \mathrm{cl}(T^*_X X)$, the assertion follows. □

**Lemma 2.1.6.** *The pull-back of $CC_{\mu}\mathcal{F}$ by the* 0*-section*

$$0^* \colon \mathrm{H}^0_{SS_{\mu}\mathcal{F}}(T^*X, e^{\vee *}K_X) \to \mathrm{H}^0(X, K_X)$$

*equals the canonical class $cc\ \mathcal{F}$.*

*Proof.* The pull-back $0^* \colon \mathrm{H}^0_{SS_{\mu}\mathcal{F}}(T^*X, e^{\vee *}K_X) \to \mathrm{H}^0(X, K_X)$ is the same as the composition of the canonical morphism $\mathrm{H}^0_{SS_{\mu}\mathcal{F}}(T^*X, e^{\vee *}K_X) \to \mathrm{H}^0(T^*X, e^{\vee *}K_X)$ and the inverse of the pull-back $e^{\vee *} \colon \mathrm{H}^0(X, K_X) \to \mathrm{H}^0(T^*X, e^{\vee *}K_X)$. Since the composition of (2.10) is the pull-back by $e^{\vee}$ of that of (2.7), the assertion follows. □

**Proposition 2.1.7.** *Let $F$ be a filtration on $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ such that $\mathrm{Gr}^F \mathcal{F} = \bigoplus_i \mathrm{Gr}^F_i \mathcal{F}$ is in $D_{\mathrm{ctf}}(X, \Lambda)$ and let $S = SS_{\mu}\mathrm{Gr}^F\mathcal{F}$. Then, we have an equality*

$$CC_{\mu}\mathcal{F} = \sum_i CC_{\mu}\mathrm{Gr}^F_i \mathcal{F} \tag{2.14}$$

*in $\mathrm{H}^0_S(T^*X, e^{\vee *}K_X)$.*

*Proof.* The filtration $F$ on $\mathcal{F}$ induces a filtration also denoted $F$ on $\mu\mathcal{H}om(\mathcal{F}, \mathcal{F})$. Since the identity $1_{\mathcal{F}}$ is in $F^0\mu\mathcal{H}om(\mathcal{F}, \mathcal{F})$, the class $CC_{\mu}\mathcal{F}$ is the image of $1_{\mathcal{F}} \in \mathrm{H}^0_S(T^*X, F^0\mu\mathcal{H}om(\mathcal{F}, \mathcal{F}))$ by the evaluation morphism. Since the restriction of the evaluation morphism is the composition of $F^0\mu\mathcal{H}om(\mathcal{F}, \mathcal{F}) \to \bigoplus_i \mu\mathcal{H}om(\mathrm{Gr}^F_i \mathcal{F}, \mathrm{Gr}^F_i \mathcal{F}) \to e^{\vee *}K_X$, the assertion follows. □

**Corollary 2.1.8.** *Let $i \colon Z \to X$ be a closed immersion and $j \colon U = X - Z \to X$ be the open immersion of the complement. Then, we have*

$$CC_{\mu}\mathcal{F} = CC_{\mu}(j_! j^* \mathcal{F}) + CC_{\mu}(i_* i^* \mathcal{F}) \tag{2.15}$$

*in $\mathrm{H}^0_{SS_{\mu}(j_! j^* \mathcal{F} \oplus i_* i^* \mathcal{F})}(T^*X, e^{\vee *}K_X)$.*

*Proof.* We define a filtration $F$ on $\mathcal{F}$ by $F^0\mathcal{F} = j_!j_*\mathcal{F} \subset \mathcal{F}$, $F^{-1} = 0$ and $F^1 = \mathcal{F}$. Then $\mathrm{Gr}^F\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$ and we obtain (2.15). $\square$

On the relations between $SS_\mu\mathcal{F}$ and $SS\mathcal{F}$ and between $CC_\mu\mathcal{F}$ and $CC\mathcal{F}$, we raise the following questions.

**Question 2.1.9.** 1. Do we have $SS_\mu\mathcal{F} \subset SS\mathcal{F}$ ?

2. Does $CC_\mu\mathcal{F}$ equal the cycle class of $CC\mathcal{F}$ ?

If $\mathcal{F}$ is locally constant, Lemmas 2.1.2.2 and 2.1.5 mean that Questions 2.1.9 has a positive answer. Lemma 2.1.2.3 means that if $\dim X = 1$, Question 2.1.9.1 has a positive answer.

## 2.2 Tamely ramified sheaves on curves

Let $X$ be a smooth irreducible curve over $k$ and let $x \in X$ be a $k$-rational point. Let $j\colon U = X - \{x\} \to X$ be the open immersion and $\mathcal{F}$ be a locally constant constructible sheaf of free $\Lambda$-modules of rank $n$ on $U$. By Lemma 2.1.2.3, we know that $SS_\mu j_!\mathcal{F} \subset T^*X$ is a subset of the union of the 0-section $T^*_XX$ and the fiber $T^*_xX$. By the semi-purity [7, Rappel 2.2.8], we have $\mathrm{H}^0_{T^*_XX\cup T^*_xX}(T^*X, e^{\vee *}K_X) = \Lambda[T^*_XX] \oplus \Lambda[T^*_xX]$. By Lemma 2.1.5, we have $CC_\mu j_!\mathcal{F} = -(n\cdot[T^*_XX] + a\cdot[T^*_xX])$ for some $a \in \Lambda$. We give a method to compute the coefficient $a \in \Lambda$ in Lemma 2.2.3.

Let $+\colon T^*_XX \cup T^*_xX \to T^*X$ be the closed immersion. Since $\mu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})$ is supported on $SS_\mu j_!\mathcal{F} \subset T^*_XX \cup T^*_xX$, the adjoint of the morphism $\mu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to e^{\vee *}K_X$ (2.10) induced by ev defines $\mu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to +^!e^{\vee *}K_X$. Here and in the following, identify $\mu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})$ with its restriction to $T^*_XX\cup T^*_xX$ by abuse of notation. By the isomorphism $F_\psi\mathcal{N}_\Lambda \to +^!e^{\vee *}K_X$ (1.37), this is induced by a morphism $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_\Lambda$.

We define a subcomplex $\mathcal{N}_\mathcal{F}$ of $e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)$ on $TX$ similarly as $\mathcal{N}_\Lambda$ and construct a factorization

$$\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_\mathcal{F} \to \mathcal{N}_\Lambda \tag{2.16}$$

such that the restriction on $TU$ is $\nu\mathcal{H}om(\mathcal{F}, \mathcal{F}) \to e_U^*(\mathcal{E}nd\,\mathcal{F} \otimes K_U) \to e_U^*K_U$ where the second arrow is induced by $\mathrm{Tr}\colon \mathcal{E}nd\,\mathcal{F} \to \Lambda_U$ and $e_U\colon TU \to U$ is the projection. The cohomology sheaves of $j_*\mathcal{E}nd\,\mathcal{F} \otimes K_X$ are $\mathcal{H}^{-2} = R^0j_*\mathcal{E}nd\mathcal{F}(1)$, $\mathcal{H}^{-1} = R^1j_*\mathcal{E}nd\mathcal{F}(1)$ and vanish otherwise. Let $I_x$ denote the inertia group at $x \in X$. The stalk $(R^1j_*\mathcal{E}nd\mathcal{F}(1))|_x$ is canonically identified with the Galois cohomology $\mathrm{H}^1(I_x, \mathrm{End}\,F(1))$. Let $P \subset I_x$ denote the wild inertia subgroup. Since $I_x/P$ is isomorphic to $\widehat{\mathbf{Z}}' = \varprojlim_{p\nmid n}\mu_n$ and $P$ is a pro-$p$ group, the group $\mathrm{H}^1(I_x, \mathrm{End}\,F(1))$ is canonically identified with the coinvariant $(\mathrm{End}\,F)_{I_x}$.

Let $T = TX \times_X x$ be the fiber at $x$ and $g\colon T^\times = T - \{0\} \to T$ be the open immersion. Define a subcomplex $\mathcal{N}_\mathcal{F}$ of $e^*(j_*\mathcal{E}nd\mathcal{F}_U \otimes K_X)$ by replacing the cohomology sheaf $\mathcal{H}^{-1} = e^*R^1j_*\mathcal{E}nd\mathcal{F}(1) = (\mathrm{End}\,F)_{I_x,T}$ at the top degree by the subsheaf $g_!(\mathrm{End}\,F)_{I_x,T^\times}$. Here and in the following $(\mathrm{End}\,F)_{I_x,T}$ denotes the geometrically constant sheaf on $T$ with stalk $(\mathrm{End}\,F)_{I_x}$. The trace morphism $e_U^*(\mathcal{E}nd\mathcal{F} \otimes K_U) \to e_U^*K_U$ induces $e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X) \to e^*(j_*\Lambda \otimes K_X)$. Define a morphism $\mathrm{Tr}\colon \mathcal{N}_\mathcal{F} \to \mathcal{N}_\Lambda$ to be the restriction.

**Lemma 2.2.1.** 1. *The isomorphism* $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})|_{TU} \to e_U^*(\mathcal{E}nd\mathcal{F} \otimes K_U)$ *induces*

$$\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_\mathcal{F}. \tag{2.17}$$

2. *The morphism* (2.17) *induces a factorization* (2.16)*.*

*Proof.* 1. The isomorphism $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})|_{TU} = \nu\mathcal{H}om(\mathcal{F}, \mathcal{F}) \to e_U^*(\mathcal{E}nd\mathcal{F} \otimes K_U)$ induces a morphism $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)$ by adjunction and the smooth base change theorem. Since $0^*\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) = \delta^*(D_X(j_!\mathcal{F}) \boxtimes j_!\mathcal{F}) = D_X(j_!\mathcal{F}) \otimes j_!\mathcal{F} = j_!\mathcal{E}nd\mathcal{F} \otimes K_X$ by Proposition 1.2.8.2, the stalk $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})_0$ at $0 \in T \subset TX$ is 0. Hence by adjunction, $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})|_{TX - \{0\}} \to e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)|_{TX - \{0\}} = \mathcal{N}_\mathcal{F}|_{TX - \{0\}}$ induces (2.17).

2. The restriction on $TU$ of $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_\Lambda$ is induced by the trace $e_U^*(\mathcal{E}nd\mathcal{F} \otimes K_U) \to e_U^* K_U$. Hence the assertion follows by adjunction as in the proof of 1. □

We have isomorphisms $j_*\mathcal{E}nd\,\mathcal{F} \to \mathcal{E}nd\, j_!\mathcal{F} \to 0^!\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})$ by Proposition 1.2.8.2 and (2.3). This and the identity $1_\mathcal{F}$ define the first two arrows in

$$\Lambda \to j_*\mathcal{E}nd\,\mathcal{F} \to 0^!\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to 0^!\mathcal{N}_\mathcal{F} \to 0^!\mathcal{N}_\Lambda. \tag{2.18}$$

The last two arrows are induced by (2.16). Since the Fourier transform of its adjoint

$$0_!\Lambda \to 0_!\mathcal{E}nd\, j_!\mathcal{F} \to \nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_\mathcal{F} \to \mathcal{N}_\Lambda$$

defines $\Lambda \to \mu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to +^! e^{\vee *} K_X$ defining $CC_\mu j_!\mathcal{F}$, the composition of (2.18) determines $CC_\mu j_!\mathcal{F}$.

Similar computations as in Proposition 1.1.12.1 and Corollary 1.1.13.1 work for $\mathcal{N}_\mathcal{F}$ and $0^!\mathcal{N}_\mathcal{F}$.

**Lemma 2.2.2.** *Let $I_x$ denote the inertia group at $x \in X$ and $I_0$ denote the inertia group at $0 \in T$. Let $F$ be the representation of $I_x$ defined by $\mathcal{F}$ and consider the coinvariant quotient $(\mathrm{End}\, F)_{I_x}$ as a trivial representation of $I_0$.*

1. *The canonical morphism $\mathcal{N}_\mathcal{F} \to e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)$ induces a distinguished triangle*

$$\mathcal{N}_\mathcal{F} \to e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X) \oplus g_!(\mathrm{End}\, F)_{I_x, T^\times}[1] \to (\mathrm{End}\, F)_{I_x, T}[1] \to . \tag{2.19}$$

2. *The distinguished triangle* (2.19) *induces a distinguished triangle*

$$0^!\mathcal{N}_\mathcal{F} \to j_*\mathcal{E}nd\mathcal{F} \oplus \Gamma(I_0, (\mathrm{End}\, F)_{I_x})_0 \to \mathrm{H}^1(I_0, (\mathrm{End}\, F)_{I_x})_0[-1] \to . \tag{2.20}$$

*The last two terms are placed at $0 \in T \cap X \subset TX$.*

*Proof.* 1. The distinguished triangle (2.19) is clear from the construction of $\mathcal{N}_\mathcal{F}$.

2. For the geometrically constant sheaf $\mathcal{R} = (\mathrm{End}\, F)_{I_x, T}$ on $T$, we have canonical isomorphisms $0^!\mathcal{R} \to \mathrm{H}^1(I_0, \mathcal{R})_0[-2]$ and $0^! g_! g^*\mathcal{R} \to \Gamma(I_0, \mathcal{R})_0[-1]$. Hence (2.19) induces (2.20). □

**Lemma 2.2.3.** *Suppose that the composition*

$$\Lambda \to j_*\mathcal{E}nd\,\mathcal{F} \to 0^!\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to 0^!\mathcal{N}_\mathcal{F} \to \Gamma(I_0, (\mathrm{End}\, F)_{I_x})_0 \tag{2.21}$$

*of* (2.18) *and the second component of the first arrow of* (2.20) *maps $1 \in \mathrm{H}^0(X, \Lambda)$ to $a \in \mathrm{H}^0(I_0, (\mathrm{End}\, F)_{I_x}) = (\mathrm{End}\, F)_{I_x}$ and let $\mathrm{Tr} \colon (\mathrm{End}\, F)_{I_x} \to \Lambda$ be the morphism induced by the trace. Then, we have*

$$CC_\mu j_!\mathcal{F} = -(\mathrm{rank}\, \mathcal{F} \cdot [T^*_X X] + \mathrm{Tr}\, a \cdot [T^*_x X]) \tag{2.22}$$

*in $\mathrm{H}^0_{T^*_X X \cup T^*_x X}(T^*X, e^{\vee *} K_X)$.*

*Proof.* The trace morphism $\mathcal{E}nd\,\mathcal{F} \to \Lambda$ induces a commutative diagram

$$\begin{tikzcd}
0^!\mathcal{N}_{\mathcal{F}} \arrow[r] \arrow[d, "\mathrm{Tr}"'] & j_*\mathcal{E}nd\mathcal{F} \oplus \Gamma(I_0, (\mathrm{End}\,F)_{I_x})_0 \arrow[d, "\mathrm{Tr}\oplus\mathrm{Tr}"] \\
0^!\mathcal{N}_{\Lambda} \arrow[r] & j_*\Lambda \oplus \Gamma(I_0, \Lambda)_0.
\end{tikzcd}$$

The composition of (2.18) and the first component of the first arrow of (2.20) maps $1 \in \mathrm{H}^0(X, \Lambda)$ to $1_{\mathcal{F}}$. The equality (2.22) follows from this and Corollary 1.1.13.2. □

In this section, we prove the following.

**Proposition 2.2.4.** *Let* $j\colon \mathbf{G}_m \to X = \mathbf{A}^1$ *be the open immersion and let* $\mathcal{F}$ *be a locally constant constructible sheaf of free* $\Lambda$*-modules of rank* $n$ *on* $\mathbf{G}_m$ *tamely ramified at* $0 \in X$*. Let* $I_x$ *denote the inertia group at* $0 \in X$ *and* $I_0$ *denote the inertia group at* $0 \in T$*. Then, the composition*

$$j_*\mathcal{E}nd\,\mathcal{F} \to 0^!\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to 0^!\mathcal{N}_{\mathcal{F}} \to \Gamma(I_0, (\mathrm{End}\,F)_{I_x})_0. \tag{2.23}$$

*of* (2.18) *and the second component of the first arrow of* (2.20) *induces the canonical morphism* $\mathrm{H}^0(U, \mathcal{E}nd\mathcal{F}) \to \mathrm{H}^0(I_0, (\mathrm{End}\,F)_{I_x}) = (\mathrm{End}\,F)_{I_x}$.

Proposition 2.2.4 implies the following equality by Lemma 2.2.3.

**Corollary 2.2.5.** *We have*

$$CC_{\mu}j_!\mathcal{F} = -n([T^*_XX] + [T^*_0X]) \tag{2.24}$$

*in* $\mathrm{H}^0_{T^*_XX\cup T^*_0X}(T^*X, e^{\vee *}K_X)$.

As a preparation of the proof of Proposition 2.2.4, first we prove a lemma on cohomology of a tame inertia group.

**Lemma 2.2.6.** *Let* $\widehat{\mathbf{Z}}' = \varprojlim_{p\nmid n} \mathbf{Z}/n\mathbf{Z}$ *and let* $I$ *be a pro-finite group isomorphic to* $\widehat{\mathbf{Z}}'^2$*. Let* $\sigma, \tau \in I$ *be elements defining an isomorphism* $\widehat{\mathbf{Z}}'^2 \to I$ *and let* $I_1 = \langle\sigma\rangle$, $I_2 = \langle\tau\rangle \subset I = I_1 \times I_2$ *be the subgroups isomorphic to* $\widehat{\mathbf{Z}}'$*. Let* $M$ *be a* $\Lambda$*-module with a continuous action of* $I$ *such that the action of* $\tau \in I$ *is trivial.*

1. *The restriction morphisms define an isomorphism*

$$\mathrm{H}^1(I, M) \to \mathrm{H}^1(I_1, M) \oplus \mathrm{H}^1(I_2, M)^{I_1}. \tag{2.25}$$

2. *Let* $\rho = \sigma\tau$ *and* $I_3 = \langle\rho\rangle \subset I = I_3 \times I_2$*. Then, we have a commutative diagram*

$$\begin{tikzcd}
\mathrm{H}^1(I, M) \arrow[r, "(2.25)"] \arrow[d, "1"'] & \mathrm{H}^1(I_1, M) \oplus \mathrm{H}^1(I_2, M)^{I_1} \arrow[d, "\begin{pmatrix} i^*_{13} & i^*_{23} \\ 0 & 1 \end{pmatrix}"] \\
\mathrm{H}^1(I, M) \arrow[r, "(2.25)"] & \mathrm{H}^1(I_3, M) \oplus \mathrm{H}^1(I_2, M)^{I_3}
\end{tikzcd} \tag{2.26}$$

*where* $i^*_{13}\colon \mathrm{H}^1(I_1, M) \to \mathrm{H}^1(I_3, M)$ *and* $i^*_{23}\colon \mathrm{H}^1(I_2, M)^{I_1} = \mathrm{H}^1(I_2, M^{I_1}) \to \mathrm{H}^1(I_3, M)$ *are induced by the projections* $I_3 \to I_1$ *and* $I_3 \to I_2$.

*Proof.* 1. Let $A = \Lambda[[I]]$ be the completed group algebra and regard $M$ as an $A$-module. By the Koszul free resolution $A \xrightarrow{(\tau-1,-(\sigma-1))} A^2 \xrightarrow{(\sigma-1,\tau-1)} A$ of the $A$-module $\Lambda$, the complex $M \xrightarrow{(\sigma-1,\tau-1)} M^2 \xrightarrow{(\tau-1,-(\sigma-1))} M$ computes the cohomology $\mathrm{H}^*(I, M)$. Since $\tau$ acts trivially on $M$, we obtain an isomorphism $\mathrm{H}^1(I, M) = \mathrm{Coker}(\sigma - 1\colon M \to M) \oplus \mathrm{Ker}(\sigma - 1\colon M \to M) \to \mathrm{H}^1(I_1, M) \oplus \mathrm{H}^1(I_2, M)^{I_1}$.

2. We have a commutative diagram

$$\begin{tikzcd}
M \arrow[r, "(\sigma-1,\tau-1)"] \arrow[d, "1"'] & M^2 \arrow[r, "(\tau-1,-(\sigma-1))"] \arrow[d, "\begin{pmatrix}\tau & 1\\ 0 & 1\end{pmatrix}"'] & M \arrow[d, "\tau"] \\
M \arrow[r, "(\rho-1,\tau-1)"] & M^2 \arrow[r, "(\tau-1,-(\rho-1))"] & M
\end{tikzcd}$$

of complexes. The middle vertical arrow induces the right vertical arrow in (2.26). □

To prove Proposition 2.2.4, we prepare a geometric construction. Let $X$ denote $\mathbf{A}^1 = \mathrm{Spec}\, k[x]$. We define blow-ups

$$(X \times X \times \mathbf{A}^1)^\sharp \to (X \times X \times \mathbf{A}^1)^\natural \to X \times X \times \mathbf{A}^1 = \mathrm{Spec}\, k[x, y, u].$$

The right arrow is the blow-up at the closed point $(0, 0, 0)$ and the left arrow is the blow-up at the inverse image of $\delta(X) \times \{0\}$, in other words, at the pull-back of the ideal $(y - x, u)$. We define an open subscheme $A_X(X \times X)^\sharp \subset (X \times X \times \mathbf{A}^1)^\sharp$ by the cartesian diagram

$$\begin{tikzcd}
A_X(X \times X)^\sharp \arrow[r] \arrow[d, "q"'] & (X \times X \times \mathbf{A}^1)^\sharp \arrow[d] \\
A_X(X \times X) \arrow[r] & (X \times X \times \mathbf{A}^1)'.
\end{tikzcd} \tag{2.27}$$

The right vertical arrow is defined by the universality of blow-up and $A_X(X \times X)^\sharp \subset (X \times X \times \mathbf{A}^1)^\sharp$ is the complement of the proper transform of the divisor $(u) = X \times X \times \{0\} \subset X \times X \times \mathbf{A}^1$.

The diagonal immersion $\delta\colon X \times \mathbf{A}^1 \to X \times X \times \mathbf{A}^1$ is lifted to the immersion $0\colon (X \times \mathbf{A}^1)' \to A_X(X \times X)^\sharp$ from the blow-up $(X \times \mathbf{A}^1)' \to X \times \mathbf{A}^1$ at the closed point $(0, 0)$. The projections $X \times X \times \mathbf{A}^1 \to X \times \mathbf{A}^1$ are lifted to smooth morphisms $p_1$, $p_2\colon A_X(X \times X)^\sharp \to (X \times \mathbf{A}^1)'$.

On $A_X(X \times X)^\sharp_0 = A_X(X \times X)^\sharp \times_{\mathbf{A}^1} 0$, we have a cartesian diagram

$$\begin{tikzcd}
\mathbf{A}^1 \arrow[r] \arrow[d, "\delta"'] & E \arrow[r] \arrow[d, "\bar\delta"'] & (X \times \mathbf{A}^1)'_0 \arrow[d, "0"] & X \arrow[l] \arrow[d, "0"] \\
A_0 = \mathbf{A}^2 \arrow[r] & P \arrow[r] & A_X(X \times X)^\sharp_0 & TX \arrow[l]
\end{tikzcd} \tag{2.28}$$

defined as follows. The immersion $TX \to A_X(X \times X)$ is lifted to a closed immersion $TX \to A_X(X \times X)^\sharp_0$. The inverse image in $A_X(X \times X)^\sharp$ of the exceptional divisor of the second blow-up $(X \times X \times \mathbf{A}^1)^\sharp \to (X \times X \times \mathbf{A}^1)^\natural$ is the complement $P = \mathbf{P}^{2\prime} - L'$ in the blow-up $\mathbf{P}^{2\prime}$ of $\mathbf{P}^2$ at $(1, 1, 0) \in L = \mathbf{P}^2 - \mathbf{A}^2$ of the proper transform $L'$ of the line $L$ at infinity. In the middle cartesian square, $(X \times \mathbf{A}^1)'_0 \to A_X(X \times X)^\sharp_0$ is the base change of the lifting $0\colon (X \times \mathbf{A}^1)' \to A_X(X \times X)^\sharp$ of the morphism $\delta\colon X \times \mathbf{A}^1 \to X \times X \times \mathbf{A}^1$. The closed fiber $A_X(X \times X)^\sharp_0$ equals the union $P \cup TX$. The intersection $P \cap TX$ is the fiber

$T = TX \times_X 0 \subset TX$ and is dense in the exceptional divisor of the blow-up $\mathbf{P}^{2\prime} \to \mathbf{P}^2$. The inverse image of $P$ is the exceptional divisor $E$ of the blow-up $(X \times \mathbf{A}^1)' \to X \times \mathbf{A}^1$. The immersion $\overline{\delta}\colon E \to P$ is an extension of the diagonal morphism $\delta\colon \mathbf{A}^1 \to \mathbf{A}^2$.

They are explicitly described as follows. The scheme $A_X(X \times X)^\sharp$ is the union $A \cup B$ of two affine spaces $A = \mathbf{A}^3 = \operatorname{Spec} k[x_1, y_1, u]$ and $B = \mathbf{A}^3 = \operatorname{Spec} k[x, w, v]$. The second morphism in

$$A_X(X \times X)^\sharp = A \cup B \xrightarrow{q} A_X(X \times X) = \operatorname{Spec} k[x, u, v] \to X \times X \times \mathbf{A}^1 = \operatorname{Spec} k[x, y, u]$$

is given by $y = x + uv$ and the first morphism is given by $x = x_1 u, v = y_1 - x_1$ and $u = xw$. The closed subscheme $TX \subset A_X(X \times X)^\sharp_0$ is identified with $\operatorname{Spec} k[x, v] \subset B$. The closed fiber $A_X(X \times X)^\sharp_0 = A_X(X \times X)^\sharp \times_{\mathbf{A}^1} 0$ is the union $A_0 \cup B_0$ of $A_0 = \mathbf{A}^2 = \operatorname{Spec} k[x_1, y_1]$ and $B_0 = \operatorname{Spec} k[x, w, v]/(xw)$.

The blow-up $(X \times \mathbf{A}^1)'$ is the union $A_1 \cup B_1$ of two affine planes $A_1 = \mathbf{A}^2 = \operatorname{Spec} k[x_1, u]$ and $B_1 = \mathbf{A}^2 = \operatorname{Spec} k[x, w]$. The morphism

$$(X \times \mathbf{A}^1)' = A_1 \cup B_1 \to X \times \mathbf{A}^1 = \operatorname{Spec} k[x, u]$$

is given by $x = x_1 u$ and $u = xw$. The immersion $0\colon (X \times \mathbf{A}^1)' = A_1 \cup B_1 \to A_X(X \times X)^\sharp = A \cup B$ is given by $y_1 = x_1$ on $A_1 \subset A$ and by $v = 0$ on $B_1 \subset B$. The morphism $p_1\colon A_X(X \times X)^\sharp \to (X \times \mathbf{A}^1)'$ is given by the projections $A = \operatorname{Spec} k[x_1, y_1, u] \to A_1 = \operatorname{Spec} k[x_1, u]$ and $B = \operatorname{Spec} k[x, w, v] \to B_1 = \operatorname{Spec} k[x, w]$.

We fix notation for immersions of subschemes of $A_X(X \times X)^\sharp_0$ as in the diagram

$$\begin{tikzcd}
\mathbf{G}_m \arrow[r, "j_A"] & \mathbf{A}^1 \arrow[r, "j_E"] \arrow[d, "\delta"'] & E \arrow[d, "\overline{\delta}"'] & T^\times \arrow[d, "g"'] & X \arrow[d, "0"] & U = X - \{0\} \arrow[l, "j"'] \arrow[d, "0"] \\
& \mathbf{A}^2 \arrow[r, "j_P"] & P & T \arrow[l, "i_T"'] \arrow[r] & TX & TU \arrow[l, "j_{TX}"']
\end{tikzcd}$$

where $T^\times = T - \{0\}$ is the complement of the origin. The middle vertical arrow and the labeled horizontal arrows except $i_T$ are open immersions. The other vertical arrows, the unlabeled horizontal arrows and $i_T$ are closed immersions. We have $\mathbf{A}^2 = P - T$, $TU = TX - T$ and $T = P \cap TX$. Let $e\colon TX \to X$ be the canonical morphism.

Let $\mathcal{F}$ be a locally constant constructible sheaf of free $\Lambda$-modules on $\mathbf{G}_m$ tamely ramified at $0 \in X = \mathbf{A}^1$. Define $\mathcal{H} = \mathcal{H}om(\mathrm{pr}_1^* j_! \mathcal{F}, \mathrm{pr}_2^! j_! \mathcal{F})$ on $X \times X$ and let $\mathcal{H}^\sharp$ be the pull-back on $A_X(X \times X)^\sharp$. Let $\Psi\mathcal{H}^\sharp$ on $A_X(X \times X)^\sharp_0$ be the nearby cycles complex with respect to $A_X(X \times X)^\sharp \to \mathbf{A}^1 = \operatorname{Spec} k[u]$ at 0. Let $\mathcal{M}$ be the restriction on the complement $\mathbf{G}_m = E - \{0, \infty\}$ of the nearby cycles complex $\Psi\mathcal{F}$ with respect to $(X \times \mathbf{A}^1)' \to \mathbf{A}^1 = \operatorname{Spec} k[u]$ at 0 as in Lemma 1.2.10 and define $\mathcal{H}_{\mathcal{M}} = \mathcal{H}om(\mathrm{pr}_1^* j_{A!} \mathcal{M}, \mathrm{pr}_2^! j_{A!} \mathcal{M})$ on $\mathbf{A}^2$.

**Proposition 2.2.7.** *The distinguished triangle* $\Psi\mathcal{H}^\sharp \to \Psi\mathcal{H}^\sharp|_{TX} \oplus \Psi\mathcal{H}^\sharp|_P \to \Psi\mathcal{H}^\sharp|_T \to$ *defines a distinguished triangle*

$$\Psi\mathcal{H}^\sharp \to e^*(j_* \mathcal{E}nd\mathcal{F} \otimes K_X) \oplus j_{P*}\mathcal{H}_{\mathcal{M}} \to (j_{P*}\mathcal{H}_{\mathcal{M}})_T \to . \tag{2.29}$$

*We have a canonical isomorphism*

$$e^*(j_* \mathcal{E}nd\mathcal{F} \otimes K_X)|_T \to (j_{P*}\mathcal{H}_{\mathcal{M}})_T. \tag{2.30}$$

*Proof.* The tensor product $\Psi\mathcal{H}^\sharp\otimes$ with the exact sequence $0\to\Lambda_{A_X(X\times X)^\sharp_0}\to\Lambda_{TX}\oplus\Lambda_P\to\Lambda_T\to 0$ defines a distinguished triangle $\Psi\mathcal{H}^\sharp\to\Psi\mathcal{H}^\sharp|_{TX}\oplus\Psi\mathcal{H}^\sharp|_P\to\Psi\mathcal{H}^\sharp|_T\to$. By identifying $A=\operatorname{Spec} k[x_1,y_1,u]$ with the fiber product $\operatorname{Spec} k[x_1,u]\times_{\operatorname{Spec} k[u]}\operatorname{Spec} k[y_1,u]$ and by applying [10, Théorème 4.7], we obtain an isomorphism $\Psi\mathcal{H}^\sharp|_{\mathbf{A}^2}\to\mathcal{H}_{\mathcal{M}}$. Since $p_1,p_2\colon A_X(X\times X)^\sharp\to(X\times\mathbf{A}^1)'$ are smooth, we obtain isomorphisms $\Psi\mathcal{H}^\sharp|_P\to j_{P*}\mathcal{H}_{\mathcal{M}}$, $\Psi\mathcal{H}^\sharp|_{TX}\to j_{TX*}e^*(\mathcal{E}nd\,\mathcal{F}\otimes K_U)=e^*(j_*\mathcal{E}nd\mathcal{F}\otimes K_X)$ and

$$\Psi\mathcal{H}^\sharp|_T\to(j_{P*}\mathcal{H}_{\mathcal{M}})|_T\to e^*(j_*\mathcal{E}nd\mathcal{F}\otimes K_X)|_T \tag{2.31}$$

by Lemma 1.2.9. Hence we obtain (2.29) and (2.30). □

The morphism $q\colon A_X(X\times X)^\sharp\to A_X(X\times X)$ induces a morphism $q_0\colon A_X(X\times X)^\sharp_0\to TX=\operatorname{Spec} k[x,v]$ to the tangent bundle. On $A_0=\operatorname{Spec} k[x_1,y_1]$, it is given by $v=y_1-x_1$, $x=0$ and on $B_0=\operatorname{Spec} k[x,w,v]/(xw)$ by the injection $k[x,v]\to k[x,w,v]/(xw)$. The restriction on $TX\subset A_X(X\times X)^\sharp_0$ is the identity. The restriction to $\mathbf{A}^2$ induces the morphism $v\colon\mathbf{A}^2=A_0=\operatorname{Spec} k[x_1,y_1]\to T=\operatorname{Spec} k[v]\subset TX$ defined by $v=y_1-x_1$. This is extended to a morphism $\overline{v}\colon P\to T$.

We rewrite the second component in (2.23) in terms of $\mathcal{M}$.

**Corollary 2.2.8.** 1. *The direct image of* (2.29) *by* $q_0\colon A_X(X\times X)^\sharp_0\to TX$ *defines a distinguished triangle*

$$\nu\mathcal{H}om(j_!\mathcal{F},j_!\mathcal{F})\to e^*(j_*\mathcal{E}nd\mathcal{F}\otimes K_X)\oplus v_*\mathcal{H}_{\mathcal{M}}\to(j_{P*}\mathcal{H}_{\mathcal{M}})_T\to . \tag{2.32}$$

*The second component*

$$v_*\mathcal{H}_{\mathcal{M}}\to(j_{P*}\mathcal{H}_{\mathcal{M}})_T \tag{2.33}$$

*of* (2.32) *is given by the restriction* $v_*\mathcal{H}_{\mathcal{M}}=\overline{v}_*j_{P*}\mathcal{H}_{\mathcal{M}}\to(j_{P*}\mathcal{H}_{\mathcal{M}})_T$.

2. *Taking* $0^!$ *of* (2.29) *and* (2.32), *we obtain distinguished triangles*

$$0^!\Psi\mathcal{H}^\sharp\to j_*\mathcal{E}nd\,\mathcal{F}\oplus(j_Ej_A)_*\mathcal{E}nd\,\mathcal{M}\to\Gamma(\mathbf{G}_{m,\bar{k}},\mathcal{E}nd\,\mathcal{M})_\infty\to, \tag{2.34}$$

$$0^!\nu\mathcal{H}om(j_!\mathcal{F},j_!\mathcal{F})\to j_*\mathcal{E}nd\,\mathcal{F}\oplus\Gamma(\mathbf{G}_{m,\bar{k}},\mathcal{E}nd\,\mathcal{M})_0\to\Gamma(\mathbf{G}_{m,\bar{k}},\mathcal{E}nd\,\mathcal{M})_0\to . \tag{2.35}$$

*The last terms denote the geometrically constant complexes placed at* $\infty=E\cap X\subset A_X(X\times X)^\sharp_0$ *and at* $0=T\cap X\subset TX$.

*Proof.* 1. Since $q\colon A_X(X\times X)^\sharp\to A_X(X\times X)$ is proper, we have a canonical isomorphism $\nu\mathcal{H}om(j_!\mathcal{F},j_!\mathcal{F})\to q_{0*}\Psi\mathcal{H}^\sharp$. Since the restriction of $q_0$ on $TX$ is the identity and that on $\mathbf{A}^2$ is $v=\overline{v}\circ j_P$, the distinguished triangle (2.29) induces (2.32).

2. By the proper base change theorem and [8, (3.2.1)], we have isomorphisms

$$\begin{aligned}&0^!j_{P*}\mathcal{H}_{\mathcal{M}}\to j_{E*}0^!\mathcal{H}_{\mathcal{M}}\to j_{E*}\mathcal{E}nd\,j_{A!}\mathcal{M}\to(j_Ej_A)_*\mathcal{E}nd\,\mathcal{M},\\&0^!v_*\mathcal{H}_{\mathcal{M}}\to\Gamma(\mathbf{A}^1,0^!\mathcal{H}_{\mathcal{M}})\to\Gamma(\mathbf{A}^1_{\bar{k}},\mathcal{E}nd\,j_{A!}\mathcal{M})\to\Gamma(\mathbf{G}_{m,\bar{k}},\mathcal{E}nd\,\mathcal{M}).\end{aligned}$$

Hence (2.34) and (2.35) are induced by (2.29) and (2.32) respectively. □

**Corollary 2.2.9.** *Let* $I_E$ *denote the tame inertia group at* $\infty\in E$. *Let* $M$ *be the representation of* $I_E$ *defined by* $\mathcal{M}$ *and consider the coinvariant quotient* $(\operatorname{End}M)_{I_E}$ *as a trivial* *representation of $I_T$. The morphism $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_{\mathcal{F}}$* (2.17) *and the isomorphism $e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)|_T \to (j_{P*}\mathcal{H}_{\mathcal{M}})_T$* (2.30) *induce commutative diagrams*

(2.36)

$$\begin{tikzcd}
\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \arrow[r] \arrow[d] & e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X) \oplus v_*\mathcal{H}_{\mathcal{M}} \arrow[r] \arrow[d] & (j_{P*}\mathcal{H}_{\mathcal{M}})_T \arrow[r] \arrow[d] & {} \\
\mathcal{N}_{\mathcal{F}} \arrow[r] & e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X) \oplus g_!g^*R^{-1}j_{P*}\mathcal{H}_{\mathcal{M}}[1] \arrow[r] & R^{-1}j_{P*}\mathcal{H}_{\mathcal{M}}[1] \arrow[r] & {},
\end{tikzcd}$$

(2.37)

$$\begin{tikzcd}
0^!\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \arrow[r] \arrow[d] & j_*\mathcal{E}nd\,\mathcal{F} \oplus \Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M})_0 \arrow[r] \arrow[d] & \Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M})_0 \arrow[r] \arrow[d] & {} \\
0^!\mathcal{N}_{\mathcal{F}} \arrow[r] & j_*\mathcal{E}nd\mathcal{F} \oplus \Gamma(I_T, (\mathrm{End}\,M)_{I_E})_0 \arrow[r] & \mathrm{H}^1(I_T, (\mathrm{End}\,M)_{I_E})_0[-1] \arrow[r] & {}
\end{tikzcd}$$

*of distinguished triangles* (2.32), (2.19), (2.35) *and* (2.20)*. The last two terms in each line of* (2.37) *are placed at $0 = T \cap X \subset TX$.*

*Proof.* By the isomorphism $e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)|_T \to (j_{P*}\mathcal{H}_{\mathcal{M}})_T$ (2.30), the distinguished triangles (2.19) and (2.20) are rewritten as the lower lines in (2.36) and (2.37) respectively. Since the morphism $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to \mathcal{N}_{\mathcal{F}}$ (2.17) is induced by $\nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F}) \to e^*(j_*\mathcal{E}nd\mathcal{F} \otimes K_X)$ and since $(v_*\mathcal{H}_{\mathcal{M}})_0 = \nu\mathcal{H}om(j_!\mathcal{F}, j_!\mathcal{F})_0 = 0$, we obtain (2.36) by adjunction. By applying $0^!$ to (2.36) we obtain (2.37). $\square$

We consider the morphism

$$\mathrm{H}^0(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) \to \mathrm{H}^0(I_T, (\mathrm{End}\,M)_{I_E}). \tag{2.38}$$

induced by the middle vertical arrow in the commutative diagram (2.37). We identify $0^!\mathcal{H}_{\mathcal{M}} = j_{A*}\mathcal{E}nd\,\mathcal{M}$ by [8, (3.2.1)]. Then for the left hand side of (2.38), we obtain the first equality in

$$\mathrm{H}^0(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) = \mathrm{H}^0(\mathbf{A}^1_{\bar{k}}, 0^!\mathcal{H}_{\mathcal{M}}) = \mathrm{H}^0(I_E, \mathrm{H}^1(I_T, \mathrm{End}\,M(1)). \tag{2.39}$$

Let $\overline{\eta}_\infty$ be the geometric generic point of the strict localization at $\infty \in E$ and identify the stalk $(0^!\mathcal{H}_{\mathcal{M}})_{\overline{\eta}_\infty}$ with $\mathrm{H}^1(I_T, \mathrm{End}\,M(1))$. Then we obtain the second equality. Since the action of $I_T$ is trivial, the right hand side of (2.38) is identified as

$$\mathrm{H}^0(I_T, (\mathrm{End}\,M)_{I_E}) = (\mathrm{End}\,M)_{I_E} = \mathrm{H}^1(I_E, \mathrm{End}\,M(1)). \tag{2.40}$$

Since we canonically identify $(\mathrm{End}\,F)_{I_X} = (\mathrm{End}\,M)_{I_E}$ as in (2.30), Proposition 2.2.4 is reduced to the following.

**Proposition 2.2.10.** *We identify $\mathrm{H}^0(I_E, \mathrm{H}^1(I_T, \mathrm{End}\,M(1))$ with a direct summand of $\mathrm{H}^1(I_E \times I_T, \mathrm{End}\,M(1))$ by the direct sum decomposition* (2.25) *and define a morphism*

$$\mathrm{H}^0(I_E, \mathrm{H}^1(I_T, \mathrm{End}\,M(1)) \subset \mathrm{H}^1(I_E \times I_T, \mathrm{End}\,M(1)) \longrightarrow \mathrm{H}^1(I_E, \mathrm{End}\,M(1)) \tag{2.41}$$

*to be the pull-back by the diagonal morphism $I_E = \widehat{\mathbf{Z}}'(1) \to I_E \times I_T = \widehat{\mathbf{Z}}'(1)^2$. Then, the diagram*

$$\begin{tikzcd}
\mathrm{H}^0(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) \arrow[r, "(2.38)"] \arrow[d, "(2.39)"'] & \mathrm{H}^0(I_T, (\mathrm{End}\,M)_{I_E}) \arrow[d, "(2.40)"] \\
\mathrm{H}^0(I_E, \mathrm{H}^1(I_T, \mathrm{End}\,M(1))) \arrow[r, "(2.41)"] & \mathrm{H}^1(I_E, \mathrm{End}\,M(1))
\end{tikzcd} \tag{2.42}$$

*is commutative.*

*Proof.* First, we construct a morphism

$$\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) \to \Gamma(I_T, \mathrm{H}^1(I_E, \mathrm{End}\, M(1))) \tag{2.43}$$

inducing the composition of (2.42) via upper right and give its characterization. Let $P_\infty$ be the strict localization of $P$ at $\infty = E \cap T$ and let $V = P_\infty \times_P (\mathbf{A}^2 - \delta(\mathbf{A}^1)) \subset P$. We show that the diagram

(2.44)

$$\begin{tikzcd}
\Gamma(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), \mathcal{H}_{\mathcal{M}}) \arrow[r] \arrow[d] & \Gamma(V, \mathcal{H}_{\mathcal{M}}) \arrow[d] \\
(0^! v_* \mathcal{H}_{\mathcal{M}})_\infty \arrow[r] \arrow[d] & (0^! g_! g_* e^* R^1 j_{P*} \mathcal{E}nd\,\mathcal{M}(1))_\infty[1] \arrow[d] \\
\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M})[1] \arrow[r] & \Gamma(I_T, \mathrm{H}^1(I_E, \mathrm{End}\, M(1)))[1]
\end{tikzcd}$$

is commutative. The top horizontal arrow is the restriction morphism for $V \to \mathbf{A}^2 - \delta(\mathbf{A}^1)$ and the middle horizontal arrow is induced by $v_*\mathcal{H}_{\mathcal{M}} \to (j_{P*}\mathcal{H}_{\mathcal{M}})_T$ (2.33). The upper square is commutative by the functoriality of the boundary morphism $j'_* \to 0^![1]$ for the open immersion $j' \colon \mathbf{A}^2 - \delta(\mathbf{A}^1) \to \mathbf{A}^2$. The lower vertical arrows are canonical isomorphisms and define the bottom horizontal arrow by the commutativity of the lower square. By construction, the bottom horizontal arrow induces the composition of (2.42) via upper right.

The distinguished triangle $\Gamma_{\mathbf{A}^1_{\bar{k}}}(\mathbf{A}^2_{\bar{k}}, -) \to \Gamma(\mathbf{A}^2_{\bar{k}}, -) \to \Gamma(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), -) \to$ induces an isomorphism $\Gamma(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), \mathcal{H}_{\mathcal{M}}) \to \Gamma(\mathbf{A}^1_{\bar{k}}, 0^!\mathcal{H}_{\mathcal{M}})[1] \to \Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M})[1]$ since we have $\Gamma(\mathbf{A}^2_{\bar{k}}, \mathcal{H}_{\mathcal{M}}) = \Gamma(\mathbf{A}^1_{\bar{k}}, D(j_!\mathcal{M})) \otimes \Gamma(\mathbf{A}^1_{\bar{k}}, j_!\mathcal{M}) = 0$ by Lemma 1.2.7.2. Hence the left vertical arrows in (2.44) are isomorphisms. Thus, the bottom horizontal arrow in (2.44) is uniquely determined by the commutativity of the large rectangle.

We identify the tame fundamental group $\pi_1(V)^{\mathrm{tame}}$ with $I_E \times I_T$ and $\Gamma(V, \mathcal{H}_{\mathcal{M}})$ with $\Gamma(I_E \times I_T, \mathrm{End}\, M(1))[2] = \Gamma(I_T, \Gamma(I_E, \mathrm{End}\, M(1)))[2]$ by the composition of the right vertical arrows. Under this identification, the composition of right vertical arrows in (2.44) is induced by $\Gamma(I_E, \mathrm{End}\, M(1)) \to \mathrm{H}^1(I_E, \mathrm{End}\, M(1))[-1]$. We consider the diagram

(2.45)

$$\begin{tikzcd}
\mathrm{H}^{-1}(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), \mathcal{H}_{\mathcal{M}}) \arrow[r] \arrow[d] & \mathrm{H}^{-1}(V, \mathcal{H}_{\mathcal{M}}) \arrow[dr] & \\
\mathrm{H}^0(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) \arrow[r, "(2.39)"] & \mathrm{H}^0(I_E, \mathrm{H}^1(I_T, \mathrm{End}\, M(1))) \arrow[r, "(2.25)"] & \mathrm{H}^1(I_E \times I_T, \mathrm{End}\, M(1)) \arrow[d] \\
 & & \mathrm{H}^1(I_E, \mathrm{End}\, M(1)).
\end{tikzcd}$$

By the characterization of (2.43), it suffices to show the following:

(a) the pentagon is commutative.

(b) the lower right vertical arrow is the pull-back by the diagonal $I_E \to I_E \times I_T$.

Recall that the low right vertical arrow is defined by the condition that the composition $\mathrm{H}^{-1}(V, \mathcal{H}_{\mathcal{M}}) \to \mathrm{H}^1(I_E, \mathrm{End}\, M(1))$ is the composition of right vertical arrows in (2.44).

Let $\mathbf{P}^2_1$ be the strict localization of $\mathbf{P}^2$ at $(1,1,0) \in L = \mathbf{P}^2 - \mathbf{A}^2$ and let $U = \mathbf{P}^2_1 \times_{\mathbf{P}^2} (\mathbf{A}^2 - \delta(\mathbf{A}^1)) \subset \mathbf{P}^2_1$. Let $I_L$ be the tame inertia group at $(1,1,0) \in L$ and let $I_{\mathbf{P}^1}$ be the tame inertia group at $(1,1,0) \in \mathbf{P}^1$ in the closure $\mathbf{P}^1$ of $\delta(\mathbf{A}^1)$. We identify the tame fundamental group $\pi_1(U)^{\mathrm{tame}}$ with $I_{\mathbf{P}^1} \times I_L$. We regard $\operatorname{End} M(1)$ as a representation of $I_{\mathbf{P}^1} \times I_L$ with the canonical action of $I_{\mathbf{P}^1}$ and the trivial action of $I_L$. Then we have an identification

$$\Gamma(U, \mathcal{H}_{\mathcal{M}}) = \Gamma(I_{\mathbf{P}^1} \times I_L, \operatorname{End} M(1))[2]. \tag{2.46}$$

The restriction $P \to \mathbf{P}^2$ of the blow-up at $(1,1,0)$ induces a morphism $V \to U$. We have a commutative diagram

$$\begin{tikzcd}
\pi_1(U)^{\mathrm{tame}} & I_{\mathbf{P}^1} \times I_L \arrow[l] \\
\pi_1(V)^{\mathrm{tame}} \arrow[u] & I_E \times I_T \arrow[l] \arrow[u, "\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}"']
\end{tikzcd} \tag{2.47}$$

of isomorphisms for the tame fundamental groups.

We consider the diagram

(2.48)

$$\begin{tikzcd}
\mathrm{H}^{-1}(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), \mathcal{H}_{\mathcal{M}}) \arrow[r] \arrow[d] & \mathrm{H}^{-1}(U, \mathcal{H}_{\mathcal{M}}) \arrow[r] \arrow[d, "(2.46)"'] & \mathrm{H}^{-1}(V, \mathcal{H}_{\mathcal{M}}) \arrow[d] \\
\mathrm{H}^0(\mathbf{G}_m, \mathcal{E}nd\,\mathcal{M}) \arrow[r] & \mathrm{H}^1(I_{\mathbf{P}^1} \times I_L, \operatorname{End} M(1)) \arrow[r] & \mathrm{H}^1(I_E \times I_T, \operatorname{End} M(1)) \arrow[d] \\
 & & \mathrm{H}^1(I_E, \operatorname{End} M(1)).
\end{tikzcd}$$

The upper horizontal arrows are induced by $V \to U \to \mathbf{A}^2 - \delta(\mathbf{A}^1)$. Similarly as (2.39), we identify $\mathrm{H}^0(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M}) = \mathrm{H}^0(I_{\mathbf{P}^1}, \mathrm{H}^1(I_L, \operatorname{End} M(1))$ and define the lower left horizontal arrow to be the inclusion of the direct summand $\mathrm{H}^0(I_{\mathbf{P}^1}, \mathrm{H}^1(I_L, \operatorname{End} M(1)) \subset \mathrm{H}^1(I_{\mathbf{P}^1} \times I_L, \operatorname{End} M(1))$ in Lemma 2.2.6.1. Then, the left square is commutative by Lemma 2.2.11.2 below. The lower right horizontal arrow is induced by the left vertical arrow of (2.47) and the right square is commutative. Hence the large rectangle and consequently (a) the pentagon in (2.45) are commutative.

By the characterization of (2.43), this implies that the composition of the lower line in (2.45) is the same as that in (2.48). The lower right vertical arrow is induced by the inclusion $I_E \to I_E \times I_T$. By Lemma 2.2.6.2 and (2.47), the composition $\mathrm{H}^1(I_{\mathbf{P}^1} \times I_L, \operatorname{End} M(1)) \to \mathrm{H}^1(I_E, \operatorname{End} M(1))$ is the morphism induced by the composition $I_E \to I_E \times I_T \to I_{\mathbf{P}^1} \times I_L$ that equals (b) the diagonal $I_E \to I_{\mathbf{P}^1} \times I_L$. Hence, it suffices to prove Lemma 2.2.11 below. □

**Lemma 2.2.11.** 1. *Let* $j_{\mathbf{P}^2} \colon \mathbf{A}^2 \to \mathbf{P}^2$ *and* $j_{\mathbf{A}^2} \colon \mathbf{A}^2 - \delta(\mathbf{A}^1) \to \mathbf{A}^2$ *be the open immersions and let* $1 \colon (1,1,0) \to L = \mathbf{P}^2 - \mathbf{A}^2$ *be the immersion of the point* $(1,1,0)$ *to the line at infinity. We identify* $\delta^! \mathcal{H}_{\mathcal{M}} = j_{\mathbf{A}^1 *} \mathcal{E}nd\,\mathcal{M}$ *and* $\Gamma(U, \mathcal{H}_{\mathcal{M}}) = \Gamma(I_{\mathbf{P}^1} \times I_L, \operatorname{End} M(1))[2]$. *Then the diagram*

(2.49)

$$\begin{tikzcd}
\Gamma(\mathbf{A}^2_{\bar{k}} - \delta(\mathbf{A}^1_{\bar{k}}), \mathcal{H}_{\mathcal{M}}) \arrow[r] \arrow[d] & \Gamma(L_{\bar{k}}, (j_{\mathbf{P}^2 *} j_{\mathbf{A}^2 *} j^*_{\mathbf{A}^2} \mathcal{H}_{\mathcal{M}})|_L) \arrow[r] \arrow[d] & \Gamma(U, \mathcal{H}_{\mathcal{M}}) \arrow[d] \\
\Gamma(\mathbf{G}_{m,\bar{k}}, \mathcal{E}nd\,\mathcal{M})[1] \arrow[r] & (1^!(j_{\mathbf{P}^2 *} \mathcal{H}_{\mathcal{M}}|_L))_{\bar{1}}[1] \arrow[r] & \Gamma(I_{\mathbf{P}^1}, \mathrm{H}^1(I_L, \operatorname{End} M(1))[1])
\end{tikzcd}$$

*is commutative. The arrows in the left square and the lower horizontal arrows are isomorphisms.*

2. *The restriction morphism*

$$\mathrm{H}^{-1}(\mathbf{A}^2_{\bar k} - \delta(\mathbf{A}^1_{\bar k}), \mathcal{H}_{\mathcal{M}}) \to \mathrm{H}^{-1}(U, \mathcal{H}_{\mathcal{M}}) = \mathrm{H}^0(I_{\mathbf{P}^1}, \mathrm{H}^1(I_L, \operatorname{End} M(1))) \oplus \mathrm{H}^1(I_{\mathbf{P}^1}, \operatorname{End} M(1))$$

*is the direct sum of the isomorphism to the first component induced by* (2.49) *and the* 0*-morphism to the second component.*

*Proof.* 1. The commutativity is clear from the construction. We show the isomorphisms. We have $\Gamma(\mathbf{A}^2_{\bar k}, \mathcal{H}_{\mathcal{M}}) = \Gamma(\mathbf{A}^1_{\bar k}, Dj_!\mathcal{M}) \otimes \Gamma(\mathbf{A}^1_{\bar k}, j_!\mathcal{M}) = 0$ by Lemma 1.2.7.2. Hence the left vertical arrow in (2.49) is an isomorphism. We have a canonical isomorphism $\Gamma(\mathbf{A}^2_{\bar k} - \delta(\mathbf{A}^1_{\bar k}), \mathcal{H}_{\mathcal{M}}) \to \Gamma(\mathbf{A}^2_{\bar k}, j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}}) \to \Gamma(\mathbf{P}^2_{\bar k}, j_{\mathbf{P}^2*}j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}})$. We have $\Gamma_c(\mathbf{A}^2_{\bar k}, \mathcal{H}_{\mathcal{M}}) = \Gamma_c(\mathbf{A}^1_{\bar k}, Dj_!\mathcal{M}) \otimes \Gamma_c(\mathbf{A}^1_{\bar k}, j_!\mathcal{M}) = \Gamma_c(\mathbf{A}^1_{\bar k}, j_*D\mathcal{M}) \otimes \Gamma_c(\mathbf{A}^1_{\bar k}, j_!\mathcal{M}) = 0$ and $\Gamma_c(\mathbf{A}^1_{\bar k}, \delta^!\mathcal{H}_{\mathcal{M}}) = \Gamma_c(\mathbf{A}^1_{\bar k}, j_{\mathbf{A}^1*}\mathcal{E}nd\,\mathcal{M}) = 0$ further by Lemma 1.2.7.2. Hence we have $\Gamma_c(\mathbf{A}^2_{\bar k}, j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}}) = 0$ and, for $A = (j_{\mathbf{P}^2*}j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}})|_L)$, the restriction morphism $\Gamma(\mathbf{P}^2_{\bar k}, j_{\mathbf{P}^2*}j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}}) \to \Gamma(L_{\bar k}, A)$ is an isomorphism. By the proper base change theorem, the lower left horizontal arrow is an isomorphism. Since the action of $I_L$ on $\operatorname{End} M$ is trivial, the composition of the lower line is an isomorphism.

2. By 1., it suffices to show that the morphism to the second component is 0. Let $0 = (0,1,0) \in L$ (resp. $1 = (1,1,0) \in L$) be the intersection of the closures of $0 \times \mathbf{A}^1$ (resp. of $\delta(\mathbf{A}^1)$) and $L$ in $\mathbf{P}^2$ and let $j_0\colon L - \{0\} \to L$ and let $j_1\colon L - \{1\} \to L$ be the open immersions. Then, the canonical morphisms $(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L \to j_{0*}j_0^*(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L$ and $A = (j_{\mathbf{P}^2*}j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}})|_L \to j_{1*}j_1^*((j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L) \to j_{1*}j_1^*\Lambda \otimes ((j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L)$ are isomorphisms. We consider the canonical filtration on $j_0^*(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L$ and the induced filtrations on $(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L$ and on $A = (j_{\mathbf{P}^2*}j_{\mathbf{A}^2*}j^*_{\mathbf{A}^2}\mathcal{H}_{\mathcal{M}})|_L$. Then we have $\mathrm{Gr}^q A = j_{1*}j_1^*\Lambda \otimes j_{0*}\mathcal{H}^q(j_0^*(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L)[-q]$. Since $\Gamma(L, j_{0*}\mathcal{H}^q(j_0^*(j_{\mathbf{P}^2*}\mathcal{H}_{\mathcal{M}})|_L)) = 0$ by Lemma 1.2.7.2, the complex $\Gamma(L, \mathrm{Gr}^q A[q])$ is acyclic except at degree 1. Since the composition $\mathrm{H}^{-1}(L_{\bar k}, A) \to \mathrm{H}^{-1}(U, \mathcal{H}_{\mathcal{M}}) \to \mathrm{H}^1(I_{\mathbf{P}^1}, \operatorname{End} M(1))$ factors through $\mathrm{H}^{-1}(L_{\bar k}, \mathrm{Gr}^{-1} A) = 0$, the assertion follows. □

## 2.3 Proper pushforward

First, we show an equality for the direct image of singular support by closed immersions.

**Lemma 2.3.1.** *Let $i\colon X \to Y$ be a closed immersion of smooth schemes over $k$. Then, we have an equality*

$$SS_\mu i_*\mathcal{F} = i_\circ SS_\mu \mathcal{F} \tag{2.50}$$

*of closed subsets of $T^*Y$.*

*Proof.* By Lemma 1.2.5.1, the morphism $A_X(X \times X) \to A_Y(Y \times Y)$ is a closed immersion. Hence the base change morphism $\nu\mathcal{H}om_Y(i_*\mathcal{F}, i_*\mathcal{F}) \to i_*\nu\mathcal{H}om_X(\mathcal{F}, \mathcal{F})$ is an isomorphism. Let

$$T^*X \xleftarrow{a} T^*Y \times_Y X \xrightarrow{b} T^*Y$$

be the canonical morphisms. Then the isomorphism $\nu\mathcal{H}om_Y(i_*\mathcal{F}, i_*\mathcal{F}) \to i_*\nu\mathcal{H}om_X(\mathcal{F}, \mathcal{F})$ induces an isomorphism $\mu\mathcal{H}om_Y(i_*\mathcal{F}, i_*\mathcal{F}) \to b_*a^*\mu\mathcal{H}om_X(\mathcal{F}, \mathcal{F})$ by Proposition 1.1.8. □

**Lemma 2.3.2.** *Let*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "c"'] & D \arrow[d, "d"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{2.51}$$

*be a commutative diagram of separated schemes of finite type over $k$.*

1. *Define $f_!c_!c^! \to d_!d^!f_!$ to be the adjoint of the composition of the canonical isomorphism $f_!c_!c^!f^! \to d_!g_!g^!d^!$ and the morphism $d_!g_!g^!d^! \to d_!d^!$ induced by $\mathrm{adj}_g$. Then the diagram*

$$\begin{tikzcd} f_!c_!c^! \arrow[r] \arrow[d, "\mathrm{adj}_c"'] & d_!d^!f_! \arrow[d, "\mathrm{adj}_d"] \\ f_! \arrow[r, equal] & f_!. \end{tikzcd} \tag{2.52}$$

*is commutative.*

2. *Define $f^*c_*c^* \leftarrow d_*d^*f_*$ to be the adjoint of the composition of the canonical isomorphism $f_*c_*c^*f^* \to d_*g_*g^*d^*$ and the morphism $d_*g_*g^*d^* \to d_*d^*$ induced by $\mathrm{adj}_g$. Then the diagram*

$$\begin{tikzcd} f_* \arrow[r, equal] \arrow[d, "\mathrm{adj}_c"'] & f_* \arrow[d, "\mathrm{adj}_d"] \\ f_*c_*c^* & d_*d^*f_* \arrow[l] \end{tikzcd} \tag{2.53}$$

*is commutative.*

*Proof.* 1. By the commutative diagram (2.51), we have a commutative diagram

$$\begin{tikzcd} f_!c_!c^!f^! \arrow[r, "\mathrm{adj}_g"] \arrow[d, "\mathrm{adj}_c"'] & d_!d^! \arrow[d, "\mathrm{adj}_d"] \\ f_!f^! \arrow[r, "\mathrm{adj}_f"] & 1_X. \end{tikzcd}$$

By taking the adjoint, we obtain (2.52).

2. is proved similarly as 1. ☐

**Proposition 2.3.3.** *Let $f\colon X \to Y$ be a proper morphism of smooth schemes over $k$ and $\mathcal{F} \in D^b_{\mathrm{ctf}}(X, \Lambda)$.*

1. ([1, Proposition 2.1.6], [8, Théorème 4.4, Corollaire 4.8]) *We have*

$$cc_Y f_*\mathcal{F} = f_* cc_X \mathcal{F} \tag{2.54}$$

*in $\mathrm{H}^0(Y, \mathcal{K}_Y)$.*

2. *We have*

$$CC_\mu f_*\mathcal{F} = f_! CC_\mu \mathcal{F} \tag{2.55}$$

*in $\mathrm{H}^0_{f_\circ(SS_\mu\mathcal{F}) \cup SS_\mu f_*\mathcal{F}}(T^*Y, e_Y^{\vee *}\mathcal{K}_Y)$.*

*Proof.* We apply Lemma 2.3.2 to the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "\delta_X"'] & X \arrow[r, "f"] \arrow[d, "\gamma_f"'] & Y \arrow[d, "\delta_Y"] \\
X \times X \arrow[r, "1_X \times f"] & X \times Y \arrow[r, "f \times 1_Y"] & Y \times Y
\end{tikzcd} \tag{2.56}$$

as follows. Set

$$\begin{aligned} \mathcal{H}_X &= \mathcal{H}om_{X\times X}(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{F}), \quad \mathcal{H}_f = \mathcal{H}om_{X\times Y}(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^! f_*\mathcal{F}), \\ \mathcal{H}_Y &= \mathcal{H}om_{Y\times Y}(\mathrm{pr}_1^* f_*\mathcal{F}, \mathrm{pr}_2^! f_*\mathcal{F}) \end{aligned}$$

and identify them with

$$\boxtimes_X = D_X(\mathcal{F}) \boxtimes \mathcal{F}, \ \boxtimes_f = D_X(\mathcal{F}) \boxtimes f_*\mathcal{F}, \ \boxtimes_Y = D_Y(f_*\mathcal{F}) \boxtimes f_*\mathcal{F}$$

by the canonical isomorphisms [8, (3.1.1)].

By applying Lemma 2.3.2.1 and by [8, (3.2.1)], we obtain commutative diagrams

$$\begin{tikzcd}
(1_X \times f)_! \delta_{X!} \mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \arrow[r] \arrow[d] & \gamma_{f!} \mathcal{H}om_X(\mathcal{F}_1, f^! f_! \mathcal{F}_2) \arrow[d] \\
(1_X \times f)_! \mathcal{H}_X \arrow[r, "\simeq"] & \mathcal{H}_f,
\end{tikzcd} \tag{2.57}$$

$$\begin{tikzcd}
(f \times 1_Y)_! \gamma_{f!} \mathcal{H}om_X(\mathcal{F}_1, f^! f_! \mathcal{F}_2) \arrow[r, "\simeq"] \arrow[d] & \delta_{Y!} \mathcal{H}om_Y(f_!\mathcal{F}_1, f_!\mathcal{F}_2) \arrow[d] \\
(f \times 1_Y)_! \mathcal{H}_f \arrow[r, "\simeq"] & \mathcal{H}_Y.
\end{tikzcd} \tag{2.58}$$

Further if $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$, defining vertical arrows by $1_{\mathcal{F}}$, $\mathrm{adj}\colon \mathcal{F} \to f^! f_! \mathcal{F}$ and $1_{f_!\mathcal{F}}$, we obtain commutative diagrams

$$\begin{tikzcd}
(1_X \times f)_! \delta_{X!} \Lambda \arrow[r, "\simeq"] \arrow[d, "1_{\mathcal{F}}"'] & \gamma_{f!} \Lambda \arrow[d, "\mathrm{adj}"] \\
(1_X \times f)_! \delta_{X!} \mathcal{H}om_X(\mathcal{F}, \mathcal{F}) \arrow[r, "\mathrm{adj}"] & \gamma_{f!} \mathcal{H}om_X(\mathcal{F}, f^! f_! \mathcal{F}),
\end{tikzcd} \tag{2.59}$$

$$\begin{tikzcd}
(f \times 1_Y)_! \gamma_{f!} \Lambda \arrow[d, "\mathrm{adj}"'] & \delta_{Y!} \Lambda \arrow[l] \arrow[d, "1_{f_!\mathcal{F}}"] \\
(f \times 1_Y)_! \gamma_{f!} \mathcal{H}om_X(\mathcal{F}, f^! f_! \mathcal{F}) \arrow[r, "\simeq"] & \delta_{Y!} \mathcal{H}om_Y(f_!\mathcal{F}, f_!\mathcal{F}).
\end{tikzcd} \tag{2.60}$$

By applying Lemma 2.3.2.2, we obtain commutative diagrams

$$\begin{tikzcd}
(1_X \times f)_* \boxtimes_X \arrow[r, "\simeq"] \arrow[d] & \boxtimes_f \arrow[d] \\
(1_X \times f)_* \delta_{X*}(D_X\mathcal{F}_1 \otimes \mathcal{F}_2) & \gamma_{f*}(D_X\mathcal{F}_1 \otimes f^* f_* \mathcal{F}_2), \arrow[l, "\mathrm{adj}"']
\end{tikzcd} \tag{2.61}$$

$$\begin{tikzcd}
(f\times 1_Y)_*\boxtimes_f \arrow[r, "\simeq"] \arrow[d] & \boxtimes_Y \arrow[d] \\
(f\times 1_Y)_*\gamma_{f*}(D_X\mathcal{F}_1\otimes f^*f_*\mathcal{F}_2) & \delta_{Y*}(D_Yf_*\mathcal{F}_1\otimes f_*\mathcal{F}_2). \arrow[l, "\simeq"']
\end{tikzcd} \tag{2.62}$$

The lower line in (2.62) is induced by the isomorphism $f_*(D_X\mathcal{F}_1\otimes f^*f_*\mathcal{F}_2)\to f_*D_X\mathcal{F}_1\otimes f_*\mathcal{F}_2$ of projection formula and the canonical isomorphism $f_*D_X\mathcal{F}_1\to D_Yf_!\mathcal{F}_1\to D_Yf_*\mathcal{F}_1$.

Further if $\mathcal{F}=\mathcal{F}_1=\mathcal{F}_2$, defining vertical arrows by $\mathrm{ev}_X\colon D_X\mathcal{F}\otimes\mathcal{F}\to\mathcal{K}_X$, $\mathrm{ev}_X\circ(1\otimes\mathrm{adj})\colon D_X\mathcal{F}\otimes f^*f_*\mathcal{F}\to D_X\mathcal{F}\otimes\mathcal{F}\to\mathcal{K}_X$ and $\mathrm{ev}_Y\colon D_Yf_*\mathcal{F}\otimes f_*\mathcal{F}\to\mathcal{K}_Y$, we obtain commutative diagrams

$$\begin{tikzcd}
(1_X\times f)_*\delta_{X*}(D_X\mathcal{F}\otimes\mathcal{F}) \arrow[d, "\mathrm{ev}"'] & \gamma_{f*}(D_X\mathcal{F}\otimes f^*f_*\mathcal{F}) \arrow[l, "\mathrm{adj}"'] \arrow[d, "\mathrm{ev}\circ(1\otimes\mathrm{adj})"] \\
(1_X\times f)_*\delta_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & \gamma_{f*}\mathcal{K}_X,
\end{tikzcd} \tag{2.63}$$

$$\begin{tikzcd}
(f\times 1_Y)_*\gamma_{f*}(D_X\mathcal{F}\otimes f^*f_*\mathcal{F}) \arrow[r] \arrow[d, "\mathrm{ev}\circ\mathrm{adj}"'] & \delta_{Y*}(D_Yf_*\mathcal{F}\otimes f_*\mathcal{F}) \arrow[d, "\mathrm{ev}"] \\
(f\times 1_Y)_*\gamma_{f*}\mathcal{K}_X \arrow[r, "\mathrm{adj}"] & \delta_{Y*}\mathcal{K}_Y.
\end{tikzcd} \tag{2.64}$$

The lower line in (2.64) is induced by the adjunction $f_*\mathcal{K}_X=f_!f^!\mathcal{K}_Y\to\mathcal{K}_Y$.

Commutative diagrams (2.57), (2.59), (2.61) and (2.63) define a commutative diagram

$$\begin{tikzcd}
(1_X\times f)_*\delta_{X!}\Lambda \arrow[r, "\simeq"] \arrow[d, "1_{\mathcal{F}}"'] & \gamma_{f!}\Lambda \arrow[d, "\mathrm{adj}"] \\
(1_X\times f)_*\mathcal{H}_X \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}"'] & \mathcal{H}_f \arrow[d, "\mathrm{ev}\circ\mathrm{adj}"] \\
(1_X\times f)_*\delta_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & \gamma_{f*}\mathcal{K}_X.
\end{tikzcd} \tag{2.65}$$

Similarly, commutative diagrams (2.58), (2.60), (2.62) and (2.64) define a commutative diagram

$$\begin{tikzcd}
(f\times 1_Y)_*\gamma_{f!}\Lambda \arrow[d, "\mathrm{adj}"'] & \delta_{Y!}\Lambda \arrow[l] \arrow[d, "1_{f_*\mathcal{F}}"] \\
(f\times 1_Y)_*\mathcal{H}_f \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}\circ\mathrm{adj}"'] & \mathcal{H}_Y \arrow[d, "\mathrm{ev}"] \\
(f\times 1_Y)_*\gamma_{f*}\mathcal{K}_X \arrow[r] & \delta_{Y*}\mathcal{K}_Y.
\end{tikzcd} \tag{2.66}$$

The diagrams (2.65) and (2.66) imply (2.54).

The commutative diagram (2.56) induces a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "1_X"] \arrow[d, "0_X"'] & X \arrow[r, "f"] \arrow[d, "0_f"'] & Y \arrow[d, "0_Y"] \\
TX \arrow[r, "a"] & TY\times_Y X \arrow[r, "b"] & TY.
\end{tikzcd} \tag{2.67}$$

The vertical arrows are the 0-sections. We apply $\nu_{X/X\times Y}$ to (2.65) and apply the morphism $a_!\nu_{X/X\times X} \to \nu_{X/X\times Y}(1_X \times f)_!$ of functors (1.55) to the left column. Then, we obtain a commutative diagram

$$\begin{tikzcd}
a_!0_{X!}\Lambda \arrow[r, "\simeq"] \arrow[d] & 0_{f!}\Lambda \arrow[d] \\
a_!\nu_{X/X\times X}\mathcal{H}_X \arrow[r] \arrow[d] & \nu_{X/X\times Y}\mathcal{H}_f \arrow[d] \\
a_!0_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & 0_{f*}\mathcal{K}_X.
\end{tikzcd}\tag{2.68}$$

Since the arrows in (2.56) are proper and since the right square is cartesian, the base change morphism $\nu_{Y/Y\times Y}(f\times 1_Y)_* \to b_*\nu_{X/X\times Y}$ (1.54) is an isomorphism. Hence applying the functor $\nu_{Y/Y\times Y}$ to (2.66) defines a commutative diagram

$$\begin{tikzcd}
b_*0_{f!}\Lambda \arrow[d] & 0_{Y!}\Lambda \arrow[l] \arrow[d] \\
b_*\nu_{X/X\times Y}\mathcal{H}_f \arrow[r, "\simeq"] \arrow[d] & \nu_{Y/Y\times Y}\mathcal{H}_Y \arrow[d] \\
b_*0_{f*}\mathcal{K}_X \arrow[r] & 0_{Y*}\mathcal{K}_Y.
\end{tikzcd}\tag{2.69}$$

Let $a^\vee\colon T^*Y\times_Y X \to T^*X$ be the dual of $a$ and $b^\vee\colon T^*Y\times_Y X\to T^*Y$ be the canonical morphism. Let $e_X^\vee\colon T^*X\to X$, $e_f^\vee\colon T^*Y\times_Y X\to X$ and $e_Y^\vee\colon T^*Y\to Y$ be the projections. By applying the Fourier transform to (2.68) and the isomorphism (1.21) to the left column and using the isomorphisms (1.4), we obtain a commutative diagram

$$\begin{tikzcd}
a^{\vee *}\Lambda \arrow[d] & \Lambda \arrow[l, "\simeq"'] \arrow[d] \\
a^{\vee *}\mu_{X/X\times X}\mathcal{H}_X \arrow[r] \arrow[d] & \mu_{X/X\times Y}\mathcal{H}_f \arrow[d] \\
a^{\vee *}e_X^{\vee *}\mathcal{K}_X & e_f^{\vee *}\mathcal{K}_X. \arrow[l, "\simeq"']
\end{tikzcd}\tag{2.70}$$

Similarly by applying the Fourier transform to (2.69) and the isomorphism (1.32) to the left column and using the isomorphisms (1.4),

$$\begin{tikzcd}
b^\vee_*\Lambda \arrow[d] & \Lambda \arrow[l] \arrow[d] \\
b^\vee_*\mu_{X/X\times Y}\mathcal{H}_f \arrow[r] \arrow[d] & \mu_{Y/Y\times Y}\mathcal{H}_Y \arrow[d] \\
b^\vee_*e_f^{\vee *}\mathcal{K}_X \arrow[r] & e_Y^{\vee *}\mathcal{K}_Y.
\end{tikzcd}\tag{2.71}$$

Combining these diagrams, we obtain the equality (2.55). □

**Corollary 2.3.4.** *Let $X$ be a proper smooth scheme over $k$ and $\mathcal{F} \in D_{\mathrm{ctf}}(X, \Lambda)$. Then, we have*

$$\chi(X_{\bar{k}}, \mathcal{F}) = (CC_\mu \mathcal{F}, T^*_X X)_{T^*X} \tag{2.72}$$

*in $\Lambda$. The right hand side denotes the image of $CC_\mu \mathcal{F}$ by the composition*

$$\mathrm{H}^0_{SS_\mu \mathcal{F}}(T^*X, e^{\vee *}K_X) \xrightarrow{0^*_X} \mathrm{H}^0(X, K_X) \xrightarrow{\mathrm{Tr}_{X/k}} \Lambda.$$

*Proof.* Let $f\colon X \to Y = \operatorname{Spec} k$ be the canonical morphism. Then, we have $\chi(X_{\bar{k}}, \mathcal{F}) = CC_\mu f_* \mathcal{F}$. Since $(CC_\mu \mathcal{F}, T^*_X X)_{T^*X} = f_! CC_\mu \mathcal{F}$, the equality (2.72) follows from Proposition 2.3.3.2. □

## 2.4 Smooth pullback

**Lemma 2.4.1.** *Let $h\colon W \to X$ be a smooth morphism of smooth schemes over $k$. Then, we have an equality*

$$SS_\mu h^* \mathcal{F} = h^\circ SS_\mu \mathcal{F} \tag{2.73}$$

*of closed subsets of $T^*W$.*

*Proof.* Let

$$T^*X \xleftarrow{b} T^*X \times_X W \xrightarrow{a} T^*W \tag{2.74}$$

be the canonical morphisms. By Lemma 1.2.5.2, the morphism $A_X(W \times W) \to A_X(X \times X)$ is smooth. Hence the base change morphism $a^! b^* \nu \mathcal{H}om_X(\mathcal{F}, \mathcal{F}) \to \nu \mathcal{H}om_W(h^! \mathcal{F}, h^! \mathcal{F})$ is an isomorphism by the smooth base change theorem. This induces an isomorphism $a^\vee_! b^{\vee *} \mu \mathcal{H}om_X(\mathcal{F}, \mathcal{F}) \to \mu \mathcal{H}om_W(h^! \mathcal{F}, h^! \mathcal{F}) = \mu \mathcal{H}om_W(h^* \mathcal{F}, h^* \mathcal{F})$ by Corollary 1.1.9.1. □

**Lemma 2.4.2.** *Let*

$$\begin{tikzcd} C \arrow[d, "c"'] & D \arrow[l, "g"'] \arrow[d, "d"] \\ X & W \arrow[l, "h"] \end{tikzcd} \tag{2.75}$$

*be a commutative diagram of separated schemes of finite type over $k$.*

1. *Define $h^! c_! c^! \leftarrow d_! d^! h^!$ to be the adjoint of the composition of the canonical isomorphism $h_! d_! d^! h^! \to c_! g_! g^! c^!$ and the morphism $c_! g_! g^! c^! \to c_! c^!$ induced by $\mathrm{adj}_g$. Then the diagram*

$$\begin{tikzcd} h^! c_! c^! \arrow[d, "\mathrm{adj}_c"'] & d_! d^! h^! \arrow[l] \arrow[d, "\mathrm{adj}_d"] \\ h^! \arrow[r, equal] & h^!. \end{tikzcd} \tag{2.76}$$

*is commutative.*

2. *Define $h^*c_*c^* \to d_*d^*h^*$ to be the adjoint of the composition of the canonical isomorphism $c_*g_*g^*c^* \to h_*d_*d^*h^*$ and the morphism $c_*c^* \to c_*g_*g^*c^*$ induced by $\mathrm{adj}_g$. Then the diagram*

$$\begin{tikzcd}
h^* \arrow[r, equal] \arrow[d, "\mathrm{adj}_c"'] & h^* \arrow[d, "\mathrm{adj}_d"] \\
h^*c_*c^* \arrow[r] & d_*d^*h^*
\end{tikzcd} \tag{2.77}$$

*is commutative.*

*Proof.* Similar to Lemma 2.3.2. □

Let $h\colon W \to X$ be a smooth morphism of relative dimension $r = \dim W - \dim X$ of smooth schemes over $k$. Let (2.74) be the canonical morphisms of cotangent bundles and let $e_X^\vee\colon T^*X \to X$, $e_h^\vee\colon T^*X \times_X W \to W$ and $e_W^\vee\colon T^*W \to W$ be the projections. For a closed conical subset $S \subset T^*X$, define a morphism

$$h^!\colon \mathrm{H}^0_S(T^*X, e_X^*K_X) \to \mathrm{H}^0_{h^\circ S}(T^*W, e_W^*K_W) \tag{2.78}$$

to be $(-1)^r$-times the composition of $b^{\vee *}\colon \mathrm{H}^0_S(T^*X, e_X^{\vee *}K_X) \to \mathrm{H}^0_{b^{-1}S}(T^*X \times_X W, b^*e_X^{\vee *}K_X)$ and $a_*^\vee\colon \mathrm{H}^0_{b^{-1}S}(T^*X \times_X W, b^*e_X^{\vee *}K_X) \to \mathrm{H}^0_{a(b^{-1}S)}(T^*W, e_W^{\vee *}K_W)$. Note that the definition of $h^!$ involves the sign $(-1)^{\dim X - \dim W}$.

**Lemma 2.4.3.** *Let $h\colon W \to X$ be a smooth morphism of relative dimension $r = \dim W - \dim X$ of smooth schemes over $k$. Define a morphism*

$$h^!\colon \mathrm{H}^0(X, K_X) \to \mathrm{H}^0(W, K_W) \tag{2.79}$$

*to be the composition of $h^*\colon \mathrm{H}^0(X, K_X) \to \mathrm{H}^0(W, h^*K_X)$ and the cup-product with the top Chern class $c_r(TW/X) = (-1)^r c_r(T^*X/W) \in \mathrm{H}^0(W, h^*\Lambda)$ of the relative tangent bundle. Then, the diagram*

$$\begin{tikzcd}
\mathrm{H}^0_S(T^*X, e_X^{\vee *}K_X) \arrow[r, "h^!"] \arrow[d] & \mathrm{H}^0_{h^\circ S}(T^*W, e_W^{\vee *}K_W) \arrow[d] \\
\mathrm{H}^0(X, K_X) \arrow[r, "h^!"] & \mathrm{H}^0(W, K_W)
\end{tikzcd} \tag{2.80}$$

*is commutative. The vertical arrows are defined by the pull-back by the 0-sections.*

*Proof.* It suffices to show that the diagram

$$\begin{tikzcd}
\mathrm{H}^0_S(T^*X, e_X^{\vee *}K_X) \arrow[r, "b^{\vee *}"] \arrow[d] & \mathrm{H}^0_{b^{-1}S}(T^*W, e_h^{\vee *}K_X) \arrow[r, "(-1)^r a_*^\vee"] \arrow[d] & \mathrm{H}^0_{h^\circ S}(T^*W, e_W^{\vee *}K_W) \arrow[d] \\
\mathrm{H}^0(X, K_X) \arrow[r, "h^*"] & \mathrm{H}^0(W, h^*K_X) \arrow[r, "c_r(TY/X)"] & \mathrm{H}^0(W, K_W)
\end{tikzcd}$$

is commutative. The commutativity of the left square is clear from the functoriality. By the exact sequence $0 \to h^*\Omega^1_X \to \Omega^1_W \to \Omega^1_{W/X} \to 0$, the morphism $a_*\colon \mathrm{H}^0_{b^{-1}S}(T^*W, e_h^{\vee *}K_X) \to \mathrm{H}^0_{h^\circ S}(T^*W, e_W^{\vee *}K_W)$ is compatible with the cup-product $\cup c_r(T^*Y/X)\colon \mathrm{H}^0(W, h^*K_X) \to \mathrm{H}^0(W, K_W)$. Hence the assertion follows. □

**Proposition 2.4.4.** *Let $h\colon W \to X$ be a smooth morphism of smooth schemes over $k$ and $\mathcal{F} \in D^b_{\mathrm{ctf}}(X, \Lambda)$. Then, we have*

$$CC_\mu h^*\mathcal{F} = h^! CC_\mu \mathcal{F} \tag{2.81}$$

*in* $\mathrm{H}^0_{h^\circ(SS_\mu \mathcal{F})}(T^*W, e_W^{\vee *}\mathcal{K}_W)$ *and*

$$cc_W h^*\mathcal{F} = h^! cc_X \mathcal{F} \tag{2.82}$$

*in* $\mathrm{H}^0(W, \mathcal{K}_W)$.

*Proof.* We apply Lemma 2.4.2 to the commutative diagram

$$\begin{tikzcd} X \arrow[d, "\delta_X"'] & W \arrow[l, "h"'] \arrow[d, "\gamma_h"'] & W \arrow[l, "1_W"'] \arrow[d, "\delta_W"] \\ X \times X & W \times X \arrow[l, "h\times 1_X"'] & W \times W \arrow[l, "1_W \times h"'] \end{tikzcd} \tag{2.83}$$

as follows. Set

$$\begin{aligned} \mathcal{H}_X &= \mathcal{H}om_{X\times X}(\mathrm{pr}_1^*\mathcal{F}, \mathrm{pr}_2^!\mathcal{F}), \qquad \mathcal{H}_h = \mathcal{H}om_{W\times X}(\mathrm{pr}_1^* h^!\mathcal{F}, \mathrm{pr}_2^!\mathcal{F}), \\ \mathcal{H}_W &= \mathcal{H}om_{W\times W}(\mathrm{pr}_1^* h^!\mathcal{F}, \mathrm{pr}_2^! h^!\mathcal{F}) \end{aligned}$$

and identify them with

$$\boxtimes_X = D_X(\mathcal{F}) \boxtimes \mathcal{F},\ \boxtimes_h = D_W(h^!\mathcal{F}) \boxtimes \mathcal{F},\ \boxtimes_W = D_W(h^!\mathcal{F}) \boxtimes h^!\mathcal{F}$$

by the canonical isomorphisms [8, (3.1.1)]. We have canonical isomorphisms $(h\times 1_X)^*\mathcal{H}_X \to \mathcal{H}_h$ and $(1_X \times h)^!\mathcal{H}_h \to \mathcal{H}_W$. We consider the commutative diagram

$$\begin{tikzcd} (1_W \times h)^! \mathcal{H}_h \arrow[r] & \mathcal{H}_W \\ (1_W \times h)^* \boxtimes_h \otimes \mathrm{pr}_2^* h^! \Lambda \arrow[r] \arrow[u] & \boxtimes_W \arrow[u] \end{tikzcd} \tag{2.84}$$

where the upper horizontal arrow is an isomorphism.

By applying Lemma 2.4.2.1 and by [8, (3.2.1)], we obtain commutative diagrams

$$\begin{tikzcd} (h\times 1_X)^! \delta_{X!} \mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \arrow[d] & \gamma_{h!}(\mathcal{H}om_W(h^!\mathcal{F}_1, h^!\mathcal{F}_2) \otimes h^!\Lambda) \arrow[l] \arrow[d] \\ (h\times 1_X)^! \mathcal{H}_X & \mathcal{H}_h \otimes \mathrm{pr}_1^* h^! \Lambda, \arrow[l] \end{tikzcd} \tag{2.85}$$

$$\begin{tikzcd} (1_W \times h)^! \gamma_{h!} \mathcal{H}om_W(h^!\mathcal{F}_1, h^!\mathcal{F}_2) \arrow[d] & \delta_{W!} \mathcal{H}om_W(h^!\mathcal{F}_1, h^!\mathcal{F}_2) \arrow[l] \arrow[d] \\ (1_W \times h)^! \mathcal{H}_h \arrow[r, "\simeq"] & \mathcal{H}_W. \end{tikzcd} \tag{2.86}$$

Since $h$ is smooth, the canonical morphism $(h \times 1_X)^*(-) \otimes \mathrm{pr}_1^* h^! \Lambda \to (h \times 1_X)^!$ of functors is an isomorphism and (2.85) defines a commutative diagram

$$\begin{tikzcd}
(h \times 1_X)^* \delta_{X!} \mathcal{H}om_X(\mathcal{F}_1, \mathcal{F}_2) \arrow[d] & \gamma_{h!} \mathcal{H}om_W(h^! \mathcal{F}_1, h^! \mathcal{F}_2) \arrow[l] \arrow[d] \\
(h \times 1_X)^* \mathcal{H}_X & \mathcal{H}_h \arrow[l]
\end{tikzcd} \tag{2.87}$$

Since $h$ is smooth, the canonical morphism $(1_X \times h)^*(-) \otimes \mathrm{pr}_2^* h^! \Lambda \to (1_X \times h)^!$ is an isomorphism and (2.86) defines a commutative diagram

$$\begin{tikzcd}
(1_W \times h)^* \gamma_{h!} \mathcal{H}om_W(h^! \mathcal{F}_1, h^! \mathcal{F}_2) \otimes \mathrm{pr}_2^* h^! \Lambda \arrow[d] & \delta_{W!} \mathcal{H}om_W(h^! \mathcal{F}_1, h^! \mathcal{F}_2) \arrow[l] \arrow[d] \\
(1_W \times h)^* \mathcal{H}_h \otimes \mathrm{pr}_2^* h^! \Lambda \arrow[r, "\simeq"] & \mathcal{H}_W.
\end{tikzcd} \tag{2.88}$$

Further if $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$, defining vertical arrows by $1_{\mathcal{F}}$ and $1_{h^! \mathcal{F}}$, we obtain commutative diagrams

$$\begin{tikzcd}
(h \times 1_X)^* \delta_{X!} \Lambda \arrow[r, "\simeq"] \arrow[d, "1_{\mathcal{F}}"'] & \gamma_{h!} \Lambda \arrow[d, "1_{h^! \mathcal{F}}"] \\
(h \times 1_X)^* \delta_{X!} \mathcal{H}om_X(\mathcal{F}, \mathcal{F}) \arrow[r, "\simeq"] & \gamma_{h!} \mathcal{H}om_W(h^! \mathcal{F}, h^! \mathcal{F}),
\end{tikzcd} \tag{2.89}$$

$$\begin{tikzcd}
(1_W \times h)^! \gamma_{h!} \Lambda \arrow[d, "1_{h^! \mathcal{F}}"'] & \delta_{W!} \Lambda \arrow[l, "\mathrm{adj}"'] \arrow[d, "1_{h^! \mathcal{F}}"] \\
(1_W \times h)^! \gamma_{h!} \mathcal{H}om_W(h^! \mathcal{F}, h^! \mathcal{F}) & \delta_{W!} \mathcal{H}om_W(h^! \mathcal{F}, h^! \mathcal{F}). \arrow[l]
\end{tikzcd} \tag{2.90}$$

The upper horizontal arrow in (2.90) is the adjoint of the isomorphism $\gamma_{h!} \leftarrow (1_W \times h)_! \delta_{W!}$.

By applying Lemma 2.4.2.3, we obtain commutative diagrams

$$\begin{tikzcd}
(h \times 1_X)^* \boxtimes_X \arrow[r, "\simeq"] \arrow[d] & \boxtimes_h \arrow[d] \\
(h \times 1_X)^* \delta_{X*}(D_X \mathcal{F}_1 \otimes \mathcal{F}_2) \arrow[r, "\simeq"] & \gamma_{h*}(D_W h^! \mathcal{F}_1 \otimes h^* \mathcal{F}_2),
\end{tikzcd} \tag{2.91}$$

$$\begin{tikzcd}
(1_W \times h)^* \boxtimes_h \otimes \mathrm{pr}_2^* h^! \Lambda \arrow[r, "\simeq"] \arrow[d] & \boxtimes_W \arrow[d] \\
(1_W \times h)^* \gamma_{h*}(D_W h^! \mathcal{F}_1 \otimes h^* \mathcal{F}_2 \otimes h^! \Lambda) \arrow[r, "\mathrm{adj}"] & \delta_{W*}(D_W h^! \mathcal{F}_1 \otimes h^! \mathcal{F}_2).
\end{tikzcd} \tag{2.92}$$

The lower line is induced by the canonical morphism $h^* \otimes h^! \Lambda \to h^!$ and the adjoint of the isomorphism $\gamma_{h*} \to (1_W \times h)_* \delta_{W*}$.

Further if $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$, defining vertical arrows by $\mathrm{ev}_X \colon D_X \mathcal{F} \otimes \mathcal{F} \to \mathcal{K}_X$, $\mathrm{ev}_X \circ (1 \otimes \mathrm{adj}) \colon D_X \mathcal{F} \otimes f^* f_* \mathcal{F} \to D_X \mathcal{F} \otimes \mathcal{F} \to \mathcal{K}_X$ and $\mathrm{ev}_Y \colon D_Y f_* \mathcal{F} \otimes f_* \mathcal{F} \to \mathcal{K}_Y$, we obtain commutative diagrams

$$\begin{tikzcd}
(h \times 1_X)^*\delta_{X*}(D_X\mathcal{F} \otimes \mathcal{F}) \arrow[d, "\mathrm{ev}"'] & \gamma_{h*}(D_W h^!\mathcal{F} \otimes h^*\mathcal{F}) \arrow[l, "\simeq"'] \arrow[d, "\mathrm{ev}"] \\
(h \times 1_X)^*\delta_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & \gamma_{h*}h^*\mathcal{K}_X,
\end{tikzcd}\tag{2.93}$$

$$\begin{tikzcd}
(1_W \times h)^*\gamma_{h*}(D_W h^!\mathcal{F} \otimes h^*\mathcal{F} \otimes h^!\Lambda) \arrow[r, "\mathrm{adj}"] \arrow[d, "\mathrm{ev}"'] & \delta_{W*}(D_W h^!\mathcal{F} \otimes h^!\mathcal{F}) \arrow[d, "\mathrm{ev}"] \\
(1_W \times h)^*\gamma_{h*}(h^*\mathcal{K}_X \otimes h^!\Lambda) \arrow[r, "\mathrm{adj}"] & \delta_{W*}\mathcal{K}_W.
\end{tikzcd}\tag{2.94}$$

The horizontal lines in (2.94) are induced by the morphism $h^* \otimes h^!\Lambda \to h^!$. The lower line is further induced by the adjoint of the isomorphism $\gamma_{h*} \to (1_W \times h)_*\delta_{W*}$ and $h^*\mathcal{K}_X \otimes h^!\Lambda \to \mathcal{K}_W$. Commutative diagrams (2.87), (2.89), (2.91) and (2.93) define a commutative diagram

$$\begin{tikzcd}
(h \times 1_X)^*\delta_{X!}\Lambda \arrow[r, "\simeq"] \arrow[d, "1_{\mathcal{F}}"'] & \gamma_{h!}\Lambda \arrow[d, "1_{h^!\mathcal{F}}"] \\
(h \times 1_X)^*\mathcal{H}_X \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}"'] & \mathcal{H}_h \arrow[d, "\mathrm{ev}"] \\
(h \times 1_X)^*\delta_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & \gamma_{h*}h^*\mathcal{K}_X.
\end{tikzcd}\tag{2.95}$$

Similarly, commutative diagrams (2.86), (2.90), (2.92) and (2.94) define a commutative diagram

$$\begin{tikzcd}
(1_W \times h)^*\gamma_{h!}h^!\Lambda \arrow[d, "1_{h^!\mathcal{F}}"'] & \delta_{W!}\Lambda \arrow[l, "\mathrm{adj}"'] \arrow[d, "1_{h^!\mathcal{F}}"] \\
(1_W \times h)^*\mathcal{H}_h \otimes \mathrm{pr}_2^*h^!\Lambda \arrow[r, "\simeq"] \arrow[d, "\mathrm{ev}"'] & \mathcal{H}_W \arrow[d, "\mathrm{ev}"] \\
(1_W \times h)^*\gamma_{h*}(h^*\mathcal{K}_X \otimes h^!\Lambda) \arrow[r] & \delta_{W*}\mathcal{K}_W.
\end{tikzcd}\tag{2.96}$$

The commutative diagram (2.83) induces a commutative diagram

$$\begin{tikzcd}
X \arrow[d, "0_X"'] & W \arrow[l, "h"'] \arrow[d, "0_h"] & W \arrow[l, "1_W"'] \arrow[d, "0_W"] \\
TX & TX \times_X W \arrow[l, "b"'] & TW. \arrow[l, "a"']
\end{tikzcd}\tag{2.97}$$

The vertical arrows are the 0-sections. Applying the functor $\nu_{W/W\times X}$ to (2.95) and the base change morphism $b^*\nu_{X/X\times X} \to \nu_{W/W\times X}(h \times 1_X)^*$ (1.53) to the left column, we obtain a commutative diagram

(2.98)
$$\begin{tikzcd}
b^*0_{X!}\Lambda \arrow[r, "\simeq"] \arrow[d] & 0_{h!}\Lambda \arrow[d] \\
b^*\nu_{X/X\times X}\mathcal{H}_X \arrow[r] \arrow[d] & \nu_{W/W\times X}\mathcal{H}_h \arrow[d] \\
b^*0_{X*}\mathcal{K}_X \arrow[r, "\simeq"] & 0_{h*}h^*\mathcal{K}_X.
\end{tikzcd}$$

Applying the functor $\nu_{W/W\times X}$ to (2.96) and the base change morphism $a^*\nu_{W/W\times W} \to \nu_{W/W\times X}(1_W \times h)^*$ (1.56) to the left column, we obtain a commutative diagram

(2.99)
$$\begin{tikzcd}
a^!0_{h!}\Lambda \arrow[d] & 0_{W!}\Lambda \arrow[l, "\mathrm{adj}_a"'] \arrow[d] \\
a^!\nu_{W/W\times X}\mathcal{H}_h \arrow[r] \arrow[d] & \nu_{W/W\times W}\mathcal{H}_W \arrow[d] \\
a^!0_{h*}h^*\mathcal{K}_X \arrow[r, "\mathrm{res}_a"] & 0_{W*}\mathcal{K}_W.
\end{tikzcd}$$

Let $b^\vee\colon T^*X \times_X W \to T^*X$ be the canonical morphism and $a^\vee\colon T^*X \times_X W \to T^*W$ be the dual of $a$. Let $e_X^\vee\colon T^*X \to X$, $e_h^\vee\colon T^*X \times_X W \to W$ and $e_W^\vee\colon T^*W \to W$ be the projections. By applying the Fourier transforms to (2.98) and the isomorphism (1.33) to the left column and using the isomorphisms (1.4), we obtain a commutative diagram

(2.100)
$$\begin{tikzcd}
b^{\vee*}\Lambda \arrow[r, "\simeq"] \arrow[d] & \Lambda \arrow[d] \\
b^{\vee*}\mu_{X/X\times X}\mathcal{H}_X \arrow[r] \arrow[d] & \mu_{W/W\times X}\mathcal{H}_h \arrow[d] \\
b^{\vee*}e_X^{\vee*}\mathcal{K}_X \arrow[r, "\simeq"] & e_h^{\vee*}h^*\mathcal{K}_X.
\end{tikzcd}$$

By applying the Fourier transforms to (2.99) and the isomorphism (1.22) to the left column and using the isomorphism (1.4), we obtain a commutative diagram

(2.101)
$$\begin{tikzcd}
a^\vee_!\Lambda \arrow[d] & \Lambda \arrow[l] \arrow[d] \\
a^\vee_!\mu_{W/W\times X}\mathcal{H}_h \arrow[d] & \mu_{W/W\times W}\mathcal{H}_W \arrow[l] \arrow[d] \\
a^\vee_!e_h^{\vee*}h^*\mathcal{K}_X \arrow[r] & e_W^{\vee*}\mathcal{K}_W.
\end{tikzcd}$$

By Proposition 1.1.11.2 and 1., the top and the bottom horizontal arrows in (2.101) are $\mathrm{res}_{a^\vee}$ and $(-1)^r\mathrm{adj}_{a^\vee}$ respectively. Hence (2.100) and (2.101) imply (2.81). □

**Corollary 2.4.5.** *Assume $\mathcal{F}$ is locally constant. Then, we have $SS_\mu\mathcal{F} \subset T^*_XX$ and $CC_\mu\mathcal{F} = (-1)^{\dim X}\mathrm{rank}\,\mathcal{F}[T^*_XX]$.*

*Proof.* We may assume that $\mathcal{F}$ is locally constant and is the pull-back from $\operatorname{Spec} k$. By Proposition 2.4.4, we may assume that $X = \operatorname{Spec} k$. Then, $CC_\mu \mathcal{F}$ equals $\operatorname{rank} \mathcal{F} = \operatorname{Tr} 1_{\mathcal{F}} \in \mathrm{H}^0(\operatorname{Spec} k, \Lambda)$. □

## 2.5 Characteristic cycles

We prove that Question 2.1.9.2 has an affirmative answer for curves.

**Proposition 2.5.1.** *If* $\dim X = 1$, *Question* 2.1.9.2 *has a positive answer.*

*Proof.* Let $j\colon U \to X$ be a dense open immersion such that $j^*\mathcal{F}$ is locally constant. Since the assertion is local on $X$ and since we may replace $k$ by an algebraic closure, we may assume that $U = X - \{x\}$ is the complement of a $k$-rational point $x \in X$. Let $i\colon x \to X$ be the closed immersion. Then, by Corollary 2.1.8, we have $CC_\mu \mathcal{F} = CC_\mu j_! j^* \mathcal{F} + CC_\mu i_* i^* \mathcal{F}$. Since $CC_\mu i_* i^* \mathcal{F} = \operatorname{rank} i^* \mathcal{F} \cdot [T^*_x X]$ by Corollary 2.4.5 and Proposition 2.3.3.2, we may assume that $j_! j^* \mathcal{F} \to \mathcal{F}$ is an isomorphism.

Since $j^*\mathcal{F} \in D^b_{\mathrm{ctf}}(U, \Lambda)$, by [4, Lemme 4.5.1], there exist a finite complex $\mathcal{L}_\bullet$ of locally constant sheaves of free $\Lambda$-modules of finite rank and a quasi-isomorphism $\mathcal{L}_\bullet \to j^*\mathcal{F}$. Hence by Proposition 2.1.7, we may further assume that $j^*\mathcal{F}$ is a locally constant sheaf of free $\Lambda$-modules of finite rank.

In the case where $X = \mathbf{A}^1$, $x = 0$ and $\mathcal{F}$ is a locally constant constructible sheaf of free $\Lambda$-module of rank $n$ on $U = \mathbf{G}_m$ tamely ramified at $x$, we have $CC_\mu \mathcal{F} = -n([T^*_X X] + [T^*_x X])$ by Corollary 2.4.5 and hence $CC_\mu \mathcal{F} = \operatorname{cl} CC\mathcal{F}$.

We show the general case by reducing to this case. Since the assertion is étale local on $X$, by [13, Theorem 4.1], we may assume that $X = \mathbf{P}^1 - \{0\}$, $x = \infty$ and $U = \mathbf{G}_m$ and that $j^*\mathcal{F}$ is a locally constant constructible sheaf of free $\Lambda$-modules of rank $n$ tamely ramified at 0. Let $\bar{j}\colon \mathbf{A}^1 \to \mathbf{P}^1$ be the open immersion. We have $SS_\mu \bar{j}_! \mathcal{F} \subset T^*_X X \cup T^*_0 X \cup T^*_\infty X$. By the tamely ramified case proved above and by the semi-purity [7, Rappel 2.2.8], we have $CC_\mu \bar{j}_! \mathcal{F} = -n([T^*_X X] + [T^*_0 X]) - a_\infty [T^*_\infty X]$ for some $a_\infty \in \Lambda$. We have $\chi(X_{\bar{k}}, \mathcal{F}) = n - a_\infty$ by Corollary 2.3.4. Since $\chi(X_{\bar{k}}, \mathcal{F}) = n - a_\infty \mathcal{F}$ by the Grothendieck–Ogg–Shafarevich formula, we obtain $a_\infty = a_\infty \mathcal{F}$ and $CC_\mu \mathcal{F} = \operatorname{cl} CC\mathcal{F}$. □

**Proposition 2.5.2.** *Suppose that* $\dim SS_\mu \mathcal{F} \leqq \dim X$ *for every* $X$ *and* $\mathcal{F} \in D^b_{\mathrm{ctf}}(X, \Lambda)$. *Then,* $CC_\mu \mathcal{F}$ *is the cycle class of* $CC\mathcal{F}$.

It suffices to show that the coefficient $a_x$ of the fiber $T^*_x X$ in $CC_\mu \mathcal{F}$ equals $-a_x(\mathcal{F})$.

*Proof.* The proof is similar to that of the functorial characterization of characteristic cycles [18, Proposition 8]. Let $C \subset T^*X$ be the union $SS_\mu \mathcal{F} \cup SS\mathcal{F}$. By the assumption and [2, 1.3 Theorem (ii)], we have $\dim C \leqq \dim X$. Hence by the semi-purity [7, Rappel 2.2.8], we have $\mathrm{H}^0_C(T^*X, e^{\vee *}K_X) = \bigoplus_a \Lambda[C_a]$ where $C_a$ runs irreducible components of $C$ of dimension $\dim X$ and hence $CC_\mu \mathcal{F} = \sum_a \mu_a [C_a]$ for some $\mu_a \in \Lambda$. Let $CC\mathcal{F} = \sum_a m_a [C_a]$. It suffices to show the equality $\mu_a = m_a$ in $\Lambda$ for each irreducible component $C_a$.

Since the question is local on $X$, we may assume that $X$ is affine. By applying Proposition 2.3.3 to an immersion $X \to \mathbf{A}^n$, we may assume $X = \mathbf{A}^n$. Further, since the question is local on $X$, we may assume that $X$ is projective. We fix a closed immersion $X \to \mathbf{P}^N$.

Let $C_a$ be an irreducible component of $C$ of dimension $\dim X$. Then, there exists a linear subvariety $A \subset \mathbf{P}^N$ of codimension 2 and $x \in X - X \cap A$ satisfying the following conditions. The intersection $X \cap A$ is transversal. Let $X' \to X$ be the blow-up at $X \cap A$

and let $f\colon X'\to Y=\mathbf{P}^1$ be the morphism to the projective line parametrizing hyperplanes containing $A$ such that the fibers are the intersections with hyperplanes. The point $x$ is an isolated characteristic point of $f\colon X'\to Y$ and $x$ is the unique characteristic point in the fiber $f^{-1}(y)$ for $y=f(x)$. The component $C_a\subset T^*X$ is the unique irreducible component of $C$ meeting the section $df$ at $x$. The intersection number $(C_a,df)_x$ equals 1, if necessary replacing $X$ by $X\times\mathbf{A}^1$ in the case $p=2$ [16, Proposition 5.19].

Let $m_a$ be the coefficient of $C_a$ in $CC\mathcal{F}$ and $\mu_a\in\Lambda$ be the coefficient of $\mathrm{cl}\,C_a$ in $CC_\mu\mathcal{F}$. Then, we have $(CC\mathcal{F},df)_x=m_a(C_a,df)_x\in\mathbf{Z}$ and $(CC_\mu\mathcal{F},df)_x=\mu_a(C_a,df)_x\in\Lambda$. By the Milnor formula, for the complex of vanishing cycles $\Phi_x(\mathcal{F},f)$, we have $\dim\mathrm{tot}\,\Phi_x(\mathcal{F},f)=-(CC\mathcal{F},df)_x=-m_a(C_a,df)_x$. Since $f_!CC_\mu\mathcal{F}=CC_\mu f_*\mathcal{F}$ by Proposition 2.3.3, the intersection product $(CC_\mu\mathcal{F},df)_x=\mu_a(C_a,df)_x$ equals the coefficient $-a_yf_*\mathcal{F}$ of $\mathrm{cl}\,T^*_yY$ in $f_!CC_\mu\mathcal{F}=CC_\mu f_*\mathcal{F}$. By the distinguished triangle $\Gamma(X_{\overline{y}},\mathcal{F})\to\Gamma(X_{\overline{\eta}},\mathcal{F})\to\Phi_x(\mathcal{F},f)\to$ for a geometric generic point $\overline{\eta}$ of the strict localization at $\overline{y}$, we have $\dim\mathrm{tot}\,\Phi_x(\mathcal{F},f)=a_yf_*\mathcal{F}$. Thus we have $\dim\mathrm{tot}\,\Phi_x(\mathcal{F},f)=-m_a(C_a,df)_x=-\mu_a(C_a,df)_x$. Since $(C_a,df)_x=1$, we obtain $m_a=\mu_a$ in $\Lambda$ and hence $CC_\mu\mathcal{F}=\mathrm{cl}\,CC\mathcal{F}$. □

The assumption of Proposition 2.5.2 is satisfied if the following question has an affirmative answer.

**Question 2.5.3** (cf. [11, Corollary 5.4.10 (i)])**.** Let $Z\to X$ be a closed immersion of smooth schemes over $k$ and $\mathcal{F}\in D_{\mathrm{ctf}}(X,\Lambda)$. Do we have

$$\mathrm{supp}\,\mu_Z(\mathcal{F})\subset SS\mathcal{F}\cap T^*_ZX\,?$$

**Lemma 2.5.4.** *If $Z\subset X$ is a divisor, then Question* 2.5.3 *has an affirmative answer.*

*Proof.* We may assume that $Z$ is defined by $f$. It suffices to show that $\nu_Z\mathcal{F}$ on $T_ZX$ is isomorphic to the pull-back $e^*\mathcal{F}|_Z$ assuming that $SS\mathcal{F}\cap T^*_ZX$ is a subset of the 0-section. We may assume that the morphism $f\colon X\to\mathbf{A}^1$ is $SS\mathcal{F}$-acyclic by the assumption. Then, since the morphism $A_ZX\to A_0\mathbf{A}^1$ is the base change of $f\colon X\to\mathbf{A}^1$, it is locally acyclic relatively to the pull-back of $\mathcal{F}$. Since $A_0\mathbf{A}^1\to\mathbf{A}^1$ is smooth, the composition $A_ZX\to A_0\mathbf{A}^1\to\mathbf{A}^1$ is also locally acyclic relatively to the pull-back of $\mathcal{F}$ by [9, Corollaire 2.7]. Hence the assertion follows. □

**Lemma 2.5.5.** *If Question* 2.5.3 *has an affirmative answer, then the assumption in Proposition* 2.5.2 *that* $\dim SS_\mu\mathcal{F}\leqq\dim X$ *for every* $X$ *and* $\mathcal{F}\in D^b_{\mathrm{ctf}}(X,\Lambda)$ *is satisifed.*

*Proof.* By Question 2.5.3 applied to $\delta\colon X\to X\times X$ and $\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F},\mathrm{pr}_1^!\mathcal{F})$, we have

$$SS_\mu\mathcal{F}=\mathrm{supp}\,\mu\mathcal{H}om(\mathcal{F},\mathcal{F})\subset T^*X\cap SS\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F},\mathrm{pr}_1^!\mathcal{F}) \tag{2.102}$$

in $T^*(X\times X)$. Since

$$SS\mathcal{H}om(\mathrm{pr}_2^*\mathcal{F},\mathrm{pr}_1^!\mathcal{F})=SS(\mathcal{F}\boxtimes D_X\mathcal{F})=SS\mathcal{F}\times SSD_X\mathcal{F}=SS\mathcal{F}\times SS\mathcal{F}$$

in $T^*X\times T^*X=T^*(X\times X)$ by [17, Theorem 2.2.3], the right hand side of (2.102) is $SS\mathcal{F}\subset T^*X$. □

## 2.6 An Artin–Schreier sheaf

In this section, we compute $CC_\mu$ for an Artin–Schreier sheaf with wild ramification, directly without using Proposition 2.5.1.

**Lemma 2.6.1.** *Let*

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "c"'] & D \arrow[d, "d"] \\ X \arrow[r, "q"] & A \end{tikzcd}$$

*be a cartesian diagram of separated schemes of finite type over $k$.*

1. *Assume that the vertical arrows are closed immersions and define a morphism $q^*d_!d^! \to c_!c^!q^*$ by $q^*\mathcal{H}om(d_*\Lambda, -) \to \mathcal{H}om(c_*\Lambda, q^*-)$. Then, the diagram*

(2.103)
$$\begin{tikzcd} c_!c^!q^* \arrow[d, "\mathrm{adj}_c"'] & q^*d_!d^! \arrow[l] \arrow[d, "\mathrm{adj}_d"] \\ q^* \arrow[r, equal] & q^* \end{tikzcd}$$

*is commutative.*

2. *The diagram*

(2.104)
$$\begin{tikzcd} q^* \arrow[r, "\mathrm{adj}_d"] \arrow[d, "\mathrm{adj}_c"'] & q^*d_*d^* \arrow[d, "\mathrm{bc}"] \\ c_*c^*q^* & c_*g^*d^* \arrow[l, "\mathrm{can}"'] \end{tikzcd}$$

*is commutative.*

*Proof.* 1. This follows from the commutative diagram

$$\begin{tikzcd} q^*\mathcal{H}om(d_*\Lambda, -) \arrow[r] \arrow[d] & \mathcal{H}om(c_*\Lambda, q^*-) \arrow[d] \\ q^*\mathcal{H}om(\Lambda, -) \arrow[r, equal] & \mathcal{H}om(\Lambda, q^*-). \end{tikzcd}$$

2. By adjunction and by the definition of $q^*d_* \to c_*g^*$, this follows from the commutative diagram

$$\begin{tikzcd} 1 \arrow[r, "\mathrm{adj}_d"] \arrow[d, "\mathrm{adj}_{qc}"'] & d_*d^* \arrow[d, "\mathrm{adj}_g"] \\ q_*c_*c^*q^* & d_*g_*g^*d^*. \arrow[l, "\mathrm{can}"'] \end{tikzcd}$$

□

**Proposition 2.6.2.** *Let $j_X \colon X - \{0\} = \operatorname{Spec} k[x^{\pm 1}] \to X = \mathbf{A}^1 = \operatorname{Spec} k[x]$ be the open immersion and let $\mathcal{F} = j_{X!}\mathcal{L}_\psi(1/x)$. Then, we have*

(2.105)
$$CC_\mu\mathcal{F} = -\mathrm{cl}([T^*_XX] + 2\cdot[T^*_0X])$$

*in $\mathrm{H}^0_{T^*_XX\cup T^*_0X}(T^*X, e^{\vee*}K_X)$.*

*Proof.* Set $X \times X = \operatorname{Spec} k[x, y]$ and let

$$A_X(X \times X) = \mathbf{A}^3 = \operatorname{Spec} k[x, u, v] \to \mathbf{A}^1 = \operatorname{Spec} k[u]$$

be the deformation to the normal bundle $T = TX = \operatorname{Spec} k[x, v]$ for $v = (y - x)/u$. Let $q \colon W = \operatorname{Spec} k[w] \to X$ be the morphism defined by $k[w] \to k[x]$ sending $w$ to $-x^2$. We construct a commutative diagram

(2.106)

$$\begin{tikzcd}
X \times \mathbf{A}^1 \arrow[r] \arrow[d, "\delta"'] & W \times \mathbf{A}^1 = \operatorname{Spec} k[w, u] \arrow[d, "0_A"] & & W \times \mathbf{A}^1 \arrow[ll, "1"'] \arrow[d, "0_B"] \\
A_X(X \times X) \arrow[r, "q_A"] & A = \operatorname{Spec} k[w, u, v] & \overline{B} \supset \arrow[l, "\overline{g}"'] & B = \operatorname{Spec} k[w, u, s].
\end{tikzcd}$$

The left vertical arrow $\delta \colon X \times \mathbf{A}^1 = \operatorname{Spec} k[x, u] \to A_X(X \times X) = \operatorname{Spec} k[x, u, v]$ is a lifting of the diagonal $X \times \mathbf{A}^1 \to X \times X \times \mathbf{A}^1$ and is defined by the surjection $k[x, u, v] \to k[x, u]$ sending $v$ to 0. The scheme $A$ is a trivial line bundle over $W \times \mathbf{A}^1$ and the morphism $0_A \colon W \times \mathbf{A}^1 \to A$ is the 0-section also defined by the surjection $k[w, u, v] \to k[w, u]$ sending $v$ to 0. The morphism $q_A \colon A = \operatorname{Spec} k[w, u, v] \to A_X(X \times X) = \operatorname{Spec} k[x, u, v]$ is defined by $w = -x(x + uv)$ and the left square is cartesian. The morphism $\overline{g} \colon \overline{B} \to A$ is the blow-up at the ideal $(w, v)$ and $B \subset \overline{B}$ is the complement of the proper transform of the divisor $w = 0$. The restriction $g \colon B \to A$ of $\overline{g}$ is a morphism of line bundles over $W \times \mathbf{A}^1$ defined by $v = ws$ and the vertical arrows in the left square are the 0-sections.

The base change of the lower line to the origin $0 \to \mathbf{A}^1$ gives a diagram

(2.107)

$$\begin{tikzcd}
TX \arrow[r, "q_T"] \arrow[d, "e"'] & E = \operatorname{Spec} k[w, v] \arrow[d, "e_E"] & \overline{F} \supset F = \operatorname{Spec} k[w, s] \arrow[l, "\overline{g}_F"'] \arrow[d, "e_F"] \\
X \arrow[r, "q"] & W & W \arrow[l, "1"']
\end{tikzcd}$$

of line bundles.

We have an isomorphism

$$(j_X \times j_X)_! \mathcal{L}(1/y - 1/x) \otimes \mathrm{pr}_1^* K_X \to \mathcal{H}om(\mathrm{pr}_1^* \mathcal{F}, \mathrm{pr}_2^! \mathcal{F}).$$

on $X \times X$. Let $\mathcal{H}_X$ on $X \times X \times \mathbf{G}_m$ be the pull-back of $\mathcal{H}om(\mathrm{pr}_1^* \mathcal{F}, \mathrm{pr}_2^! \mathcal{F})$. Let $e_A \colon A \to W$ be the projection and $j_A \colon \operatorname{Spec} k[w^{\pm 1}, u, v] \to A$ be the open immersion. Define $\mathcal{H}_A$ on $A$ by

$$\mathcal{H}_A = j_{A!} \mathcal{L}_\psi(uv/w) \otimes e_A^* K_W$$

and let $\mathcal{H}_{A^\circ}$ be the restriction on $A^\circ = A - E$. We extend the canonical isomorphism $(q^* K_W)|_{X - \{0\}} \to K_X|_{X - \{0\}}$ uniquely to an isomorphism $q^* K_W \to K_X$ and identify them and let $q_{A^\circ} \colon X \times X \times \mathbf{G}_m \to A^\circ$ be the restriction of $q_A$. Then, since

$$1/y - 1/x = -(y - x)/xy = -(y - x)/(x(x + uv)) = uv/w,$$

we have a canonical isomorphism

$$\mathcal{H}_X \leftarrow q_{A^\circ}^* \mathcal{H}_{A^\circ}. \tag{2.108}$$

By Lemma 2.6.1 applied to $q_{A^\circ}\colon X \times X \times \mathbf{G}_m \to A^\circ$, the isomorphism (2.108) defines a commutative diagram

$$\begin{tikzcd}
\delta^\circ_! \delta^{\circ !} \mathcal{H}_X \arrow[d] & q^*_{A^\circ} 0_{A^\circ !} 0^!_{A^\circ} \mathcal{H}_A \arrow[l] \arrow[d] \\
\mathcal{H}_X \arrow[d] & q^*_{A^\circ} \mathcal{H}_{A^\circ} \arrow[l] \arrow[d] \\
\delta^\circ_* \delta^{\circ *} \mathcal{H}_X & q^*_{A^\circ} 0_{A^\circ *} 0^*_{A^\circ} \mathcal{H}_{A^\circ}. \arrow[l]
\end{tikzcd} \tag{2.109}$$

Here and in the following, the superscript ° indicates the base change by $\mathbf{G}_m \to \mathbf{A}^1$.

Let $e_B\colon B \to W$ be the projection and let $j_B\colon B \to \overline{B}$ be the open immersion. Define $\mathcal{H}_B$ on $B$ by

$$\mathcal{H}_B = \mathcal{L}_\psi(us) \otimes e_B^* K_W.$$

and set $\mathcal{H}_{\overline{B}} = j_{B!}\mathcal{H}_B$. Let $g^\circ\colon B^\circ = B - F \to A^\circ = A - E$ be the restriction of $g$ and $\overline{g}^\circ\colon \overline{B}^\circ = \overline{B} - \overline{F} \to A^\circ$ be the restriction of $\overline{g}$. Let $\mathcal{H}_{B^\circ}$ and $\mathcal{H}_{\overline{B}^\circ}$ be the restrictions of $\mathcal{H}_B$ and $\mathcal{H}_{\overline{B}}$. Then, since $g^\circ_!(\mathcal{L}(us)|_{B^\circ}) = \mathcal{L}(uv/w)|_{A^\circ}$, we have an isomorphism

$$\mathcal{H}_{A^\circ} \to \overline{g}^\circ_* \mathcal{H}_{\overline{B}^\circ} = g^\circ_! \mathcal{H}_{B^\circ}. \tag{2.110}$$

We consider the cartesian diagram

$$\begin{tikzcd}
A & B \arrow[l, "g"'] \\
W \times \mathbf{A}^1 \arrow[u, "0_A"] & g^{-1}(W \times \mathbf{A}^1) \arrow[l] \arrow[u, "+_B"']
\end{tikzcd}$$

and the extension $+_{\overline{B}}\colon \overline{g}^{-1}(W \times \mathbf{A}^1) \to \overline{B}$ of the right vertical arrow. Similarly as (2.109), by Lemma 2.3.2 applied to $\overline{g}^\circ\colon \overline{B}^\circ \to A^\circ$, the isomorphism (2.110) defines a commutative diagram

$$\begin{tikzcd}
0_{A^\circ !} 0^!_{A^\circ} \mathcal{H}_{A^\circ} \arrow[r] \arrow[d] & \overline{g}^\circ_* +^\circ_{\overline{B}!} +^{\circ !}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \arrow[d] \\
\mathcal{H}_{A^\circ} \arrow[d] & \overline{g}^\circ_* \mathcal{H}_{\overline{B}^\circ} \arrow[l, "\simeq"'] \arrow[d] \\
0_{A^\circ *} 0^*_{A^\circ} \mathcal{H}_{A^\circ} & g^\circ_{\overline{B}*} +^\circ_{\overline{B}*} +^{\circ *}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \arrow[l]
\end{tikzcd} \tag{2.111}$$

on $A^\circ$. By proper base change theorem, the horizontal arrows are isomorphisms.

We compute the top and the bottom of the right column of (2.111). Let $j_W\colon \mathbf{G}_m \to W$ be the open immersion. The canonical morphism $+^*_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \otimes +^!_{\overline{B}} \Lambda \to +^!_{\overline{B}} \mathcal{H}_{\overline{B}^\circ}$ is an isomorphism. Since $\overline{g}^\circ_*(\mathcal{L}(us)|_{B^\circ}) = \mathcal{L}(uv/w)|_{A^\circ}$, we have canonical isomorphisms

$$\overline{g}^\circ_* +^\circ_{\overline{B}!} +^{\circ !}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \to 0_{A^\circ !} j_{W*}\Lambda = \overline{g}^\circ_* 0_{B^\circ !} j_{W*}\Lambda, \quad \overline{g}^\circ_* +^\circ_{\overline{B}*} +^{\circ *}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \to 0_{A^\circ *} j_{W!} K_W = \overline{g}^\circ_* 0_{B^\circ !} j_{W!} K_W. \tag{2.112}$$

Let $\nu_A\colon D_{\mathrm{ctf}}(A^\circ, \Lambda) \to D_{\mathrm{ctf}}(E \times_k \eta_0, \Lambda)$ be the specialization functor. Let $j_F\colon F \to \overline{F}$ be the open immersion. By Proposition 1.1.3.1, the projection $\overline{B} \to W \times \mathbf{A}^1$ is locally acyclic relatively to $\mathcal{H}_{\overline{B}}$. Hence by [9, Corollaire 2.7], the composition $\overline{B} \to \mathbf{A}^1 = \operatorname{Spec} k[u]$ is locally acyclic relatively to $\mathcal{H}_{\overline{B}}$ and we have $\Psi \mathcal{H}_{\overline{B}} = \mathcal{H}_{\overline{B}}|_{\overline{F}} = j_{F!} e_F^* \mathcal{K}_W$. Let $g_F \colon F \to E$ be the restriction of $g \colon B \to A$. Then, by proper base change theorem, the isomorphisms (2.112) induce isomorphisms

$$\nu_A \overline{g}^{\circ}_{*} \overline{+}^{\circ}_{!} \overline{+}^{\circ !} \mathcal{H}_{\overline{B}^\circ} \to g_{F!} 0_{F!} j_{W*} \Lambda, \quad \nu_A \overline{g}^{\circ}_{*} +^{\circ}_{\overline{B}*} +^{\circ *}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \to g_{F!} 0_{F!} j_{W!} K_W. \tag{2.113}$$

Hence the specialization of (2.111) defines a commutative diagram

$$\begin{tikzcd}
\nu_A 0_{A^\circ !} 0^!_{A^\circ} \mathcal{H}_{A^\circ} \arrow[r] \arrow[d] & g_{F!} 0_{F!} j_{W*} \Lambda \arrow[d] \\
\nu_A \mathcal{H}_{A^\circ} \arrow[d] & g_{F!} e_F^* K_W \arrow[l, "\simeq"'] \arrow[d] \\
\nu_A 0_{A^\circ *} 0^*_{A^\circ} \mathcal{H}_{A^\circ} & g_{F!} 0_{F*} j_{W!} K_W \arrow[l]
\end{tikzcd} \tag{2.114}$$

on $E$ where the horizontal arrows are isomorphisms.

Let $0_{\overline{B}^\circ} \colon W \times \mathbf{G}_m \to \overline{B}^\circ$ be the 0-section and consider canonical morphisms

$$0_{\overline{B}^\circ !} 0^!_{\overline{B}^\circ} \mathcal{H}_{\overline{B}^\circ} \to +^{\circ}_{\overline{B}!} +^{\circ !}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \to \mathcal{H}_{\overline{B}^\circ} \to +^{\circ}_{\overline{B}*} +^{\circ *}_{\overline{B}} \mathcal{H}_{\overline{B}^\circ} \to 0_{\overline{B}^\circ *} 0^*_{\overline{B}^\circ} \mathcal{H}_{\overline{B}^\circ}. \tag{2.115}$$

where the middle arrows induce the right columns of (2.111) and (2.114). Then, the specialization of compositions on (2.115) defines

$$0_{F!} \Lambda \to e_F^* K_W \to 0_{F*} K_W \tag{2.116}$$

where the first arrow is the adjoint of the canonical isomorphism $\Lambda \to 0_F^! e_F^* K_W$.

Let $\mu_A \colon D_{\mathrm{ctf}}(A, \Lambda) \to D_{\mathrm{ctf}}(E \times_k \eta_0, \Lambda)$ denote the composition of $\nu_A$ with the Fourier transform $F_\psi$. Let $g_F^\vee \colon E^\vee \to F^\vee$ be the dual of $g_F \colon F \to E$ and define a cartesian diagram

$$\begin{tikzcd}
g_F^{\vee -1}(W) \arrow[r] \arrow[d, "+_{E^\vee}"'] & W \arrow[d, "0_{F^\vee}"] \\
E^\vee \arrow[r, "g^\vee"] & F^\vee.
\end{tikzcd}$$

Let $e_{E^\vee} \colon E^\vee \to W$ and $e_{F^\vee} \colon F^\vee \to W$ be the canonical morphisms. Since $F_\psi g_{F!} e_F^* K_W = g_F^{\vee *} 0_{F^\vee *} \Lambda = +_{E^\vee *} \Lambda$ by Proposition 1.1.8, the Fourier transform of (2.114) gives

$$\begin{tikzcd}
\mu_A 0_{A^\circ !} 0^!_{A^\circ} \mathcal{H}_{A^\circ} \arrow[r] \arrow[d] & e^*_{E^\vee} j_{W*} \Lambda \arrow[d] \\
\mu_A \mathcal{H}_{A^\circ} \arrow[d] & +_{E^\vee *} \Lambda \arrow[l, "\simeq"'] \arrow[d] \\
\mu_A 0_{A^\circ *} 0^*_{A^\circ} \mathcal{H}_{A^\circ} & e^*_{E^\vee} j_{W!} K_W \arrow[l]
\end{tikzcd} \tag{2.117}$$

on the dual $E^\vee$ where the horizontal arrows are isomorphisms (1.4) and (1.21). We consider the compositions with $\Lambda \to j_{W*}\Lambda$ and $j_{W!}K_W \to K_W$ and the commutative diagram

(2.118)

$$\begin{tikzcd}
e^*_{E^\vee}\Lambda \arrow[d] & F_\psi 0_{E!}\Lambda \arrow[l] \arrow[d] & g^{\vee *}_F\Lambda \arrow[l] \arrow[d] \\
+_{E^\vee *}\Lambda \arrow[d] & F_\psi g_{F!} e^*_F K_W \arrow[l] \arrow[d] & g^{\vee *}_F 0_{F^\vee !}\Lambda \arrow[l] \arrow[d] \\
e^*_{E^\vee}K_W & F_\psi 0_{E*}K_W \arrow[l] & g^{\vee *}_F e^*_{F^\vee}K_W. \arrow[l]
\end{tikzcd}$$

The left column is obtained from the right column of (2.117) and the right half is obtained by applying the isomorphism $F_\psi g_{F!} \leftarrow g^{\vee *}_F F_\psi$ (1.21) to (2.116). By the description above of (2.116) and Proposition 1.1.11.1, the composition of the right column is $g^{\vee *}_F$ of $-1$-times the cycle class of the 0-section. By the commutativity of the diagram, the left column is $-1$-times the cycle class of $g^{\vee -1}_F(W)$.

Let $\nu_X\colon D_{\rm ctf}(X \times X \times \mathbf{G}_m, \Lambda) \to D_{\rm ctf}(TX \times_k \eta_0, \Lambda)$ be the specialization functor and let $\mu_X$ be the composition with the Fourier transform. Let $0_X\colon X \to TX$ be the 0-section. The specialization of the commutative diagram (2.109) defines a commutative diagram

(2.119)

$$\begin{tikzcd}
0_{X!}0^!_X \nu_X \mathcal{H}_X \arrow[d] & q^*_T \nu_A 0_{A!} 0^!_A \mathcal{H}_A \arrow[l] \arrow[d] \\
\nu_X \mathcal{H}_X \arrow[d] & q^*_T \nu_A \mathcal{H}_A \arrow[l] \arrow[d] \\
0_{X*}0^*_X \nu_X \mathcal{H}_X & q^*_T \nu_A 0_{A*} 0^*_A \mathcal{H}_A. \arrow[l]
\end{tikzcd}$$

Let $q_{T^*}\colon T^*X \to E^\vee$ denote the morphism induced by $q_T$ on the dual. Then the Fourier transform of (2.119) gives

(2.120)

$$\begin{tikzcd}
e^{\vee *}\Lambda \arrow[d] & q^*_{T^*}\mu 0_{A!} 0^!_A \mathcal{H}_A \arrow[l] \arrow[d] \\
\mu_X \mathcal{H}_X \arrow[d] & q^*_{T^*}\mu_A \mathcal{H}_A \arrow[l] \arrow[d] \\
e^{\vee *}K_X & q^*_{T^*}\mu 0_{A*} 0^*_A \mathcal{H}_A \arrow[l]
\end{tikzcd}$$

on $E^\vee$. Since the ramification index of $q\colon W \to X$ is 2, the pull-back $q^*_{T^*}\colon {\rm H}^0_+(E^\vee, e^{\vee *}_E K_W) \to {\rm H}^0_+(T^*X, e^{\vee *}_X K_X)$ maps the class of of the composition of the left column in (2.118) to the class $-([T^*_X X] + 2 \cdot [T^*_0 X])$. Thus, combining (2.117) and (2.120), we obtain (2.105). □

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