RAMIFICATION GROUPS AND LOCAL CONSTANTS

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Let K be a complete discrete valuation field with a perfect residue field k of characteristic $ch\ k = p \neq 0$. The filtration by the ramification groups G^v of the absolute Galois group Gal (K^{sep}/K) has been known for long time [S3]. We define a canonical isomorphism $\pi_1(N^v) \simeq G^v/G^{v+}$ of pro-finite $\mathbb{F}_p$-vector spaces for $v \in \mathbb{Q} \geq 0$ where $G^{v+} = \overline{\bigcup_{v'>v} G^{v'}}$. Here $N^v = \{x \in K^{\text{sep}}; \text{ord } x \geq v\}/\{x \in K^{\text{sep}}; \text{ord } x > v\}$ is a k^{sep}-vector space of dimension 1 and $\pi_1(N^v)$ denotes the fundamental group [S1] of an affine line N^v as an algebraic group. In particular, the Pontragin dual of G^v/G^{v+} has a natural structure of k^{sep}-vector space of dimension 1. We give an interpretation of a refinement of the Swan conductor defined by Kato [K] in terms of the isomorphism. We also study a relation with an explicit formula for local constants modulo roots of unity of p-power order by Henniart [He].

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1. The isomorphism.

Let

$$M = \bigoplus_{n\in\mathbb{Z}} M^n = \bigoplus_{n\in\mathbb{Z}} m_K^n/m_K^{n+1} = \text{Gr}^\bullet(K)$$
$$\subset N = \bigoplus_{v\in\mathbb{Q}} N^v = \text{Gr}^\bullet(K^{\text{sep}})$$

be the associate graded with respect to the filtrations defined by the valuation. They have natural structures of $\mathbb{Z}$-graded k-algebra and $\mathbb{Q}$-graded k^{sep}-algebra respectively. They are non-canonically isomorphic to $k[t,t^{-1}]$ and to $k^{\text{sep}}[t^{\frac{1}{m}}, t^{-\frac{1}{m}} (m \geq 1)]$ respectively. The natural action of G_K on N factors the quotient $G/P =$ Gal (K^{tr}/K) where the maximal tamely ramified extension $K^{tr} = K^{nr}(\pi^{\frac{1}{m}}, p \nmid m)$ is the union of the extensions with prime-to-p ramification index. In fact, N is the perfect closure of the subalgebra $\text{Gr}^\bullet(K^{tr})$. The action of the inertia $I/P =$ Gal $(K^{tr}/K^{nr}) = \varprojlim_{p\nmid m} \mu_m$ on N^v is given by a character $[v] \in \text{Hom}(I/P, k^{\text{sep}\times})$ defined as follows. Identify $\text{Hom}(I/P, k^{\text{sep}\times}) = \varinjlim_{p\nmid m} \text{Hom}(\mu_m, \mu_m) = \bigcup_{p\nmid m} \frac{1}{m}\mathbb{Z}/\mathbb{Z}$ $\subset \mathbb{Q}/\mathbb{Z}$ and let p^e be the p-part of the denominator of v. Then a character $[v] \in \text{Hom}(I/P, k^{\text{sep}\times})$ is characterized by $p^e[v] = p^e v \in \bigcup_{p\nmid m} \frac{1}{m}\mathbb{Z}/\mathbb{Z}$. From this, we see that the fixed part N^{G_K} is the perfect closure of the subalgebra M. For a finite separable extension L of K in K^{sep}, we naturally identify N for L and K but the grading is multiplied by the ramification index $e_{L/K}$. When we want to distinguish N for K and for L, we write N_K and N_L.

To define the isomorphism $G^v/G^{v+} \to \pi_1(N^v)$, we briefly recall the definition of a refinement of the different by Kato [K] (2.1). Let L be a finite separable extension of K. Let $d_{L/K}$ be the smallest fractional ideal of L satisfying $\text{Tr}_{L/K}(d_{L/K}) = \mathcal{O}_K$.

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The refined different $D_{L/K} \in L^\times/1+m_L$ is defined to be the class of an element $\widetilde{D}$ of $L^\times$ such that $d_{L/K} = \widetilde{D}^{-1}\mathcal{O}_L$ and that the diagram

$$\begin{array}{ccccc} d_{L/K} & \xrightarrow{\mathrm{Tr}_{L/K}} & & & \mathcal{O}_K \\ {\scriptstyle \widetilde{D}\times}\downarrow & & & & \downarrow{\scriptstyle \mathrm{mod}\ m_K} \\ \mathcal{O}_L & \xrightarrow[\mathrm{mod}\ m_L]{} & \ell & \xrightarrow[\mathrm{Tr}_{\ell/k}]{} & k \end{array}$$

is commutative where ℓ is the residue field of L. For a tamely ramified extension L, the refined different $D_{L/K}$ is the class of the ramified index $e_{L/K} \in \ell^\times \subset L^\times/1+m_L$. For $L \supset K' \supset K$, we have the chain rule $D_{L/K} = D_{L/K'} \times D_{K'/K}$.

Let $\psi = \psi_{L/K} = \varphi_{L/K}^{-1} : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be Herbrand's function for a finite separable extension L over K [D2] A.4. We say that the ramification of L over K is bounded by v if ψ is linear on the half line $(v-\varepsilon, \infty)$ for some $\varepsilon > 0$. Assume that the ramification of L over K is bounded by v. Then we have $\psi(v) = ev - \delta$ for $\delta = ord_L D_{L/K}$. Hence the multiplication by $D_{L/K}$ induces an isomorphism

$$N_L^{\psi(v)} \xrightarrow{\times D_{L/K}} N_L^{ev} = N_K^v .$$

Further if $\psi(v)$ is an integer, it induces an isomorphism of ℓ-vector spaces of dimension 1

$$\alpha_{L/K} : 1 + m_L^{\psi(v)}/1 + m_L^{\psi(v)+1} \to (N_K^v)^{G_L}$$

where ^{G_L} denotes the fixed part. Here and in the following, we naturally identify $1 + m_K^n/1 + m_K^{n+1} = M_K^n$ for $n \in \mathbb{N}$.

LEMMA 1. *Let $L \supset K'$ be finite separable extensions of K such that the ramifications over K are bounded by v and assume $w = \psi_{L/K}(v)$ and $w' = \psi_{K'/K}(v)$ are integers. Then the norm map $N_{L/K'}$ induces a map $N_{L/K'} : 1 + m_L^w/1 + m_L^{w+1} \to 1 + m_{K'}^{w'}/1 + m_{K'}^{w'+1}$ and the diagram*

$$\begin{array}{ccc} 1 + m_L^w/1 + m_L^{m+1} & \xrightarrow{\alpha_{L/K'}} & (N_K^v)^{G_L} = \ell \otimes_{k'} (N_K^v)^{G_{K'}} \\ {\scriptstyle N_{L/K'}}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_{\ell/k'}\otimes \mathrm{id}} \\ 1 + m_{K'}^{w'}/1 + m_{K'}^{w'+1} & \xrightarrow[\alpha_{K'/K}]{} & (N_K^v)^{G_{K'}} \end{array}$$

is commutative.

Proof. By the chain rules $D_{L/K} = D_{L/K'} \cdot D_{K'/K}$, $\psi_{L/K} = \psi_{L/K'} \cdot \psi_{K'/K}$ and by the definition of α, we may assume $K' = K$ and hence $w' = v$. By the assumption that the ramification is bounded by v, the trace map $\mathrm{Tr}_{L/K} : L \to K$ induces a map $T_{L/K} : m_L^w/m_L^{w+1} \to m_K^v/m_K^{v+1}$ and the diagram

$$\begin{array}{ccc} 1 + m_L^w/1 + m_L^{w+1} & \longrightarrow & m_L^w/m_L^{w+1} \\ {\scriptstyle N_{L/K}}\downarrow & & \downarrow{\scriptstyle T_{L/K}} \\ 1 + m_K^v/1 + m_K^{v+1} & \longrightarrow & m_K^v/m_K^{v+1} \end{array}$$

is commutative. Now the assertion follows immediately from the definition of the different $D_{L/K}$.

We briefly recall the fundamental group $\pi_1(V)$ of an affine line $V \simeq \mathbb{A}^1$ over the algebraically closed field k^{sep}. It is a pro-finite $\mathbb{F}_p$-vector space and there exists a one-to-one correspondence between a surjection $\pi_1(V) \to g$ to a finite group and an (isomorphism class of) exact sequence

$$0 \to g \to V' \to V \to 0$$

of algebraic groups where V' is isomorphic to $\mathbb{A}^1$. It is known that the k^{sep}-vector space $\mathrm{Hom}_{cont}(\pi_1(\mathbb{A}^1), \mathbb{F}_p)$ is of dimension 1 and the exact sequence

$$\begin{array}{c} 0 \to \mathbb{F}_p \to \mathbb{A}^1 \to \mathbb{A}^1 \to 0 \\ \qquad\quad x \mapsto x - x^p \end{array}$$

gives a basis ([S1] 8.3 Prop. 3). By the basis, we identify

$$\mathrm{Hom}_{\mathrm{cont}}(\pi_1(V), \mathbb{F}_p) = \mathrm{Hom}_{k^{\mathrm{sep}}}(V, k^{\mathrm{sep}})$$

which determines $\pi_1(V)$ as the Pontrjagin dual.

For a positive rational number $v \in \mathbb{Q}, > 0$, let $G^v \subset G$ be the filtration by the ramification group [S3] Chap IV. Sect. 3 Remarque 1 and let G^{v+} be the closure of the union $\bigcup_{v'>v} G^{v'}$.

THEOREM 1. *There is a canonical G/P-equivariant isomorphism*

$$\pi_1(N^v) \simeq G^v/G^{v+}.$$

Proof. Let L be a finite Galois extention of K in K^{sep} with Galois group $\bar{G}$ such that $\bar{G}^{v+\varepsilon} = 1$ for any $\varepsilon > 0$. Let K' be the subextention corresponding to $\bar{G}^v$. Then the ramification of K' over K is bounded by v and $\psi_{K'/K}(v)$ is an integer. By [S3] Chap. V Sect. 6 Prop.9, we have an exact sequence

$$1 \to \bar{G}^v \to M_L^{\psi_{L/K}(v)} \to M_{K'}^{\psi_{K'/K}(v)} .$$

Since $D_{K'/K}$ induces an isomorphism $N^v \simeq M_{K'}^{\psi_{K'/K}(v)} \otimes k^{\text{sep}}$, it defines an extention of an affine line N^v by the group $\bar{G}^v$ which is a finite abelian group killed by p. By passing to the limit, we obtain a surjection $\pi_1(N^v) \to G^v/G^{v+}$.

We prove the injectivity. First we reduce to the case where $v = n$ is an integer. By the definition above, if the ramification of a finite separable extension L over K is bounded by v, we have a commutative diagram

$$\begin{array}{ccc} \pi_1(N_L^{\psi(v)}) & \longrightarrow & G_L^{\psi(v)}/G_L^{\psi(v)+} \\ {\scriptstyle (\times D_{L/K})_*}\downarrow & & \downarrow \\ \pi_1(N_K^v) & \longrightarrow & G_K^v/G_K^{v+}. \end{array}$$

Here the vertical arrows are isomorphisms. By taking such an L with $e_{L/K}\cdot v$ integer, we can make $\psi(v)$ integer. Thus we may assume $v=n$ is an integer.

Further we may assume k is algebraically closed. Then the induced map $\pi_1(N^n)\to G^n/G^{n+}\to G^{ab\ n}/G^{ab\ n+1}$ coincide with the graded piece of the isomorphism [S2] 2.4

$$\theta:\pi_1(U)\to G^{ab}$$

since θ is defined by the covering defined by the norm. Hence it is injective by Theorem 1 loc. cit. 4.1 and is an isomorphism.

By the proof above, the isomorphism in Theorem is characterised by the following

COROLLARY 1. 1. *For a finite separable extension L over K with ramification bounded by v, we have a commutative diagram*

$$\begin{array}{ccc}\pi_1(N_L^{\psi(v)}) & \longrightarrow & G_L^{\psi(v)}/G_L^{\psi(v)+}\\ {\scriptstyle\alpha_{L/K*}}\downarrow & & \downarrow\\ \pi_1(N_K^v) & \longrightarrow & G_K^v/G_K^{v+}\end{array}$$

2. *For an integer $v=n$, the composition*

$$\pi_1(N^n)\to G_K^n/G_K^{n+}\to G_{K^{nr}}^{ab\ n}/G_{K^{nr}}^{ab\ n+1}$$

is the graded piece of the isomorphism

$$\pi_1(U_{K^{nr}})\to G_{K^{nr}}^{ab}$$

of [S2].

For finite residue field case, we have

COROLLARY 2. *If k is finite of order q, the isomorphism for an integer $v=n$ satisfies the commutative diagram*

$$\begin{array}{ccc}\pi_1(N^n) & \longrightarrow & G_K^n/G_K^{n+}\\ \downarrow & & \downarrow\\ M_K^n & \longrightarrow & (G_K^{ab})^n/(G_K^{ab})^{n+1}.\end{array}$$

Here the left vertical arrow is defined by the exact sequence

$$0\to M_K^n\to N^n\xrightarrow{1-\varphi}N^n\to 0$$

where the Frobenius φ is induced by $x\mapsto x^q$, the right vertical is the canonical one and the bottom horizontal arrow is induced by the reciprocity map $K^\times\to G_K^{ab}$ of the local class field theory sending a prime element to a geometric Frobenius.

Proof. It follows from Corollary 1.2 and the commutative diagram

$$\begin{array}{ccc}\pi_1(U) & \longrightarrow & G_{K^{nr}}^{ab}\\ \downarrow & & \downarrow\\ U_K\subset K^\times & \longrightarrow & G_K^{ab}.\end{array}$$

Here the left vertical arrow is defined by the exact sequence

$$0 \to U_K \to U \xrightarrow{1-\varphi} U \to 0$$

the horizontal arrows are the reciprocity maps of the local class field theory for algebraically closed field and finite residue field respectively and the right vertical arrow is the canonical map. The commutativity follows from the definition of the reciprocity map in [Ha] which is the inverse of ours.

2. Refined break and refined conductor.

We define a refinement of the notion of break of a continuous representation of G and give its relation with the refined Swan conductor defined by Kato.

Let C be an algebraically closed field of characteristic $\neq p$ and $\rho : G \to GL_C(V)$ be a continuous representation of G on a C-vector space V of finite dimension. We say that a rational number v is a break of V if $V^{G^v} \subsetneqq V^{G^{v+}}$. We put $m_v(V) = \dim V^{G^{v+}}/V^{G^v}$. Then v is a break of V if and only if $m_v(V) \neq 0$. We say V is pure of break v if $m_v(V) = \dim V \neq 0$ namely $V = V^{G^{v+}} \neq V^{G^v} = 0$. We know that the Swan conductor is given by the formula

$$\mathrm{sw}(V) = \sum_{v \in \mathbb{Q}} v \cdot m_v(V)$$

and is an integer.

We define a refinement. Let $\mu = \mu_p(C) = \{\zeta \in C; \zeta^p = 1\}$ and N^{v*} denote the k^{sep}-vector space $\mathrm{Hom}_{k^{\mathrm{sep}}}(N^v, k^{\mathrm{sep}}) \otimes_{\mathbb{F}_p} \mu$. By the isomorphism $\pi_1(N^v) \to G^v/G^{v+}$ defined in section 1 and by the canonical isomorphism $\mathrm{Hom}_{k^{\mathrm{sep}}}(N, k^{\mathrm{sep}}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1(N^v), \mathbb{F}_p)$, we identify $N^{v*} = \mathrm{Hom}_{\mathrm{cont}}(G^v/G^{v+}, \mu)$. A natural action of G/P on N^{v*} is induced by that on N^v, or equivalently by that on G^v/G^{v+}. We consider an orbit Σ of $N^{v*} - \{0\}$ with respect to the action of G/P. We say that an orbit σ is a refined break of a representation V, if v is a break of V and if an (and hence any) element σ of Σ appears in the representation $V^{G^{v+}}$ of G^v/G^{v+}. We put $m_\Sigma(V)$ to be the multiplicity of a character $\sigma \in \Sigma$ in the representation $V^{G^{v+}}$ of G^v/G^{v+}. It is independent of the choice of σ and is well-defined. Obviously we have

$$m_v(V) = \sum \mathrm{Card}\,(\Sigma) \cdot m_\Sigma(V)$$

where Σ runs the orbits of $N^{v*} - \{0\}$. We say V is pure of refined break Σ if $\dim V = \mathrm{Card}\,(\Sigma) m_\Sigma(V) \neq 0$ namely if V is pure of break v and if the representation of G^v/G^{v+} on $V = V^{G^{v+}}$ is $(\bigoplus_{\sigma \in \Sigma} \sigma)^{\dim V/\mathrm{Card}\,\Sigma}$. An irreducible wildly ramified representation has a pure refined break.

We briefly recall the definition of the refined Swan conductor $\mathrm{rsw}(V)$ of a continuous representation $\rho : G \to GL_C(V)$ [K] (3.1). For a finite totally ramified Galois extension L over K with Galois group $\bar{G}$ of degree n a power of p, let $s_{\bar{G}} : \bar{G} \to L^\times/1 + m_L$ be the function

$$s_{\bar{G}}(\sigma) = \begin{cases} (1 - (\sigma(\pi)/\pi))^{-1} & \sigma \neq 1 \\ D_{L/K} & \sigma = 1 \end{cases}$$

which is independent of choice of a prime element π of L. Since $D_{L/K} = f'(\pi)/\pi^{n-1} = \prod_{\sigma\neq 1\in\bar{G}} (1-(\sigma(\pi)/\pi))$ for the minimal polynomial of π over K, we have $\prod_{\sigma\in\bar{G}} s_{\bar{G}}(\sigma) = 1$.

Let $\rho : G_K \to GL_C(V)$ be a continuous representation. Take a finite Galois extension L such that ρ factors the finite quotient $\bar{G} = \mathrm{Gal}\,(L/K)$. Then the refined Swan conductor is almost defined by the formula

$$rsw'_{L/K}(\rho) = \sum_{\sigma\in P_{\bar{G}}} s_{P_{\bar{G}}}(\sigma) \otimes \mathrm{Tr}\rho(\sigma) \in (L^\times/1+m_L)\otimes_{\mathbb{Z}} \mathcal{O} \subset N^\times \otimes_{\mathbb{Z}} \mathcal{O}$$

upto the slight modification below. Here $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}$ in C and $P_{\bar{G}}$ is the subgroup of $\bar{G}$ corresponding to the maximal tamely ramified extension in L.

Let $N(\mu)^\times = \amalg_{v\in\mathbb{Q},i\in\mathbb{Z}}(N^v\otimes_{\mathbb{F}_p}\mu^{\otimes i}-\{0\})$ be the multiplicative group of the bases of the $k^{\,\mathrm{sep}}$-vector spaces $N^v \otimes_{\mathbb{F}_p} \mu^{\otimes i}$ for $v \in \mathbb{Q}$ and $i \in \mathbb{Z}$ with natural (tensor) product. It is an extension of $\mathbb{Z}$ by $N^\times$. We define a morphism $N(\mu)^\times \to N^\times \otimes_{\mathbb{Z}}\mathcal{O}$ to be the identity on $N^\times$ and $1\otimes\zeta \mapsto \varepsilon(\zeta) = \sum_{r\in\mathbb{F}_p^\times} [r]\otimes\zeta^r$ for a basis ζ of μ. It is well-defined since $\varepsilon(\zeta^r) = [r]\cdot\varepsilon(\zeta)$.

THEOREM K (Theorem (3.4) [K]). *There exists a unique non-zero element*

$$rsw(\rho) \in M^{\mathrm{sw}(\rho)} \otimes \mu^{\otimes -\dim V/V^P}$$

whose image is equal to

$$rsw'_{L/K}(\rho)\cdot m^{-\dim V/V^P}\ .$$

Here L is as above and m is the prime-to-p part of the ramification index $e_{L/K}$,

We call $\mathrm{rsw}(\rho)$ the refined Swan conductor of ρ.

For an orbit $\Sigma \subset N^{v*} - \{0\}$, we define its norm $N(\Sigma) \in N^{-v\cdot\mathrm{Card}\,\Sigma} \otimes \mu^{\otimes\mathrm{Card}\,\Sigma}$ to be the product $\prod_{\sigma\in\Sigma}\sigma$. It is invariant under the action of G_K.

THEOREM 2. *Let $\rho : G_K \to GL_C(V)$ be a continuous representation such that $V^P = 0$. Then we have*

$$rsw\rho = (\prod_{\Sigma} N(\Sigma)^{m_\Sigma(V)})^{-1}$$

where Σ runs the refined breaks of V.

COROLLARY 1. *Let V be as in Theorem 2 and L be a finite separable extension whose ramification is bounded by any break v of V. Then we have*

$$rsw_K\rho = rsw_L(\mathrm{Res}^{G_K}_{G_L}\rho)\cdot D_{L/K}^{\deg\rho}\ .$$

Proof of Theorem 2. First, we prove the case where $\rho = \chi$ is a character of order p. Let L be the corresponding totally ramified cyclic extension of degree p and

$t = ord_L(a)$ where $\sigma(\pi)/\pi = 1 + a$ for a prime element π of L and a generator σ of Gal (L/K). By [K] Lemma (3.9), the refined Swan conductor is given by the formula

$$\mathrm{rsw}(\chi) = -N_{L/K}a \otimes \chi(\sigma)^{\otimes -1} .$$

We compute the refined break. Let $f \in K[T]$ be the minimal polynomial of π. Since the refined different is given by

$$D_{L/K} = \frac{f'(\pi)}{\pi^{p-1}} = \prod_{p=1}^{p-1}\left(1 - \frac{\sigma^i(\pi)}{\pi}\right) \equiv (p-1)!a^{p-1} = -a^{p-1} ,$$

we have

$$Tr_{L/K}(xa) \equiv -xa^p \equiv x(-N_{L/K}a) .$$

Hence we have

$$\begin{aligned} N_{L/K}(1 + xa) &\equiv 1 + Tr_{L/K}(xa) + N_{L/K}(xa) \\ &\equiv 1 + (x - x^p)(-N_{L/K}a) . \end{aligned}$$

Namely we have an isomorphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{F}_p & \longrightarrow & \mathbb{A}^1 & \xrightarrow{x \mapsto x - x^p} & \mathbb{A}^1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Gal}\,(L/K) & \xrightarrow[\sigma \mapsto \sigma(\pi)/\pi]{} & N_L^t & \xrightarrow[N_{L/K}]{} & N_K^t . \end{array}$$

Here the vertical arrows are $1 \mapsto \sigma$, $x \mapsto xa$, $y \mapsto y(-N_{L/K}a)$ respectively and $N_L^t \to N_K^t$ is the induced map by the norm $N_{L/K}$. It means that the refined break is given by

$$(-N_{L/K}a)^{\otimes -1} \otimes \chi(\sigma) \in \mathrm{Hom}(N_K^t, k^{\,\mathrm{sep}}) \otimes \mu = N^{t*} .$$

Thus theorem is proved in this case.

We reduce the general case to the case of characters of order p. Let $\bar{G} =$ Gal $(L/K) = \mathrm{Im}\rho$ be the finite image of $\rho : G_K \to \mathrm{GL}(V)$. First we reduce to the case where L is totally ramified of order a power of p. Let K' be the maximal tamely ramified extension in L so that L over K' satisfies the condition. By [K] Lemma 3.2 (3), we have

$$\mathrm{rsw}_K(\rho) = \mathrm{rsw}_{K'}(\mathrm{Res}\rho) \cdot D_{K'/K}^{\deg \rho}$$

where the different $D_{K'/K}$ is the class of the ramification index $e_{K'/K} \in k^\times$. We show that

$$\prod_{\Sigma_K} N(\Sigma_K)^{m_{\Sigma_K}(V)} = \prod_{\Sigma_{K'}} N(\Sigma_{K'})^{m_{\Sigma_{K'}}(\mathrm{Res}V)} \times D_{K'/K}^{-\deg \rho} .$$

For $v \in \mathbb{Q}, > 0$ and a non-trivial character $\sigma : G_K^v/G_K^{v+} \to \mu$, let $\sigma_{K'}$ denote the composite $G_{K'}^{\psi(v)}/G_{K'}^{\psi(v)+} \to G_K^v/G_K^{v+} \to \mu$. Then the multiplicity of σ in $V^{G_K^{v+}}$ is equal to that of $\sigma_{K'}$ in $V^{G_{K'}^{\psi(v)+}}$ and σ regarded as in $N_{K'}^{v*}$ is $D_{K'/K}^{-1}$-times $\sigma_{K'}$ regarded as in $N_{K'}^{\psi(v)*}$ by Corollary 1 of Theorem 1. Thus the equality is proved and we may assume $K' = K$ and L is totally ramified of order a power of p.

We reduce to the case of character. We may assume ρ is irreducible. Since $\bar{G}$ is a p-group, it is monomial and there exists a subextension $K' \subset L$ and a character of χ of Gal (L/K') such that $\rho = \text{Ind}\ \chi$. Then by loc. cit. Prop. (3.3) (2), we have

$$\text{rsw}_K(\rho) = (\text{rsw}_{K'}(\chi) \cdot D_{K'/K})^{[K';K]} .$$

We consider the right hand side of the theorem. Let v be the break of ρ. Since ρ is irreducible and $\bar{G}^v$ is in the center of $\bar{G}$, the restriction of ρ to $\bar{G}^v$ is the direct sum of copies of an isomorphism $\sigma : \bar{G}^v \to \mu$. Further $\bar{G}^v \subset$ Gal (K'/K) and the ramification of K' is bounded by v. Hence V is purely of refined break σ and, similarly as above, we have $\sigma = \sigma_{K'} \cdot D_{K'/K}^{-1}$ where $\sigma_{K'}$ is the refined break of χ. Thus the right hand side is

$$\sigma^{\deg V} = (\sigma_{K'} \times D_{K'/K}^{-1})^{\deg V} ,$$

and it is reduced to the case of characters.

Finally we assume $\rho = \chi$ is a character. Let $K' \subset L$ be the subextension with $[L : K'] = p$. By [K] Lemma (3.10), we have

$$\text{rsw}_K(\chi) = \text{rsw}_{K'}(\text{Res}\chi) \cdot D_{K'/K} .$$

By a similar argument as above we have

$$\sigma_K = \sigma_{K'} \cdot D_{K'/K}^{-1}$$

and it is reduced to the case where χ is of order p and the proof is completed.

Proof of Corollary 1. By a similar argument as in the proof of Theorem 2 for $v \in \mathbb{Q}$ bounding the ramification of L, for $\sigma \in N^{v*}, \neq 0$, the multiplicity of σ in $V^{G_K{}^{v+}}$ is equal to that of $\sigma_L = \sigma_K \cdot D_{L/K}$ in $V^{G_L{}^{\psi(v)+}}$. Now the assertion follows by Theorem 2.

COROLLARY 2. *Assume k is finite and let $\psi_0 : k \to \mu$ be a non-trivial additive character. Let $\chi : G_K^{ab} \to C^\times$ be a wildly ramified character and, by the reciprocity map of the local class field theory sending a prime element to a geometric Frobenius, we identify it with a character $\chi : K^\times \to C^\times$. Then the refined Swan conducter is given by*

$$\text{rsw}(\chi) = y^{-1} \otimes \psi_0^{\otimes -1}$$

where $y \in m_K^{-\text{sw}(\chi)}$ is characterized by

$$\chi(1 + x) = \psi_0(yx)$$

for $x \in m_K^{\mathrm{sw}(\chi)}$.

Proof. By Theorem 2, it is enough to compute the refined break of χ. Hence it follows from Corollary 2 of Theorem 1.

3. Relation with ε-factors.

We study the relation with formulas for local epsilon-factors in [D-He] and [He]. First we study the multiplicative group $N^\times$ of N. We have an exact sequence

$$0 \to k^{\,\mathrm{sep}\times} \to N^\times \xrightarrow{ord} \mathbb{Q} \to 0\ .$$

We show that the canonical map $K^\times/1+m_K \to N^\times$ induces an isomorphism $K^\times/1+m_K \otimes \mathbb{Z}[\frac{1}{p}] \to (N^\times)^{G_K}$ onto the G_K-fixed part. In fact, we have a commutative diagram of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k^\times & \longrightarrow & K^\times/1+m_K \otimes \mathbb{Z}[\frac{1}{p}] & \longrightarrow & \mathbb{Z}[\frac{1}{p}] & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (k^{\,\mathrm{sep}\,\times})^{G_K} & \longrightarrow & (N^\times)^{G_K} & \longrightarrow & \mathbb{Q}. & & \end{array}$$

The image of $(N^\times)^{G_K} \to \mathbb{Q}$ is in $\mathbb{Z}[\frac{1}{p}]$ since the action of I/P on N^v is non-trivial if $v \notin \mathbb{Z}[\frac{1}{p}]$. For a finite separable extension L, we identify $L^\times/1+m_L \otimes \mathbb{Z}[\frac{1}{p}] = (N^\times)^{G_L}$.

Assume the residue field k is finite of order q. Let $\psi_0 : k \to C^\times$ be a non-trivial character and μ be a C-valued Haar measure of K such that $\mu(\mathcal{O}_K) = q$. For a continuous representation ρ of G_K on a C-vector space V, we consider the image

$$\varepsilon_K(V, \psi_0) \in C^\times \otimes \mathbb{Z}[\frac{1}{p}] = C^\times/\mu_{p^\infty}$$

of the local epsilon-factor $\varepsilon_K(V, \psi, \mu)$ [D1]. Here μ_{p^∞} is the group of roots of unity with p-power order and $\psi : K \to C^\times$ is an additive character annihilating the maximal ideal m_K and inducing ψ_0 on $k = \mathcal{O}_K/m_K$. It is well-defined since ψ is unique modulo multiplication by a pro-p group $1 + m_K$.

Since the epsilon-factor $\varepsilon_K(V, \psi, \mu)$ is well understood for tamely ramified V, we assume $V^P = 0$ in the following. Further without any loss of generality, we assume V is pure of break Σ for some orbit Σ of $N^{v*} - \{0\}$. To state a formula for $\varepsilon_K(V, \psi_0)$, we introduce invariants $\widetilde{\det}\ \rho(\Sigma)$ and $\tau(\Sigma)$ in $C^\times \otimes \mathbb{Z}[\frac{1}{p}]$. First we note that the basis $\psi_0 \in \mathrm{Hom}_{\mathbb{F}_p}(k, \mu)$ of a k-vector space of dimension 1 defines an isomorphism

$$\begin{aligned} N^{v*} &= \mathrm{Hom}_{\mathbb{F}_p}(\pi_1(N^v), \mu) \to \mathrm{Hom}_{k^{\,\mathrm{sep}}}(N^v, k^{\,\mathrm{sep}}) \otimes_k \mathrm{Hom}(k, \mu) \\ &\xrightarrow{\psi_0} \mathrm{Hom}_{k^{\,\mathrm{sep}}}(N^v, k^{\,\mathrm{sep}}) = N^{-v}. \end{aligned}$$

Hence by Theorem 1 $\pi_1(N^v) \simeq G^v/G^{v+}$, we identify

$$\mathrm{Hom}_{\mathbb{F}_p}(G^v/G^{v+}, \mu) = N^{-v}$$

and we regard Σ as an orbit of $N^{-v} - \{0\}$.

Take an element $\sigma \in \Sigma$ and let L be the tamely ramified extension of K corresponding to the fixing subgroup of σ. Let ρ_σ be the subrepresentation of the restriction $\mathrm{Res}^{G_L}_{G_K} \rho$ on which G^v_K/G^{v+}_K acts by σ. By local class field theory, the determinant $\det \rho_\sigma : G^{ab}_L \to C^\times$ defines a character $L^\times \to C^\times$ and hence induces $\det \rho_\sigma : L^\times/U^1 \otimes \mathbb{Z}[\frac{1}{p}] \to C^\times \otimes \mathbb{Z}[\frac{1}{p}] = C^\times/\mu_{p^\infty}$. We define $\widetilde{\det}\ \rho(\Sigma) \in C^\times/\mu_{p^\infty}$ to be the value of $\det \rho_\sigma$ evaluated at $\sigma \times D_{L/K} \in (N^\times)^{G_L} = L^\times/U^1 \otimes \mathbb{Z}[\frac{1}{p}]$. It is independent of choice of $\sigma \in \Sigma$.

We define $\tau(\Sigma)$. Let m be the prime-to-p part of the denominator of v so that $w = mv \in \mathbb{Z}[\frac{1}{p}]$. Let $\tau = \sum_{x \in k} \psi_0(x^2/2)$ be the quadratic Gauss sum. We define the square residue symbol

$$(\frac{\ }{K}) : \{s \in K^\times/1 + m_k \otimes \mathbb{Z}[\frac{1}{p}]; \mathrm{ord}\ s \in 2 \cdot \mathbb{Z}[\frac{1}{p}]\} \to \{\pm 1\}$$

to be the unique surjection such that $(\frac{s^2}{K}) = 1$ for $s \in K^\times/U^1 \otimes \mathbb{Z}[\frac{1}{p}]$. Let an orbit Σ and a tamely ramified extension L be as above and $f = f_{L/K}$ be the residual degree. Then we define

$$\tau(\Sigma) = \begin{cases} -(-(\frac{m}{K})\tau^{m-1}q^{\frac{1+w}{2}}) & \text{if } \mathrm{ord}_2 v \le 0 \\ (\frac{N\Sigma}{K})(\tau^m q^{\frac{w}{2}}) & \text{if } \mathrm{ord}_2 v > 0. \end{cases}$$

In the first (resp. second) case, $\frac{1+w}{2}$ (resp. $\frac{w}{2}$) $\in \mathbb{Z}[\frac{1}{p}]$ and hence $q^{\frac{1+w}{2}}$ (resp. $q^{\frac{w}{2}}$) $\in C^\times/\mu_{p^\infty} = C^\times \otimes \mathbb{Z}[\frac{1}{p}]$ is well-defined.

THEOREM 3. (cf. Theorem [He]) *Let the notation be as above. Then we have*

$$\varepsilon_K(\rho, \psi_0) = \widetilde{\det}\ \rho(\Sigma)^{-1} \cdot \tau(\Sigma)^{m_\Sigma(V)}$$

in $C^\times/\mu_{p^\infty} = C^\times \otimes \mathbb{Z}[\frac{1}{p}]$.

COROLLARY (cf. Th. 4.2 [D-He]). *Assume* $V^P = 0$. *Then for a tamely ramified representation* $\pi : G_K \to GL_C(W)$, *we have*

$$\varepsilon_K(\rho \otimes \pi, \psi_0) = \det \pi(\mathrm{rsw}\rho) \cdot \varepsilon_K(\rho, \psi_0)^{\deg \pi}$$

in $C^\times/\mu_{p^\infty} = C^\times \otimes \mathbb{Z}[\frac{1}{p}]$. *Here* $\det \pi(\mathrm{rsw}\rho)$ *is the value of the character* $\det \pi : K^\times/1 + m_K \subset K^\times \otimes \mathbb{Z}[\frac{1}{p}] \to C^\times \otimes \mathbb{Z}[\frac{1}{p}]$ *at* $\mathrm{rsw}\rho \in K^\times/1 + m_K$.

Proof of Corollary. We may assume ρ is pure of refined break Σ. Then $\rho \otimes \pi$ is also pure of refined break Σ and we have

$$\widetilde{\det}\ \rho \otimes \pi(\Sigma) = \widetilde{\det}\ \rho(\Sigma)^{\deg \pi} \cdot \det \pi(\mathrm{rsw}\rho)$$

by Theorem 2. Now it follows immediately from Theorem 3.

Proof of Theorem 3. First we check that it is exactly Theorem [He] if we assume $\Sigma = \{\sigma\}$ for $\sigma \in (N^\times)^{G_K} = K^\times/1 + m_K \otimes \mathbb{Z}[\frac{1}{p}]$. Note that the assumption means that ρ is homogeneous in the terminology loc. cit. We show that the g–invariant g loc. cit. is equal to σ. Let $\bar{G} = \mathrm{Im}\rho = \mathrm{Gal}\ (L/K)$ and $K' \subset L$ be the subextension corresponding to $\bar{G}^v$ for the break v of ρ. Let $\chi : \bar{G}^v = \mathrm{Gal}\ (L/K') \to \mu$ be the central character of ρ. Then the equality

$$\chi(1+x) = \psi(\mathrm{Tr}_{K'/K}gx)$$

for $x \in m_{K'}^{\psi(v)-1}$, which is equivalent to that in l.-3 p.120 loc. cit., implies that

$$\mathrm{rsw}_{K'}(\chi) = (g \cdot D_{K'/K})^{-1}$$

by Corollary 2 of Theorem 2 . Thus we have $g = \sigma$.

Since $\varepsilon'(\rho)$ loc. cit. is $\varepsilon(\rho \cdot w_{\frac{1}{2}}, \psi, q^{-\frac{1}{2}}\mu)$ in our notation, we have

$$\varepsilon'(\rho) = q^{\frac{1}{2}(\mathrm{sw}(\rho)+\deg\rho)}\varepsilon(\rho,\psi,\mu)$$

using ord $\psi = -1$. Therefore to complete the proof, it is enough to check that

$$\tau(\sigma) = \begin{cases} q^{\frac{1+v}{2}} & \mathrm{ord}_2 v = 0 \\ (\frac{\sigma}{K}) \cdot q^{\frac{v}{2}} \cdot \tau & \mathrm{ord}_2 v > 0. \end{cases}$$

is $G(\sigma) \times q^{\frac{1+v}{2}}$ loc.cit.. It is quite elementary and left to the leader.

We reduce the general case to the case proved above. Let L be the fixed field corresponding to the fixing group of an element $\sigma \in \Sigma$ and ρ_L be the subrepresentation of $\mathrm{Res}^{G_K}_{G_L}\rho$ consisting on which G_K^σ acts by σ. Then we have $\rho = \mathrm{Ind}\rho_L$ and the break of ρ_L consists of $\{\sigma_L\}$ where σ_L is the composite $N_L^{\psi(v)} \xrightarrow{\times D_{L/K}} N_K^v \xrightarrow{\sigma} \mu$. We prove

$$\frac{\varepsilon_K(\rho,\psi_0)}{\varepsilon_L(\rho_L, \psi_0 \circ Tr_{E/F}) \times \det\rho_L(D_{L/K})} = ((-1)^{f-1}((\frac{m}{F}) \cdot \tau_F^{m-1})^f)^{\deg\rho_L}) \ .$$

We show it completes the proof. In fact Theorem for ρ_L which is already proved gives

$$\varepsilon_L(\rho_L, \psi_0 \circ Tr_{E/F}) = \det\rho_L(\sigma_L)^{-1} \times \tau_L(\sigma_L)^{\deg\rho_L} \ .$$

By $\sigma^{-1} = \sigma_L^{-1}D_{L/K}$, it is enough to show

$$\tau(\Sigma) = (-1)^{f-1}((\frac{m}{F})\tau_F^{m-1})^f \times \tau_L(\sigma_L) \ .$$

By considering the action of inertia, we have $e_{L/K} = m$. Hence ord ${}_2v \le 0$ (resp. > 0) is equivalent to ord ${}_2w = 0$ (resp. > 0). Now it is easily deduced from $\tau_E = (-1)^{f-1}\tau_F$ and $(\frac{\sigma_L}{L}) = (\frac{N\Sigma}{K}) \cdot (\frac{m}{F})^F$.

Finally we prove the equality above. By the induction property of the epsilon factor, we have

$$(\text{left hand side}) = \Big(\frac{\varepsilon_K(\mathrm{Ind}1, \psi_0)}{\varepsilon_L(\rho_L \cdot \psi_0 \mathrm{Tr}_{L/K})}\Big)^{\deg \rho}.$$

We give a proof of the equality

$$\frac{\varepsilon_K(\mathrm{Ind}1, \psi_0)}{\varepsilon_L(\rho_L, \psi \circ \mathrm{Tr}_{L/K})} = (-1)^{f-1}((\frac{m}{F})\tau_F^{m-1})^f$$

by mimicing an argument by Henniart [He]. Let χ be a wildly ramified character of $K^\times$ of order p-power such that $v = \mathrm{sw}(\chi)$ is even and $s = \mathrm{rsw}(\chi)$ is a square. Applying Theorem for $\chi_L = \mathrm{Res}^{G_K}_{GL}\chi$ and $\rho = \mathrm{Ind}^{G_K}_{GL}\chi_L$ which is already proved and using the induction property, we have

$$(\text{left hand side}) = \frac{\varepsilon_K(\rho \cdot \psi_0)}{\varepsilon_L(\chi_L, \psi \circ \mathrm{Tr}_{L/K})} = \frac{\det\rho(s)^{-1}}{\chi_L(\rho \cdot m^{-1})^{-1}} \frac{((\frac{s}{K})q_F^{\frac{v}{2}}\tau_F)^{[L:K]}}{((\frac{s \cdot m^{-1}}{L})q_E^{\frac{mv}{2}}\tau_E)}.$$

By $\deg\rho = \chi^{[L:K]} \otimes \det \mathrm{Ind}1$ and by $\det \mathrm{Ind}1(s) = 1$ since s is a square, the first term in the right hand side is a root of unity of p-power order. Hence the right hand side is $(\frac{m}{E}) \cdot \tau_F^{[L:k]} \cdot \tau_E^{-1} = (-1)^{f-1}((\frac{m}{F})\tau_F^{m-1})^f$ and the proof is completed.

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