Betti and de Rham epsilon lines/unfinished

Hélène Esnault, j.w. A. Beilinson, S. Bloch, P. Deligne

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caution

- The article underlying the content of the talk was written during the years 2004-5. Since then, no one has reviewed it. Even if we do not hope so, there might be inaccuracies in it, perhaps more.
- The aim of the lecture, the subject of which has been explicitely requested by Ahmed and Takeshi, is to give the Leitfaden of our idea. Details are written, but not checked carefully.

global periods

- $X, M = (E, \nabla)$ smooth, not necessarily proper curve with an algebraic connection defined over $k \subset \mathbb{C}$. Irregular Riemann-Hilbert correspondence: $\mathcal{M}^{mg} \subset (E \otimes \mathbb{C})^{\nabla,*}$ *moderate growth* constructible subsheaf of extension of local system on real blow up X^*_{an} of poles of ∇: then $H^i_{DR}(X, M) \otimes_k \mathbb{C} \cong H^i(X^*_{\mathrm{an}}, \mathcal{M}^{mg})$. Equivalently, define *rapid decay homology* $H^{\mathrm{rd}}_i(X_{\mathrm{an}}, M)$ via pairs (γ, ϵ) of (i-cycle with boundary in the poles of ∇, solution), where the solution rapidly decays along γ.
- Then has *period* exact pairing

$$\mathfrak{per}: \; H^{\mathrm{rd}}_i(X_{\mathrm{an}}, M^\vee) \times H^i_{DR}(X, M) \otimes_k \mathbb{C} \xrightarrow{\int} \mathbb{C},$$
$$((\gamma, \epsilon^\vee), \alpha) \mapsto \int_\gamma \langle \epsilon^\vee, \alpha \rangle$$

- yields: $\begin{cases} k - \mathrm{line}\ \epsilon_{DR}(X, M) := \det H^*_{DR}(X, M) \\ \mathbb{C} - \mathrm{line}\ \epsilon_B(X, M) := \det H^*(X^*_{\mathrm{an}}, \mathcal{M}^{mg}) \\ \mathrm{iso}\ \gamma : \epsilon_{DR}(X, M) \cong \epsilon_B(X, M) \end{cases}$

global periods: example

- $X = \mathbb{G}_m = \mathbb{P}^1 \setminus \{0, \infty\}$ $M = (\mathcal{O}, \nabla)$ defined by $\nabla(1) = -dt + s\frac{dt}{t}$ with $s \in \mathbb{C} \setminus \mathbb{Z}$
- The de Rham structure is defined over $k = \mathbb{Q}(s)$.
- The horizontal sections are spanned by $\exp(t)t^{-s}$, so the monodromy field is $K = \mathbb{Q}(\exp(2\pi is))$.
- H^1_{DR} is one dimensional, spanned by dt/t. If one takes as Betti structure on the dual local system the reduction to K given by $K \cdot \exp(-t)t^s$, then $H_{1,B}$ is spanned by $\exp(-t)t^s|_\sigma$ where σ is the keyhole shaped path running from $+\infty$ to $\epsilon > 0$, then counterclockwise around a circle of radius ϵ about 0, and finally returning to $+\infty$.
- One gets $\epsilon_{DR}(\mathbb{G}_m, M)/\epsilon_B(\mathbb{G}_m, M) = k^\times \cdot K^\times \cdot \int_0^\infty exp(-t)t^s \frac{dt}{t} = k^\times \cdot K^\times \cdot \Gamma(s)$.

Stokes structures

- Riemann-Hilbert (reg. sing. conn) $\leftrightarrow$ (loc. syst.) extends to (mero. conn) $\leftrightarrow$ (loc. syst. + Stokes str.)
- *polar part local system:* $\forall x \in \bar{X} \setminus X$ (assume $x \in \bar{X}(k)$), $D^\times$ analytic disk at x, define local system $\Omega :=$(polar parts of 1 forms in x in all coverings modulo the log ones), i.e. $\Omega = \{\sum_{i=-n}^{\infty} a_i z^{i/p} dz\}/\{\sum_{i/p \geq -1}\}$: local system on S^1
- *Stokes lines of $\eta \in \Omega$:* for almost all $\theta \in S^1$, in small sector around θ, solution of either $(\mathcal{O}, d - \eta)$ or $(\mathcal{O}, d + \eta)$ has rapid decay or rapid growth, i.e. $\exp(\int^z \eta) = O(|z|^{-N})$ for all N. $\rightsquigarrow \exists$ finitely many (*Stokes*) lines θ for which no such comparison exists.
- *Turrittin-Levelt decomposition:* $M/k((t))$, then after a finite field extension $\pi : F \supset k((t))$, has $\pi^* M = \oplus$(rank 1)$\otimes \pi^*$(reg. sing.) Recall: rank 1$\cong L_\alpha := (\mathcal{O}, d + \alpha), \alpha \in \omega_{k[[t]]}[t^{-1}]$

Stokes structures 2

- Levelt decomposition is for formal connections, also for connections on small sectors, but does not extend to $D^\times$.
- *Stokes structure of* $M/D^\times$: $F = k'((u)) \supset k((t)), u^p = t$; for $\theta \in S^1$, fix lifting $\theta_u \in (S^1, u)$, $\eta \in \Omega$; Levelt implies: $\mathcal{M}^\eta_{\theta_u} = \oplus_\alpha(\mathcal{M}^\alpha_{\theta_u})$ with $\alpha - \eta$ moderate growth: filtration at θ_u independent of lifting θ_u. It defines a filtration $\mathcal{M}^\eta_\theta \subset \mathcal{M}_\theta$, and *a filtration of* $\mathcal{M}$ in a sector around θ by constructible subsheaves.
- Choose $K \subset \mathbb{C}$ containing the monodromy field (i.e. monodromy in $GL(n, K)$), and such that the subvectorspaces $\mathcal{M}^\eta_\theta$ are also defined over K. Then $H^i(X^*_{\mathrm{an}}, \mathcal{M}^{mg})$ or equivalently $H^{\mathrm{rd}}_i(X, M^\vee)$ are defined over K $\rightsquigarrow \epsilon_B(X_{\mathrm{an}}, M) = \epsilon_B(X_{\mathrm{an}}, M) \otimes_K \mathbb{C}$ (**abuse of notations!**)

Stokes structures 3

- $\mathfrak{per}$ becomes

  $\mathfrak{per}: \ H_i^{\mathrm{rd}}(X_{\mathrm{an}}, M^\vee) \otimes_K \mathbb{C} \times H^i_{DR}(X, M) \otimes_k \mathbb{C} \xrightarrow{\int} \mathbb{C}$
- In particular, $\mathfrak{per}$ yields *determinant of period* as a number in $\mathbb{C}^\times / k^\times \cdot K^\times$ (see T. Saito-Terasoma for the reg. sing. case, J. AMS 10 (1997), no 4, 865-937).

program

- given $\nu \in \omega^{\times}_{k(X)}$, define for all closed points $x \in \bar{X}$ lines
$$\begin{cases} k - \text{line } \epsilon_{DR}(x, M, \nu) \\ K - \text{line } \epsilon_B(x, M, \nu) \\ \text{iso } \gamma(x, M, \nu) : \epsilon_{DR}(x, M, \nu) \cong \epsilon_B(x, M, \nu) \end{cases}$$
- *functorially* in (X, M, ν), i.e. projection formula between $(\pi : Y \to X, M = \pi_* N, \nu)$ and $(Y, N, \pi^*\nu)$, N of virtual degree 0, and
- with *reciprocity formula*

$$\mathfrak{reciprocity}: \ \left(\epsilon_{DR}(X, M), \epsilon_B(X_{\rm an}, M), \gamma\right) =$$

$$(2\pi\sqrt{-1})^{\mathrm{rk}(M)(1-g)} \otimes_{x \in \bar{X}} \left(\epsilon_{DR}(x, M, \nu), \epsilon_B(x, M, \nu), \gamma(x, M, \nu)\right)$$

- *degree:* in fact, even super-lines, degree defined by some Euler characteristic (we don't discuss this here)

DR theory: polarized determinants

- Reference: Beilinson/Bloch/E.: Moscow Math. J. 2 (3) (2002), 1-56
- for short: given $\nu \in \omega^{\times}_{k((t))}$, assume $\exists L_1 \subset L_2 \subset E$ 2 lattices (i.e. $k[[t]]$-submodules of rank $r =$dim $E/k((t))$ and $t^{-^{\exists}N} k[[t]]^{\oplus r} \subset L_i \subset t^N k[[t]]^{\oplus r}$) with $L_1 \xrightarrow{\nu^{-1}\circ\nabla} L_2$ such that $E/L_1 \xrightarrow{\nu^{-1}\circ\nabla \ \cong} E/L_2$. Then $\epsilon_{DR}(t=0, M, \nu) = \det(L_2/L_1)$
- $\epsilon_{DR}(t=0, M, \nu)$ is a k-line
- $\mathfrak{reciprocity}$ holds for this DR definition.

DR theory: Fourier transform

- inspired by Laumon's seminal work, Publ. IHES 65 (1987), 131-210. First steps/examples: Deligne IHES seminar 1984. Try to imitate Laumon's method: develop the theory on $\mathbb{A}^1$, together with transformation rules for change of coordinates and projection formula (see above). The general case reduces to this.
- from now on M/k on $\mathbb{A}^1 \setminus S$, with coordinate t, with no singularities at ∞ (for simplicity, $S \subset \mathbb{A}^1(k)$).
- *Fourier transform* $\mathcal{F}(M) := \mathrm{Coker}(E[t'] \xrightarrow{\Phi := \nabla + t' dt} E[t'])$. As Φ lifts to the absolute connection $\nabla + d(t't)$, Gauß-Manin calculus yields a connection in t' on $\mathcal{F}(M)$. It has poles only at $t' = \infty$, and $t' = 0$ where it has reg. sing., and is smooth on $(\mathbb{A}^1 \setminus \{0\}, t')$
- *Deligne's good lattice pair:* $L_1 \subset L_2 \subset j_*E, j : \mathbb{A}^1 \setminus S \to \mathbb{P}^1$ with $\big(L_1 \xrightarrow{\nabla} \omega(S+\infty) \otimes L_2\big) \xrightarrow{\mathrm{qis}} \big(L_1(N(S+\infty)) \xrightarrow{\nabla} \omega(S+\infty) \otimes L_2(N(S+\infty))\big)$

DR theory: Fourier transform 2

- set $z = (t')^{-1}$ then $L_1 \subset L_2(\infty)$ good for $k(z)$-linear connection $\nabla + dt/z$ on $E \otimes_k k(z)$
- **Prop.:** $N >> 0$, then
  • $\mathcal{F}(E) = \mathbb{H}^1(\mathbb{P}^1_{k[z,z^{-1}]}, L_1[z,z^{-1}] \xrightarrow{z\Phi} \omega(S+\infty) \otimes L_2(\infty)[z,z^{-1}])$
  • and
  $\Psi(L_1, L_2) := \mathbb{H}^1(\mathbb{P}^1_{k[[z]]}, L_1[[z]] \xrightarrow{z\Phi} \omega(S+\infty) \otimes L_2(\infty)[[z]])$
  lattice in $\mathcal{F}(E) \otimes k((z))$.
- One defines the *local Fourier transform* by the same formula if M only defined locally $/k((t-s))$ (details in Bloch/E.: Asian J. Math. 8 (4) (2004), 587-606):
- $M/k((t-s))$, then $VC(s, M) := VC(s, L_1 \subset L_2) \otimes_{k[[z]]} k((z))$ connection over $k((z))$ with $k[[z]]$-lattice $VC(s, L_1 \subset L_2) = \mathbb{H}^1(\mathbb{P}^1_{k[[z]]}, L_1[[z]] \xrightarrow{z\Phi} \omega(S+\infty) \otimes L_2(\infty)[[z]])$, where now $L_1 \subset L_2$ good lattices on $k((t-s))$.

DR theory: Fourier transform 3

- **Facts** (as in Laumon) $\mathcal{F}(M) \otimes k((z)) = \oplus_{s \in S} VC(s, M) +$ formulae for rank, irregularity etc.
- $VC(s, M) \otimes (d - d((r + n_s)s/z))$ regular singular ($n_s =$ irregularity M at $s \in S = \dim_k L_2/L_1$) at s
- $\det \mathcal{F}(M) \otimes_{s \in S} (d - d((r + n_s)s/z))$ trivial connection.

DR theory: pol. det.= Fourier

- **Thm (DR comparison):**
  $\epsilon_{DR}(s, M, dt) = \det VC(s, L_1 \subset L_2) \otimes_{k[[z]]} k$

Deligne-Katz-Gabber extension

- $F \cong k((z))$
- Deligne: $\exists \otimes$-functor
  $RS/F \to RS/T^\times := \mathrm{Spec}(\oplus_{-\infty}^{\infty} \mathfrak{m}^n/\mathfrak{m}^{n+1})$
- choice of $z \Leftrightarrow$ iso $T^\times \cong \mathbb{G}_m \Leftrightarrow$ choice of a point in $T^\times(k)$ (then point: $z = 1$) $\rightsquigarrow \otimes$-fiber functor $RS/F \xrightarrow{\text{rest in } 1} Vec_k$
- Katz-Gabber extend to $(Conn/F, z) \to (Conn/\mathbb{G}_m) \rightsquigarrow$ fiber functor $Conn/F \xrightarrow{\text{rest in } 1} Vec_k$.
- **Fact:** $\epsilon_{DR}(s, M, dt) = \det VC(s, L_1 \subset L_2) \otimes_{k[[z]]} k =$ rest. in 1 of Katz extension of $\det(VC(s, M))$ (recall: z given in the construction)

complex Betti line

- Set $L(s, M)$ for the determinant of the Katz extension for z on $\mathbb{G}_m$ of $VC(s, M)$, $\mathcal{L}(s, M)$ for its local system.
- **Defn:** $\epsilon_B(s, M, dt) = \mathcal{L}(s, M)|_1$
- $\epsilon_B(s, M, dt)$ is a $\mathbb{C}$-line
- as $L(s, M)$ smooth on $\mathbb{G}_m$, one has immediately the comparison iso
  $\gamma(s, M, dt) : \epsilon_{DR}(s, M, dt) \otimes_k \mathbb{C} \cong \epsilon_B(s, M, dt)$
- no K-structure as it stands.
- **Goal:** show that a given K-Stokes structure on M over $\mathbb{A}^1 \setminus S$ yields a K-structure on $\epsilon_B(x, M, dt)$.

asymptotic expansions

- for f an analytic function on $D^\times$ (punctured disk), one says that $g(z) = \sum_{n \geq -N} a_n z^n \in \mathbb{C}((z))$ is its *asymptotic expansion* in a given open sector if $|f - \sum_{n=-N}^{p} a_n z^n| \leq^{\exists} C_p |z|^{p+1}$ for all $p >> 0$
- functions with an asymptotic expansion on an open sector form a ring, and those with asymptotic expansion $= 0$ form an ideal.
- $\rightsquigarrow$ one can perform with them algebraic operations.

K-Betti line

- recall $\mathcal{F}(M)$, then $\mathcal{F}(M) \otimes_k k((z)) = \oplus_{s \in S} VC(s, M)$ and can consider the local system $loc.\ sys.(\mathcal{F}(M))$ on z-disk: not a $\oplus_S$.
- choose basis $\{\epsilon_j^\vee \exp(t/z), \gamma_j\}$ for $H_1^{rd}(\mathbb{A}^1 \setminus S, \nabla + (dt)/z)$ (so $\partial\gamma_i \subset (S \cup \infty)$) and η_k basis of $\mathcal{F}(M)$, then define the period matrix
- $\mathcal{P}\mathrm{er}(z) := \Big(\int_{\gamma_j} \langle \epsilon_j^\vee \exp(t/z), \eta_k \rangle \Big)_{j,k}$
- *subtlety:* rapid decay homology defined over $\mathrm{Spec}(\mathbb{C})$. One shows $z \mapsto \mathcal{P}\mathrm{er}(z) \in \mathbb{C}$ is analytic in a small sector in z.
- **Thm:** $\mathcal{P}\mathrm{er}(z) = D \cdot Q$ with:
  asymptotic expansion of Q=asymptotic expansion of block diagonal matrix $(Q_s)_{s \in S}$ of size $r + n_s$
  $D = (D_s)_{s \in S}$ diagonal; diagonal piece corresponding to s:
  $\mathrm{Tr}(D_s) = \exp((n_s + r)s/z) \cdot z^{\alpha_s} \cdot g_s(z)$, g_s holomorphic in $z = 0$

K-Betti line 2

- proof involves the Stokes structure (filtration on the stalks $\mathcal{M}_\theta$) but not the field K
- asymptotic expansion computed with the *steepest descent* method (yields for $s \in S$ a non-degerate quadratic form)
- **Defn:** K-Betti line at s defined by $K \cdot \exp((n_s + r)s) \cdot g_s(0) \subset \epsilon_B(s, M, dt)$
- Note: this is really a K-structure as the Katz extension is compatible here with the $\otimes$ of the purely irregular piece $\exp((n_s + r)s/z)$ with the regular singular one $z^{\alpha_s} \cdot g_s(z)$.

reciprocity

- follows Laumon's idea:
- $\det H^*_{DR}(\mathbb{A}^1 \setminus S, M) \otimes \det E_\infty = \det \mathcal{F}(M)_0$ (recall M was smooth at ∞)
- $\det \mathcal{F}(M)_0 = \det \mathcal{F}(M)_0 \otimes (\otimes_{s\in S}(d - \sum(n_s + r)sdt'))_0$ =(as $\otimes$ trivial)=$\det \mathcal{F}(M)_\infty \otimes (\otimes_{s\in S}(d - \sum(n_s + r)sdt'))_\infty$
- follow now de Rham via $VC(s, M)$, and K-Betti lines defined via the global periods.
- conclusion:

$$\mathfrak{reciprocity}: \ \big(\epsilon_{DR}(X, M), \epsilon_B(X_{\rm an}, M), \gamma\big) =$$

$$(2\pi\sqrt{-1})^{\mathrm{rk}(M)} \otimes_{x\in \bar{X}} \big(\epsilon_{DR}(x, M, dt), \epsilon_B(x, \mathcal{M}, dt), \gamma(x, M, dt)\big)$$

criticism

- one uses rapid decay homology (or moderate growth constructible sheaves) in the z-family, while it is written only over $\mathrm{Spec}(\mathbb{C})$. Not nice, even if it "somehow" works.
- one uses global periods to define local K-Betti structure
- one uses the Turrittin-Levelt decomposition which is unnatural
- the very intricate (and difficult) asymptotic expansions used deal in some way with the previous criticism: one does not have a theory of periods moving with z, but has what would be the asymptotic expansion of it if one had it.
- to define $\epsilon_B(s, M, dt)$, one goes to M, then to $VC(s, M)$, and then comes back to local systems. So not nice, as one would wish a definition directly out of the moderate growth local system without any de Rham mention.
- in this respect, see Beilinson's work for on complex Betti lines: so far no comparison theorem with this definition, but one can hope for one.